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On r-equichromatic lines with few points in $\mathbb{C}^2$

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Abstract

Let P be a set of n green and $n-k$ red points in $\mathbb{C}^2$. A line determined by i green and j red points such that $i+j \geq 2$ and $|i-j| \leq r$ is called r-equichromatic. We establish lower bounds for 1-equichromatic and 2-equichromatic lines. In particular, we show that if at most $2n-k-2$ points of P are collinear, then the number of 1-equichromatic lines passing through at most six points is at least $\frac{1}{4}(6n-k(k+3))$, and if at most $\frac{2}{3}(2n-k)$ points of P are collinear, then the number of 2-equichromatic lines passing through at most four points is at least $\frac{1}{6}(10n-k(k+5))$.

1 Introduction

In this paper we study sets of n green points and $n-k$ red points in the complex plane. Let P be such a set. A line containing two or more points of P is said to be *determined* by P. A line determined by at least one green and one red point is called *bichromatic*. Otherwise, it is called *monochromatic*. A line determined by i green and j red points such that $i+j \geq 2$ and $|i-j| \leq r$ is called *r-equichromatic*. Note that every 1-equichromatic line is a bichromatic line.

In [8], Purdy and Smith studied lower bounds on the number of bichromatic lines and on the number of 1-equichromatic lines in $\mathbb{C}^2$ and $\mathbb{R}^2$. For brevity, we will mention only the results on 1-equichromatic lines and we refer interested readers to [7, 8] for some other results.

Theorem 1 (Purdy and Smith [8]) *Let P be a set of n green and $n-k$ red points in $\mathbb{R}^2$ such that the points of P are not all collinear. Let t be the total number of lines determined by P. Then the number of 1-equichromatic lines is at least $\frac{1}{4}(t+2n+3-k(k+1))$.*

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Theorem 2 (Purdy and Smith [8]) *Let P be a set of n green and $n-k$ red points in $\mathbb{R}^2$ such that the points of P are not all collinear. Then the number of 1-equichromatic lines determined by at most four points is at least $\frac{1}{4}(2n+6-k(k+1))$.*

Theorem 3 (Purdy and Smith [8]) *Let P be a set of n green and $n-k$ red points in $\mathbb{C}^2$ such that no $2n-k-2$ points of P are collinear. Then the number of 1-equichromatic lines determined by at most five points is at least $\frac{1}{4}(6n-k(k+3))$.*

Theorem 4 (Purdy and Smith [8]) *Let P be a set of n green and $n-k$ red points in $\mathbb{R}^2$ such that the points of P are not all collinear. Let t be the total number of lines determined by P. Then the number of 1-equichromatic lines determined by at most six points is at least $\frac{1}{12}(t+6n+15-3k(k+1))$.*

Purdy and Smith [8] asked whether one can prove a tight lower bound on the number of 1-equichromatic or bichromatic lines determined by at most four points in $\mathbb{C}^2$. This question motivated the current study. Unfortunately, the closest we have come is 2-equichromatic lines. Table 2 in Purdy and Smith [8] contains the summary of their results on 1-equichromatic lower bounds. In that table there is a lower bound for the number of 1-equichromatic lines determined by at most six points in $\mathbb{C}^2$, but there is no result in their paper justifying this claim. So, we prove a lower bound for the number of 1-equichromatic lines determined by at most six points in $\mathbb{C}^2$. Our lower bound is the same as the one claimed by Purdy and Smith [8] .

2 Incidence Inequalities

The main ingredients used by Purdy and Smith [8] and which also will be used in the present paper, are incidence inequalities. We list some well-known incidence inequalities. Let t_k denote the number of lines that pass through exactly k points.

Theorem 5 (Melchior's Inequality [4]) *Let S be a set of n non-collinear points in the plane. Then*

$$\sum_{k\geq 2}(3-k)t_k \geq 3. \tag{1}$$

The proof for (1) uses Euler's polyhedral formula. In [6], Langer proved this inequality by working with pairs $(\mathbb{P}^2_{\mathbb{C}}, \alpha D)$ where $\mathbb{P}^2_{\mathbb{C}}$ is the complex projective plane with a $\mathbb{Q}$-effective (boundary) divisor D such that $(\mathbb{P}^2_{\mathbb{C}}, \alpha D)$ is log canonical and effective.

Theorem 6 (Langer's Inequality [6]) *Let S be a set of n points in $\mathbb{P}^2_{\mathbb{C}}$, with at most $\frac{2}{3}n$ points collinear. Then*

$$\sum_{k\geq 2} kt_k \geq \frac{n(n+3)}{3}.$$

Theorem 7 (Hirzebruch's Inequality [2]) *Let S be a set of n points in $\mathbb{P}^2_{\mathbb{C}}$, with at most $n-2$ points collinear. Then*

$$t_2 + t_3 \geq n + \sum_{k\geq 5}(k-4)t_k. \tag{2}$$

Theorem 8 (Hirzebruch's Inequality [3]) *Let S be a set of n points in $\mathbb{P}^2_{\mathbb{C}}$, with at most $n-3$ points collinear. Then*

$$t_2 + \frac{3}{4}t_3 \geq n + \sum_{k\geq 5}(2k-9)t_k. \tag{3}$$

Hirzebruch's inequalities do not follow from Euler's formula as one would suspect. Instead, Hirzebruch's inequalities were derived from the Bogomolov–Miyaoka–Yau inequality, a deep result in algebraic geometry, and it is true for arrangements of points in the complex plane.

Bojanowski [1] and Pokora [5] used Langer's work [6] to prove the following theorem.

Theorem 9 (Bojanowski–Pokora Inequality) *Let S be a set of n points in $\mathbb{P}^2_{\mathbb{C}}$, with at most $\frac{2}{3}n$ points collinear. Then*

$$t_2 + \frac{3}{4}t_3 \geq n + \sum_{k\geq 5}(\frac{1}{4}k^2 - k)t_k. \tag{4}$$

Note that (4) is equivalent to

$$\sum_{k\geq 2}(4k-k^2)t_k \geq 4n. \tag{5}$$

Remark 1 One should note that these inequalities (except (1)) were originally proved for an arrangement of lines in the complex projective plane such that t_k is the number of intersection points where exactly k lines of the arrangement are incident.

Remark 2 Purdy and Smith [8] proved Theorems 1, 2 and 4 using Melchior's inequality (1) and proved Theorem 3 using Hirzebruch's inequality (3).

3 Lower Bounds for Lines in $\mathbb{C}^2$

The identities below can be found in [7, 8] and will be used in this section. Let $t_{i,j}$ be the number of lines determined by P with exactly i green points and j red points, where we always assume $i+j\geq 2$. Assume that the number of green points is n and the number of red points is n. Then the number of bichromatic point pairs is

$$\sum_{\substack{i,j\geq 0\\ i+j\geq 2}} ijt_{i,j} = n^2$$

and the number of monochromatic point pairs is

$$\sum_{\substack{i,j\geq 0\\ i+j\geq 2}}\left[\binom{i}{2}+\binom{j}{2}\right]t_{i,j}=2\binom{n}{2}=n^2-n.$$

In general, if we assume that the number of green points is n and the number of red points is $n-k$, then the above identities become

$$\sum_{\substack{i,j\geq 0\\ i+j\geq 2}} ijt_{i,j}=n(n-k)=n^2-nk \tag{6}$$

and

$$\sum_{\substack{i,j\geq 0\\ i+j\geq 2}}\left[\binom{i}{2}+\binom{j}{2}\right]t_{i,j}=\binom{n}{2}+\binom{n-k}{2}=n^2-n-nk+\frac{k^2+k}{2}. \tag{7}$$

We subtract (6) from (7) and then split the summation to get the following identity:

$$\sum_{\substack{i,j\geq 0\\ i+j\geq 2}}(i+j)t_{i,j}=\sum_{\substack{i,j\geq 0\\ i+j\geq 2}}(i-j)^2t_{i,j}+2n-(k^2+k). \tag{8}$$

3.1 A Lower Bound for 1-Equichromatic lines through at most six points

As stated before, we are not able to find the claimed result of Purdy and Smith [8] on 1-equichromatic lines through at most six points in $\mathbb{C}^2$. Below we will prove the result.

Theorem 10 *Let P be a set of n green and $n-k$ red points in $\mathbb{C}^2$ such that at most $2n-k-2$ points of P are collinear. Then the number of 1-equichromatic lines passing through at most six points is at least $\frac{1}{4}(6n-k(k+3))$.*

Proof. First, we express (2) as

$$-(t_{0,2}+t_{2,0})-t_{1,1}-(t_{0,3}+t_{3,0})-(t_{1,2}+t_{2,1})+\sum_{\substack{i,j\geq 0\\ i+j\geq 5}}((i+j)-4)\,t_{i,j}\leq-(2n-k). \tag{9}$$

We subtract (6) from (7) and unwind the first few terms of the summation to get

$$\begin{aligned}(t_{0,2}+t_{2,0})-t_{1,1}+3(t_{0,3}+t_{3,0})-(t_{1,2}+t_{2,1})+6(t_{0,4}+t_{4,0})\\ -2t_{2,2}+\sum_{\substack{i,j\geq 0\\ i+j\geq 5}}\left[\binom{i}{2}+\binom{j}{2}-ij\right]t_{i,j}=-n+\frac{k^2+k}{2}.\end{aligned} \tag{10}$$

Adding (9) and (10) produces

$$\begin{aligned}&-2t_{1,1}+2(t_{0,3}+t_{3,0})-2(t_{1,2}+t_{2,1})+6(t_{0,4}+t_{4,0})\\&\quad-2t_{2,2}+\sum_{\substack{i,j\geq 0\\ i+j\geq 5}}\left[\binom{i}{2}+\binom{j}{2}-ij\right]t_{i,j}\\&\quad+\sum_{\substack{i,j\geq 0\\ i+j\geq 5}}((i+j)-4)\,t_{i,j}\leq-(2n-k)-n+\frac{k^2+k}{2}.\end{aligned}\tag{11}$$

Let $\alpha_{i,j}$ be the coefficient corresponding to $t_{i,j}$ produced by the left-hand side of the inequality above. One can check that the only negative coefficients are $\alpha_{1,1}=\alpha_{1,2}=\alpha_{2,1}=\alpha_{2,2}=-2$, and $\alpha_{2,3}=\alpha_{3,2}=\alpha_{3,3}=-1$. Thus

$$-2(t_{1,1}+t_{1,2}+t_{2,1}+t_{2,2}+t_{2,3}+t_{3,2}+t_{3,3})\leq\frac{-6n+k(k+3)}{2}.$$

The result follows immediately. □

3.2 A Lower Bound for 2-Equichromatic lines through at most four points

We now consider 2-equichromatic lines through at most four points. To begin with, we write (5) within our context and add that to (8) to obtain

$$\sum_{\substack{i,j\geq 0\\ i+j\geq 2}}\left(5(i+j)-(i-j)^2-(i+j)^2\right)t_{i,j}\geq 10n-k(k+5).\tag{12}$$

Let $\alpha_{i,j}$ be the coefficient corresponding to $t_{i,j}$ in (12). One can check that the only positive coefficients are $\alpha_{0,2}=\alpha_{2,0}=2,\alpha_{1,1}=6,\alpha_{1,2}=\alpha_{2,1}=5$, and $\alpha_{2,2}=4$, and therefore,

$$6(t_{0,2}+t_{2,0}+t_{1,1}+t_{1,2}+t_{2,1}+t_{2,2})\geq 10n-k(k+5).$$

This gives us the following:

Theorem 11 *Let P be a set of n green and $n-k$ red points in $\mathbb{C}^2$ such that at most $\frac{2}{3}(2n-k)$ points of P are collinear. Then the number of 2-equichromatic lines passing through at most four points is at least $\frac{1}{6}(10n-k(k+5))$.*

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