AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **90(3)** (2024), Pages 313–325

The feasibility problem: the family $\mathcal{F}(G)$ of all induced G-free graphs

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Abstract

An infinite family of graphs $\mathcal{F}$ is called feasible if for any pair of integers (n, m), $n \geq 1$, $0 \leq m \leq \binom{n}{2}$, there is a member $G \in \mathcal{F}$ such that G has n vertices and m edges. We prove that given a graph G, the family $\mathcal{F}(G)$ of all induced G-free graphs is feasible if and only if G is not K_k, $K_k \backslash K_2$, $\overline{K_k}$, or $\overline{K_k \backslash K_2}$, for $k \geq 2$.

1 Introduction

The *Feasibility Problem* is an umbrella for various specific problems in extremal combinatorics. Let $\mathcal{F}$ be an infinite family of graphs. Then $\mathcal{F}$ is called feasible if for every $n \geq 1$, $0 \leq m \leq \binom{n}{2}$, there is a graph $G \in \mathcal{F}$ having exactly n vertices and m edges.

If $\mathcal{F}$ is not feasible, it is of interest to find the set of all feasible pairs, $FP(\mathcal{F}) =$
$\{(n, m) :$ there is a graph $G \in \mathcal{F}$ having exactly n vertices and m edges$\}$,
as well as the complementary set,
$\overline{FP}(\mathcal{F}) = \{(n, m) :$ no member of $\mathcal{F}$ has precisely n vertices and m edges$\}$.

If it is not possible to exactly determine these sets, we look for good estimates of $h(n, \mathcal{F}) = \frac{|FP(\mathcal{F})|}{\binom{n}{2}}$ and $g(n, \mathcal{F}) = \frac{|\overline{FP}(\mathcal{F})|}{\binom{n}{2}}$.

Also, in many cases in extremal graph theory, it is of interest to find

$$f(n, \mathcal{F}) = \min\{m : (n, m) \text{ is not a feasible pair for the familiy } \mathcal{F}\}$$

ISSN: 2202-3518

as well as

$$F(n,\mathcal{F}) = \max\{m : (n,m) \text{ is not a feasible pair for the family } \mathcal{F}\}.$$

A simple example is $\mathcal{F}$, the family of all connected graphs. Clearly every connected graph on n vertices must have at least $n-1$ edges and it is trivial to see that with the above notation, $f(n,\mathcal{F}) = 0$ and $F(n,\mathcal{F}) = n-2$. Another example is the family $\mathcal{F}$ of all planar graphs. Here it is well known that $f(n,\mathcal{F}) = 3n-5$ for $n \geq 4$ (since a maximal planar graph can have at most $3n-6$ edges for $n \geq 3$) and $F(n,\mathcal{F}) = \binom{n}{2}$ for $n \geq 5$. In both of these examples, the exact determination of $FP(\mathcal{F})$ and $\overline{FP}(\mathcal{F})$ as well as $h(n,\mathcal{F})$ and $g(n,\mathcal{F})$ is easy.

A further important example is the celebrated problem of Turán numbers $ex(n,G)$ [12, 13], which is the maximum number of edges in a graph on n vertices which does not have G as a subgraph.

Clearly, with the notation above where $\mathcal{F}$ is the family of all G-free graphs, $ex(n,G) = \min\{f(n,\mathcal{F}) - 1, \binom{n}{2}\}$.

Also for the class $\mathcal{F}$ of G-free graphs, $g(n,\mathcal{F}) \to 1$ if G is a bipartite graph, while $g(n,\mathcal{F}) \to \frac{1}{2(\chi(G)-1)}$ otherwise (by the Erdős–Stone–Simonvits theorem [7, 11]), where $\chi(G)$ is the chromatic number of G. For references to extremal graph theory we refer to [1, 4, 8].

Erdős, Füredi, Rothschild and Sós [6] initiated a study of classes of graphs that forbid every induced subgraph on a given number m of vertices and number f of edges. They used the notation $(n,e) \to (m,f)$ if every graph G on n vertices and e edges has an induced subgraph on m vertices and f edges, and they looked for pairs for which this relation does not hold, calling them avoidable pairs.

So, if we define $Q = Q(m,f) = \{G : v(G) = m, e(G) = f\}$, then, in our notation, the family $\mathcal{F}$ considered above is the family $\mathcal{F}(Q)$ of all G-free graphs where $G \in Q$. We emphasize here that the main interest in this line of research is to estimate a density measure defined by

$$\sigma(m,f) = \lim_{n\to\infty} \frac{|\{e : (n,e) \to (m,f)\}|}{\binom{n}{2}}$$

(along the lines indicated by the above definition of $g(n,\mathcal{F})$), and the proofs incorporate number theoretic arguments.

It is known that if $(m,f) \in \{(2,0),(2,1),(4,3),(5,4),(5,6)\}$, then $\sigma(m,f) = 1$; otherwise, $\sigma(m,f) \leq 1/2$ (see the references above). Also, Erdős et al. gave a construction that shows that for most pairs (m,f) we have $\sigma(m,f) = 0$. For recent papers on this highly active subject we refer to [2, 3, 9, 14].

Yet another example is given in the paper [5] by the authors—the feasibility problem for line graphs—where we solved completely $\overline{FP}(\mathcal{F})$ and hence $FP(\mathcal{F})$ when $\mathcal{F}$ is the family of all line graphs. The line graph $L(G)$ of a graph G is obtained by associating a vertex with each edge of the graph G and connecting two vertices with an edge if and only if the corresponding edges of G have a vertex in common. In particular the values of $f(n, \mathcal{F})$ and $F(n, \mathcal{F})$ are exactly determined for the family $\mathcal{F}$ of all line graphs.

Reznick [10] solved asymptotically, via a number theoretic approach, the value of $f(n, \mathcal{F}) = \frac{n^2}{2} - \sqrt(2)n^{3/2} + O(n^{5/4}))$ where $\mathcal{F}$ is the family of all induced P_3-free graphs (corresponding to $(m, f) = (3, 2)$ where clearly $\sigma(3, 2) = 0$), which are graphs represented as a vertex disjoint union of cliques, and his method is still in use in the research about the family $Q(m, f)$ defined above.

The same order of magnitude is proved for $f(n, \mathcal{F})$ in the case where $\mathcal{F}$ is the family of all line graphs of acyclic graphs, as well as the family of all line graphs [5].

Here, inspired in part by the problem launched by Erdős et al. concerning $\mathcal{F}(Q)$, and as a counterpart to the Turán problem concerning families with no subgraph isomorphic to G, we consider the case where $\mathcal{F} = \mathcal{F}(G)$ is the family of all induced G-free graphs. Clearly, if $G \in \{K_k, K_k\backslash K_2, \overline{K_k}, \overline{K_k\backslash K_2}\}$ for $k \geq 2$, then $\mathcal{F}(G)$ is trivially non-feasible, and hence we use in the sequel TNF $= \{K_k, K_k\backslash K_2, \overline{K_k}, \overline{K_k\backslash K_2}\}$ for $k \geq 2$. We prove the following, our Main Theorem, using only graph theoretic arguments:

Theorem (Main). *Let G be a graph. Then the family $\mathcal{F}(G)$ of all induced G-free graphs is feasible if and only if G is not a member of*

$$\text{TNF } = \{K_k, K_k\backslash K_2, \overline{K_k}, \overline{K_k\backslash K_2}\}$$

for $k \geq 2$.

In other words, if G is not a member of TNF, then, for every pair (n, m), $n \geq 1$, $0 \leq m \leq \binom{n}{2}$, there is an induced G-free graph with exactly n vertices and m edges.

While $f(n, \mathcal{F})$, $F(n, \mathcal{F})$, $\overline{FP}(\mathcal{F})$ and $FP(\mathcal{F})$ are determined by Turán 's Theorem for the cases $\mathcal{F} = \mathcal{F}(K_k)$ and $\mathcal{F} = \mathcal{F}(\overline{K_k})$, determining $f(n, \mathcal{F})$ and $\overline{FP}(\mathcal{F})$ for $\mathcal{F} = \mathcal{F}(K_k\backslash K_2)$, and $F(n, \mathcal{F})$ and $\overline{FP}(\mathcal{F})$ for $\mathcal{F} = \mathcal{F}(\overline{K_k\backslash K_2})$i, is not yet solved.

We have already started considering the general feasibility problem in our paper [5] where we proved that several natural families of graphs are feasible, namely $K_{1,r}$-free graphs for $r \geq 3$, P_r-free graphs for $r \geq 4$, rK_2-free graphs for $r \geq 2$, as well the family of chordal graphs, and the family of paw-free graphs.

In the rest of the paper, when we say G-free we mean induced G-free.

Our paper is organized as follows: in Section 2 we discuss some basic properties of feasible families with regards to containment and complementation. We then introduce the two main constructions crucial for the proof of the Main Theorem. The first is the UEP (Universal Elimination Process), first introduced in [5], and the second is the $\{K_3, K_2\}$-elimination process. We shall discuss some consequences of these constructions.

In Section 3 we prove the Main Theorem of this paper.

In Section 4 we offer interesting examples and questions for further research.

2 Feasible Families under containment and complementation and elimination procedures

2.1 Basic properties

The following are simple basic properties concerning feasibility subject to containment and complementation. The proofs are easy but we include them for the sake of completeness, except for the trivial proof of Proposition 2.1.

Proposition 2.1. *Let $\mathcal{F}$ and $\mathcal{H}$ be two families of graphs such that $\mathcal{H} \subset \mathcal{F}$. Then*

1. *If $\mathcal{H}$ is a feasible family then $\mathcal{F}$ is feasible family.*
2. *If $\mathcal{F}$ is not a feasible family, then $\mathcal{H}$ is not feasible family.*

Proposition 2.2.

1. *Let $\mathcal{F}$ be a family of graphs and $\overline{\mathcal{F}} = \{\overline{G} : G \in \mathcal{F}\}$. Then $\mathcal{F}$ is feasible if and only if $\overline{\mathcal{F}}$ is feasible.*
2. *Let $\mathcal{F}(G)$ be the family of all induced G-free graphs. Then $\overline{\mathcal{F}} = \mathcal{F}(\overline{G})$, the family of all induced $\overline{G}$-free graphs, is feasible if and only if $\mathcal{F}(G)$ is feasible.*

Proof.

1. Suppose $\mathcal{F}$ is feasible: given a pair (n, m) with $n \geq 1$, $0 \leq m \leq \binom{n}{2}$, there is a graph $G \in \mathcal{F}$ having n vertices and m edges. Clearly $\overline{G}$ has n vertices and $\binom{n}{2} - m$ edges. Then as m increases from 0 to $\binom{n}{2}$, $\binom{n}{2} - m$ decreases from $\binom{n}{2}$ to 0. Hence for every pair (n, m) there exists $\overline{G} \in \overline{\mathcal{F}}$ having n vertices and m edges. The other direction follows by symmetry.
2. Suppose $H \in \mathcal{F}(G)$ is induced G-free. Consider $\overline{H}$: if it contains an induced copy of $\overline{G}$, then H would contain an induced copy of G. The other direction is symmetric. □

Proposition 2.3. *Let G and H be two graphs with H an induced subgraph of G. Let $\mathcal{F}(G)$ and $\mathcal{F}(H)$ be, respectively, the families of all induced G-free and induced H-free graphs.*

1. *If $\mathcal{F}(H)$ is feasible then $\mathcal{F}(G)$ is feasible.*
2. *If $\mathcal{F}(G)$ is not feasible then $\mathcal{F}(H)$ is not feasible.*

Proof. Observe that since H is an induced subgraph of G, a graph P which is induced H-free is also induced G-free because if P contains an induced copy of G then it must contain induced copy of H in the induced copy of G. Hence $\mathcal{F}(H) \subset \mathcal{F}(G)$ and we apply Proposition 2.1. □

2.2 The Universal Elimination Process (UEP) and its consequences

The Universal Elimination Process (UEP), introduced in [5], is a method which is used to delete edges systematically from a complete graph. We describe UEP here again for the sake of being self-contained. We start with K_n and order the vertices $v_1, \ldots, v_n$. We now delete at each step an edge incident with v_1 until v_1 is isolated. We then repeat the process of step-by-step deletion of the edges incident with v_2, and continue until we reach the empty graph on n vertices.

Along the process, for any pair (n, m), $0 \leq m \leq \binom{n}{2}$, we have a graph G with n vertices and m edges.

Lemma 2.4 ([5]). *The maximal induced subgraphs of K_n obtained when applying UEP on K_n are of the form $H(p, q, r) = (K_p \backslash K_{1,q}) \cup rK_1$, $p - 1 \geq q \geq 0$ and $p + r = n$.*

Proof. This is immediate from the definition and description of UEP. □

Figure 1 shows examples of $H(p, q, r)$ graphs.

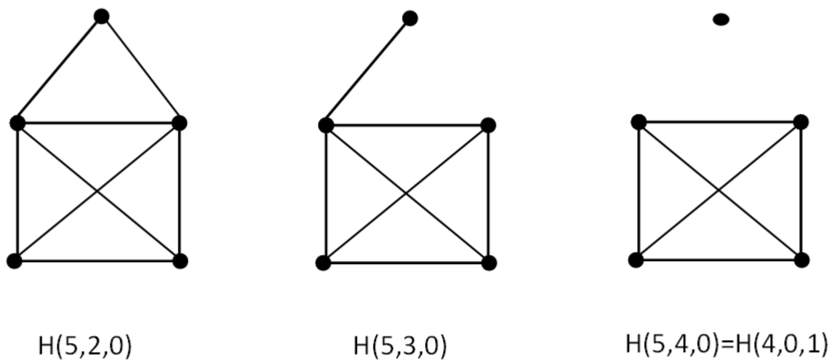

Figure 1: Examples of $H(p, q, r)$ graphs

Already the UEP supplies many feasible families, as summarized in the following corollary.

Corollary 2.5 ([5]). *The following families of graphs obtained by applying the UEP are feasible:*

1. *induced $K_{1,r}$-free for $r \geq 3$, where $K_{1,r}$ is the star with r leaves;*

2. *induced P_r-free for $r \geq 3$, where P_r is the path with r edges;*
3. *induced rK_2-free for $r \geq 2$ where rK_2 is the union of r disjoint edges.*

Proof. This is immediate from the definition and description of UEP. □

Definition 2.1. For non-negative integers p and r, with $p + r \geq 2$, let $S(p, r)$ denote the complete split graph $K_p + \overline{K_r}$, namely, a clique K_p and an independent set $\overline{K_r}$, and all edges between the vertices in K_p and $\overline{K_r}$.

Observe that $S(p, r) = \overline{H(r, 0, p)}$. We give some results related to the feasibility of $F(S(p, r))$.

Lemma 2.6. *The feasibility of $\mathcal{F}(S(p, r))$:*

1. *For $p = 0$ or $r = 0$, $\mathcal{F}(S(p, r))$ is not feasible.*
2. *For $p \geq 1$, $r \in \{1, 2\}$, $\mathcal{F}(S(p, r))$ is not feasible.*
3. *For $p \geq 1$, $r \geq 3$, $\mathcal{F}(S(p, r))$ is feasible.*

Proof.

1. This is because $S(p, r)$ is either K_p or $\overline{K_r}$ which and clearly $\mathcal{F}(G)$ is not feasible when $G = K_k$ or $G = \overline{K_k}$.
2. This is because $S(p, 1) = K_{p+1}$ and $S(p, 2) = K_{p+1} \backslash K_2$, and again $\mathcal{F}(G)$ is clearly not feasible when $G = K_k \backslash K_2$.
3. This is because $S(1, r) = K_{1,r}$ and we already proved in [5] (and mentioned before) that $\mathcal{F}(K_{1,r})$ is feasible for $r \geq 3$. Also since the UEP produces $K_{1,r}$-free graphs for $r \geq 3$, it follows that for $p \geq 2$, $r \geq 3$, $S(p, r)$ is not an induced subgraph in any graph obtained by the UEP. □

The following result is an immediate application of Proposition 2.3 and the fact that $S(p, r) = \overline{H(r, 0, p)}$, as well as Lemma 2.6 by replacing the role of r and p due to complementation.

Corollary 2.7. *The feasibility of $\mathcal{F}(H(p, 0, r))$.*

1. *For $p = 0$ or $r = 0$, $\mathcal{F}(H(p, 0, r))$ is not feasible.*
2. *For $p \in \{1, 2\}$ and $r \geq 1$, $\mathcal{F}(H(p, 0, r))$ is not feasible.*
3. *For $p \geq 3$, $r \geq 1$, $\mathcal{F}(H(p, 0, r))$ is feasible.*

Proof.

1. This is because $H(p, 0, r)$ in this case is a clique or an independent set.
2. This is because $H(p, 0, r)$ in this case is the independent set $\overline{K_{r+1}}$ for $p = 1$, and $K_2 \cup \overline{K_r}$ for $p = 2$, both members of TNF.
3. This follows by complementation. □

This elimination method, however, does not work in the case of a family $\mathcal{F}(G)$ of induced G-free graphs when G is of the form $H(p,q,r)$. In [5], the authors prove that the family of paw-free graphs is feasible, where the paw graph is isomorphic to $H(4,2,0)$. They use a different edge elimination technique, which we develop and extend in the next section.

2.3 $\{K_3, K_2\}$-elimination and its consequences.

Lemma 2.8. *For $n \geq 2$ and $0 \leq t \leq n-2$, there are integers $x, y \geq 0$ such that $3x + y = t$ and $xK_3 \cup yK_2$ is a subgraph of K_n.*

Proof. Clearly this is true by direct checking for $n = 2$ with $(x,y) = (0,0)$, $n = 3$ with $(x,y) \in \{(0,0),(0,1)\}$, $n = 4$ with $(x,y) \in \{(0,0),(0,1),(0,2)\}$ and $n = 5$ with $(x,y) \in \{(0,0),(0,1),(0,2)(1,0)\}$.

So assume $n \geq 6$ and write $n = 3k + r$, $0 \leq r \leq 2$ and $k \geq 2$. We consider three cases:

1. When $r = 0$, $n = 3k$. For $0 \leq t \leq n-3$ we shall consider K_{n-1} in K_n and by induction any t in this range can be represented by $xK_3 \cup yK_2$ as a subgraph of K_{n-1} hence of K_n.
   So we only need to consider $t = n-2$. We take $(k-1)K_3$ $(k \geq 2)$ that covers $3k-3$ vertices; hence from the remaining three vertices forming K_3 we can choose K_2 and we get $(k-1)K_3 \cup K_2$ on $3k-2 = n-2$ edges.
2. When $r = 1$, $n = 3k+1$. As above, for $0 \leq t \leq n-3$ we shall apply induction on $n-1$ vertices. So we need consider only $t = n-2 = 3k-1$. We take $(k-1)K_3$ that cover $3k-3$ vertices and from the remaining 4 vertices forming K_4 we choose $2K_2$ and get $(k-1)K_3 \cup 2K_2$ on $3k-1 = n-2$ edges.
3. When $r = 2$, $n = 3k+2$. As above, for $0 \leq t \leq n-3$ we shall apply induction on $n-1$ vertices. So we need consider only $t = n-2 = 3k$. We take kK_3 that cover $3k$ vertices and get $3k = n-2$ edges.

□

The $\{K_3, K_2\}$-elimination process is described as follows: starting from K_n, for every $0 \leq t \leq n-2$, delete edges in the form $xK_3 \cup yK_2$ such that $3x + y = t$. Once this is done, we have covered all the range $[\binom{n-1}{2} + 1, \ldots, \binom{n}{2}]$. Consider now $K_{n-1} \cup K_1$ (obtained by deleting a star $K_{1,n-1}$) and apply the $\{K_3, K_2\}$-elimination process on K_{n-1} and continue until all edges are deleted. Once again observe that this process covers all possible numbers of edges in the range $[0, \binom{n}{2}]$.

Observe that the graphs obtained through this elimination process are of the form $Q(p,r,x,y) = (K_p \backslash \{xK_3 \cup yK_2\}) \cup \overline{K_r}$ for $p, r, x, y \geq 0$ and $0 \leq 3x + y \leq p$ and $p + r = n$.

Lemma 2.9. *The graphs obtained through the* (K_3, K_2)*-elimination process are* $H(p,q,r)$*-free for* $p \geq 4$, $r \geq 0$, $2 \leq q \leq p-2$. *In particular, the family* $\mathcal{F}$ *of all* $H(p,q,r)$*-free graphs is feasible whenever* $p \geq 4$, $r \geq 0$, *and* $2 \leq q \leq p-2$.

Proof. The proof is by comparing the structure of $Q(p,r,x,y)$ versus $H(p,q,r)$ graphs.

The only cases where $Q(p,r,x,y) = H(p,q,r)$, $(3x+y=q)$ are $q = 0,1$, where the connected part is either K_p or $K_p \backslash K_2$, since the q edges in $H(p,q,r)$ are deleted via deletion of a star on q edges. In all other cases, $Q(p,r,x,y)$ graphs are $H(p,q,r)$-free graphs, and since $Q(p,r,x,y)$ with $p+r=n$ covers all possible values of m in the range $0 \leq m \leq \binom{n}{2}$ via the $\{K_3, K_2\}$-elimination process, it follows that for fixed $p \geq 4$, r, $2 \leq q \leq p-2$, $r \geq 0$, the family $\mathcal{F}$ of all $H(p,q,r)$-free graphs is feasible. □

3 Concluding the proof of the Main Theorem

We shall now complete the proof of the Main Theorem. Observe that by the UEP and $\{K_3, K_2\}$-elimination process together with the determination of the feasibility of $\mathcal{F}(H(p,0,r))$ in Section 2, what remains to be considered is the feasibility of $\mathcal{F}(G)$ where G is an $H(p,q,r)$ graph with $p \in \{2,3\}$, and the case when $p \geq 4$ and $q = 1$ or $q = p-1$ with $r \geq 0$ (and their complements which follow by Proposition 2.2).

Proposition 3.1. *The case* $p = 2$.

Proof. Observe that $p = 2$ gives either $H(2,0,r) = K_2 \cup \overline{K_r}$, or $H(2,1,r) = \overline{K_{r+1}}$ which belong to the family TNF. □

Proposition 3.2. *The case* $p = 3$.

Proof. Observe that $p = 3$ gives $H(3,0,r)$, $H(3,1,r)$, $H(3,2,r) = H(2,0,r+1)$. We consider each of these graphs:

1. If $G = H(3,0,r)$ then, if $r = 0$, $G = K_3$ belongs to TNF and $\mathcal{F}(G)$ is not feasible, while if $r \geq 1$ then by Corollary 2.7 part 2, $\mathcal{F}(G)$ is feasible.

2. If $G = H(2,0,r+1)$, $\mathcal{F}(G)$ is not feasible by Proposition 3.1.

3. If $G = H(3,1,r)$ then, if $r = 0$, $G = K_3 \backslash K_2$ which is a member of TNF and hence not feasible. If $r \geq 1$ then $G = \overline{K_3 \backslash K_2 \cup \overline{K_r}}$. When $r = 1$, G is the complement of the paw-graph, that is, $G = \overline{H(4,2,0)}$, which is feasible by Proposition 2.2, and hence $\mathcal{F}(G)$ is feasible. For $r \geq 2$, $G = \overline{K_{p+r} \backslash K_{1,2}}$, which is feasible by $\{K_3, K_2\}$-elimination since $p+r \geq 5$ and we can delete $2K_2$. □

Proposition 3.3. *The case $p \geq 4$.*

1. *For $p \geq 4$ and $q = p - 1$, $H(p, p-1, r)$ is feasible for $r \geq 0$.*
2. *For $p \geq 4$ and $q = 1$, $H(p, 1, r)$ is feasible for $r \geq 1$ and not feasible for $r = 0$ (a member of TNF).*

Proof.

1. If $p \geq 4$ and $q = p - 1$ then $G = H(p, p-1, r) = H(p-1, 0, r+1) = K_{p-1} \cup \overline{K_{r+1}}$ and we are done by Corollary 2.7.
2. If $p \geq 4$ and if $q = 1$ then $G = H(p, 1, r) = K_p \backslash K_2 \cup \overline{K_r}$. If $r = 0$, $H(p, 1, 0) = K_p \backslash K_2$ is a member of TNF hence $\mathcal{F}(G)$ is not feasible. So we may assume that $r \geq 1$. Recall that the family of claw-free graphs is feasible by UEP. For $p \geq 4$, the complement of $H(p, 1, r)$ with $r \geq 1$, H, contains an induced claw. So a claw-free graph cannot have H as an induced graph, hence it is in particular H-free. Since the family of all claw-free graphs is feasible and H-free, it follows that the family of all H-free graphs (containing the family of claw-free graphs) is feasible. Therefore, applying Proposition 2.3 we get that the family of all $H(p, 1, r)$-free graphs with $r \geq 1$ is feasible. □

Hence we have proved that $\mathcal{F}(G)$ is feasible if and only if G is not one of the graphs K_k, $K_k \backslash K_2$, $\overline{K_k}$, or $\overline{K_k \backslash K_2}$.

4 Further Examples and Open Problems

After the proof of the Main Theorem, a natural question is the following: Suppose G and H are graphs such that $\mathcal{F}(G)$ and $\mathcal{F}(H)$ are both feasible families. Is $\mathcal{F}(G, H)$, the family of all graphs which are simultaneously induced G-free and induced H-free, necessarily feasible ?

We know that if neither G nor H are $H(p, q, r)$ graphs then $\mathcal{F}(G, H)$ is feasible by UEP. Since $H(p, q, r)$ graphs on three vertices belong to the TNF family, it follows that the smallest interesting case is $H(4, 2, 0)$, the Paw graph.

The following answers the above question negatively, despite the fact that $\mathcal{F}(Paw)$ and $\mathcal{F}(K_{1,3})$ are both feasible families as proved in [5].

Theorem 4.1. $\mathcal{F}(Paw, Claw) = \mathcal{F}(H(4, 2, 0), K_{1,3})$ *is not a feasible family. Also* $\mathcal{F}(P_3 \cup K_1, K_3 \cup K_1)$ *is not feasible.*

Proof. These families are complementary and hence by Proposition 2.2 the two statements are equivalent. It is rather easy to check that the pair $(n, m) = (5, 3)$ forces an induced $P_3 \cup K_1$ or $K_3 \cup K_1$ and hence the pair $(n, m) = (5, 7)$ forces an induced Paw or $Claw$. It is also still easy to check that the pair $(n, m) = (6, 4)$ forces an induced member of $P_3 \cup K_1$ or $K_3 \cup K_1$ and hence the pair $(n, m) = (6, 11)$ forces an induced Paw or $Claw$.

So clearly $\mathcal{F}(Paw, Claw)$ and $\mathcal{F}(P_3 \cup K_1, K_3 \cup K_1)$ are not feasible.

A more general argument is the following: Suppose we consider a graph G on $n \geq 5$ vertices and m edges, with $\lfloor \frac{n}{2} \rfloor + 1 \leq m \leq n - 2$. Then G contains P_3 as we cannot pack mK_2. Clearly no such graph on m edges is connected for $n \geq 5$.

Consider the connected component B containing this P_3, and a vertex $v \in V \backslash B$. If B is not a complete graph, it must contain an induced P_3 together with v forming the induced subgraph $P_3 \cup K_1$. If B is a clique it must be of order at least 3 because B contains P_3. But in this case, together with v we have $K_3 \cup K_1$ as an induced subgraph. Hence for $n \geq 5$, the pair (n, m) where $\lfloor \frac{n}{2} \rfloor + 1 \leq m \leq n - 2$ is not a feasible pair for $\mathcal{F}(P_3 \cup K_1, K_3 \cup K_1)$.

Hence by considering the complement, for $n \geq 5$ the pair $(n, \binom{n}{2} - m)$, where $\lfloor \frac{n}{2} \rfloor + 1 \leq m \leq n - 2$, is a non-feasible pair for $\mathcal{F}(Paw, Claw)$.

However, the pairs (n, m) where $0 \leq m \leq \lfloor \frac{n}{2} \rfloor$ are feasible for $\mathcal{F} = \mathcal{F}(P_3 \cup K_1, K_3 \cup K_1)$, since the graph mK_2 is both $P_3 \cup K_1$ and $K_3 \cup K_1$ induced free. Also observe that the pair $(n, n-1)$ is feasible for $\mathcal{F} = \mathcal{F}(P_3 \cup K_1, K_3 \cup K_1)$ by taking the graph $K_{1,n-1}$. □

Observe that graphs which are made up of a union of cliques belong to $\mathcal{F} = F(Paw, Claw)$, which forces that

$$f(n, \mathcal{F}) \geq \frac{n^2}{2} - \sqrt{2}n^{3/2} + O(n^{5/4}),$$

as mentioned in the introduction, and

$$F(n, \mathcal{F}) = \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ for } n \geq 5.$$

Hence by considering the complement, for $\mathcal{F} = \mathcal{F}(P_3 \cup K_1, K_3 \cup K_1)$ we obtain

$$F(n, \mathcal{F}) \leq \sqrt{2}n^{3/2} + O(n^{5/4}),$$

$$f(n, \mathcal{F}) = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ for } n \geq 5,$$

as proved above.

Problem: It would be interesting to improve the lower bound on $f(n, \mathcal{F})$ for $\mathcal{F} = F(Paw, Claw)$ and the corresponding value of $F(n, \mathcal{F})$ for $\mathcal{F} = \mathcal{F}(P_3 \cup K_1, K_3 \cup K_1)$. In particular, is $F(n, \mathcal{F})$ linear in n for $\mathcal{F} = \mathcal{F}(P_3 \cup K_1, K_3 \cup K_1)$?

Another interesting question is: since $\mathcal{F}(Paw, Claw)$ is not a feasible family, is $\mathcal{F}(Paw, K_{1,4})$ a feasible family or not, considering the fact that $\mathcal{F}(Paw, K_{1,3}) \subset$

$\mathcal{F}(Paw, K_{1,4}) \subset \mathcal{F}(Paw, K_{1,r})$ for $r \geq 5$. We prove the following theorem to answer this question.

Theorem 4.2. *$\mathcal{F}(Paw, K_{1,4})$ is a feasible family and so is $\mathcal{F}(Paw, K_{1,r})$ for $r \geq 5$.*

Proof. By Proposition 2.1, if $\mathcal{F}(Paw, K_{1,4})$ is a feasible family then so is $\mathcal{F}(Paw, K_{1,r})$ for $r \geq 5$.

We shall work with the complementary family $\mathcal{F} = \mathcal{F}(P_3 \cup K_1, K_4 \cup K_1)$ and show that it is feasible. Then by Proposition 2.2 $\mathcal{F}(Paw, K_{1,4})$ is also feasible.

We shall use the split graphs $K_p + \overline{K_{n-p}}$ for $0 \leq p \leq n-3$ to construct graphs in $\mathcal{F}(P_3 \cup K_1, K_4 \cup K_1)$ which cover the range $0 \leq m \leq \binom{n}{2} - 2$. When $p = 0$, the graph is $\overline{K_n}$ with 0 edges. We can pack a graph with k edges of the form $aK_3 \cup bK_2 \cup cK_1$ with $n = 3a + 2b + c$ and $k = 3a + b$ for $0 \leq k \leq n-2$. This has been shown in the $\{K_3, K_2\}$-elimination process in Lemma 2.8. For $p = 1$, the split graph is $K_1 + \overline{K_{n-1}} = K_{1,n-1}$ which has exactly $n-1$ edges. Again, we can pack, in the independent part $\overline{K_{n-1}}$ of order $n-1$, graphs with k edges of the form $aK_3 \cup bK_2 \cup cK_1$ with $n-1 = 3a+2b+c$ and $k = 3a+b$ for $0 \leq k \leq n-3$. With these graphs we cover the range $n-1, \ldots, n-1+n-3 = 2n-4$. All these graphs are in $\mathcal{F}(P_3 \cup K_1, K_4 \cup K_1)$.

In general, the split graph $K_p + \overline{K_{n-p}}$ has $\frac{p(n-1)+(n-p)p}{2}$ edges and we can pack, in the independent part $\overline{K_{n-p}}$ of order $n-p$, graphs with k edges of the form $aK_3 \cup bK_2 \cup cK_1$ with $n-p = 3a+2b+c$ and $k = 3a+b$ for $0 \leq k \leq n-p-2$, and cover the range of values of m from $\frac{p(n-1)+(n-p)p}{2}$ to $\frac{p(n-1)+(n-p)p}{2} + n - p - 2$. Again, all these graphs are in $\mathcal{F}(P_3 \cup K_1, K_4 \cup K_1)$.

For the last value $p = n-3$, we cover the range

$$\frac{(n-3)(n-1)+3(n-3}{2} = \frac{n^2-n-6}{2} = \binom{n}{2} - 3$$

up to

$$\binom{n}{2} - 3 + n - (n-3) - 2 = \binom{n}{2} - 3 + 1 = \binom{n}{2} - 2$$

as discussed.

The final two values of m, which are $\binom{n}{2} - 1$ and $\binom{n}{2}$, are covered by the graph $K_n \backslash K_2$ (which is in fact $K_{n-2} + \overline{K}_2$), and K_n itself, both graphs being in $\mathcal{F}(P_3 \cup K_1, K_4 \cup K_1)$. Thus the whole range of edges $0 \leq m \leq \binom{n}{2}$ is covered and the family $\mathcal{F}(P_3 \cup K_1, K_4 \cup K_1)$ is feasible, as well as the family $\mathcal{F}(Paw, K_{1,4})$, and $\mathcal{F}(Paw, K_{1,r})$ for $r \geq 5$ by Proposition 2.1. □

Another problem concerns the family $\mathcal{F} = \mathcal{F}(K_4 \backslash K_2)$, the smallest case in TNF for which the order of $f(n, \mathcal{F})$ is interesting. Clearly, graphs which are union of cliques belong to $\mathcal{F}(K_4 \backslash K_2)$ and hence

$$f(n,\mathcal{F}) \geq \frac{n^2}{2} - \sqrt{2}n^{3/2} + O(n^{5/4}),$$

which also holds for $f(n,\mathcal{F})$ when $\mathcal{F} = \mathcal{F}(K_k \backslash K_2)$ and $k \geq 3$. This is asymptotically sharp for $k = 3$ as we have already seen.

Note that this result is equivalent to the result in [6], showing that $(n,e) \to (4,5)$ is an avoidable pair.

The following arguments give some more information on the non-feasible pairs (n,m) for $\mathcal{F}(K_k \backslash K_2)$. Clearly a trivial upper bound for $n \geq k$ is $m = \binom{n}{2} - 1$, since the graph $K_n \backslash K_2$ contains an induced $K_k \backslash K_2$, implying that the pair (n,m) is not feasible. Hence $F(n,\mathcal{F}) = \binom{n}{2} - 1$ for $\mathcal{F} = \mathcal{F}(K_k \backslash K_2)$, $k \geq 3$.

Now suppose G is a graph on n vertices and $\binom{n}{2} - t$ edges where $t \geq 1$ and $n - 2t \leq k - 2$. The missing t edges can cover (in the complement) at most $2t$ vertices; hence in G there are at least $k-2$ vertices forming a clique and adjacent to all vertices of G. Choose a missing edge $e = xy$; then the $k-2$ vertices and $\{x,y\}$ form an induced $K_k \backslash K_2$. So with $t = \left\lfloor \frac{n-k+2}{2} \right\rfloor$ and $\binom{n}{2} - \left\lfloor \frac{n-k+2}{2} \right\rfloor \leq m \leq \binom{n}{2} - 1$, all the pairs (n,m) are non-feasible for the family $\mathcal{F}(K_k \backslash K_2)$, proving that $f(n,\mathcal{F}) \leq \binom{n}{2} - \left\lfloor \frac{n-k+2}{2} \right\rfloor$ for this family, and $F(n,\mathcal{F}) \geq \left\lfloor \frac{n-k+2}{2} \right\rfloor$ for the complementary family $\mathcal{F} = \mathcal{F}(K_2 \cup (k-2)K_1)$.

Problem: It would be interesting to improve upon the lower bound for $f(n,\mathcal{F})$ for $\mathcal{F} = \mathcal{F}(K_k \backslash K_2)$, as well as the corresponding upper bound $F(n,\mathcal{F})$ for $\mathcal{F} = \mathcal{F}(K_2 \cup (k-2)K_1)$.

Acknowledgements

We would like to thank the referees whose careful reading of the paper and feedback have helped us to improve this paper considerably.

References

[1] N. Alon and J. H. Spencer, *The Probabilistic Method*, John Wiley & Sons, 2016.

[2] M. Axenovich, J. Balogh, F. C. Clemen and L. Weber, Unavoidable order-size pairs in Hypergraphs—positive forcing density, *Combinatorial Theory* **3**(3) (2023), 15.

[3] M. Axenovich and L. Weber, Absolutely avoidable order-size pairs for induced subgraphs, *J. Combin.* **15**(1) (2024),41–57.

[4] B. Bollobás, *Extremal Graph Theory*, Courier Corporation, 2004.

[5] Y. Caro, J. Lauri and C. Zarb, The feasibility problem for line graphs, *Discrete Applied Math.* **324** (2023), 167–180.

[6] P. Erdős, Z. Füredi, B. L. Rothschild and V. T. Sós, Induced subgraphs of given sizes, *Discrete Math.* **200** (1999), 61–77.

[7] P. Erdős and M. Simonovits, A limit theorem in graph theory, *Studia Sci. Math. Hungar.* **1** (1966), 51–57.

[8] Z. Füredi and M. Simonovits, *The history of degenerate (bipartite) extremal graph problems*, Erdős Centennial, Springer, 2013, 169–264.

[9] J. He, J. Ma and L. Zhao, Improvements on induced subgraphs of given sizes, *Communications Math. Stats.* (2023), 1–20.

[10] B. Reznick, The sum of the squares of the parts of a partition, and some related questions, *J. Number Theory* **33**(2) (1989), 199–208.

[11] A. H. Stone and P. Erdös, On the structure of linear graphs, *Bull. Amer. Math. Soc.* **52** (1946), 1089–1091.

[12] P. Turán, On an external problem in graph theory, *Mat. Fiz. Lapok* **48** (1941), 436–452.

[13] P. Turán On the theory of graphs, Colloquium Mathematicum **1** (1954), 19–30.

[14] L. Weber, Unavoidable order-size pairs in hypergraphs, *J. Combin.* **15**(2) (2024),179-195.

(Received 7 Nov 2023; revised 1 Sep 2024)