ON RADIALLY MAXIMAL GRAPHS

FERDINAND GLIVIAK
Comenius University, Faculty of Mathematics and Physics,
Mlynská dolina, 842 15 Bratislava, Slovakia

MARTIN KNOR
Department of Mathematics, SvF, Slovak Technical University,
Radlinského 11, 813 68 Bratislava, Slovakia

ĽUBOMÍR ŠOLTÉS
Department of Mathematics, ChTF, Slovak Technical University,
Radlinského 9, 812 37 Bratislava, Slovakia

ABSTRACT. A graph is radially maximal if its radius decreases after the addition of any edge of its complement. It is proved that any graph can be an induced subgraph of a regular radially maximal graph with a prescribed radius $r \geq 3$. For $r \geq 4$, $k \geq 1$, radially maximal graphs with radius r containing k cut-nodes are constructed.

1. INTRODUCTION

All graphs considered in this paper are finite, undirected, without loops and multiple edges. Let $k = k(G)$ be an invariant of a graph G, e.g. the radius $r(G)$, the diameter $diam(G)$ etc. A graph G is called:

minimal by k, if $G = \overline{K_n}$ or $k(G - h) \neq k(G)$ for every edge h of G;

critical by k, if $G = K_1$ or $k(G - v) \neq k(G)$ for every node v of G;

maximal by k, if $G = K_n$ or $k(G + h) \neq k(G)$ for every edge h of the complement of G.

Minimal, critical and maximal graphs by radius (diameter) are called radially (diametrically) minimal, critical, and maximal, respectively.

It is known (see [1, p. 101 and 105]) that any graph can be an induced subgraph of some diametrically minimal and diametrically critical graph of diameter $d \geq 2$, respectively. On the other hand the structure of connected diametrically maximal graphs can be described in terms of a sequential join (see Ore [11] and also [1, p. 37]):

Lemma 1. *A graph G with diameter d is diametrically maximal if and only if G has the form $K_1 + K_{a_1} + K_{a_2} + \cdots + K_{a_{d-1}} + K_1$ for some positive integers $a_1, a_2, \ldots a_{d-1}$.*

Here K_n denotes a complete graph on n nodes, and $G_1 + G_2 + \cdots + G_l$ arises from $G_1 \cup G_2 \cup \cdots \cup G_l$ by adding all edges uv, with $u \in G_i$ and $v \in G_{i+1}$, $1 \leq i \leq l-1$.

Trees are the only radially minimal graphs, as shown in [5].

The study of radially critical graphs was reduced to radially decreasing graphs, i.e. graphs G in which $r(G - v) = r(G) - 1$ for every node v of G, see Gliviak [6]. Here we recall a characterization of radially decreasing graphs, see [5, Theorem 5]:

Lemma 2. *A graph G is radially decreasing if and only if there exists a decomposition of $V(G)$ into pairs v, v' such that $d_G(v, v') = r(G) > max\{d_G(v, u), d_G(u, v')\}$ for all $u \in V(G) - \{v, v'\}$.*

Again, any graph can be an induced subgraph of some radially decreasing graph with radius $r \geq 3$, see [5]. A special subclass of radially critical graphs was characterized by Fajtlowicz [3].

We note that $r(G + h) \leq r(G)$ for any edge h of the complement of G. Thus, in the definition of radially maximal graphs the symbol $\neq$ can be replaced by $<$.

Clearly, a single node is the only graph with radius zero and it is radially maximal, too. The complete graphs K_n, $n \geq 2$, are the only radially maximal graphs with radius one since their complement contains no edges. Moreover, every disconnected, radially maximal graph is a union of two complete graphs.

The study of radially maximal graphs was started by Nishanov [9, 10] and also by Harary and Thomassen [8]. Harary and Thomassen characterized radially maximal graphs with radius two, see [8, p. 15]:

Lemma 3. *A graph G with radius two is radially maximal if and only if the complement of G, $\overline{G}$, is disconnected and each component of $\overline{G}$ is a star (i.e. the complete bipartite graph $K_{1,s}$, $s \geq 1$).*

An upper bound on the number of edges in radially maximal graphs was found by Vizing [12] and its lower bound by Nishanov [10]. The following inequalities were stated in [8] and [9]:

Lemma 4. *Let G be a radially maximal graph with radius at least two. Then $r(G) \leq diam(G) \leq 2r(G) - 2$.*

In [9] some properties of radially maximal graphs with radius $r \geq 3$ and diameter $2r - 2$ are studied.

Results on directed radially maximal graphs can be found in [4] and [7].

Let G be a graph with radius r. By adding edges that do not decrease the radius, we obtain a radially maximal graph from G. Thus, one can expect that radially maximal graphs are rather dense. This idea is also supported by the fact, that radially maximal graphs with radius $r \leq 2$ are blocks (see Lemma 3). However, this is not the case. As shown in this paper, radially maximal graphs with radius $r \geq 4$ may have arbitrarily many cut-nodes. Moreover, we show that each graph may be an induced subgraph of a radially maximal graph with a prescribed radius $r \geq 3$. Hence, the structure of radially maximal graphs seems to be rather complicated.

The outline of this paper is as follows. In section 2 so-called node-extension of radially maximal graphs is examined. In section 3 it is proved that any graph G with n nodes can be an induced subgraph of a regular radially decreasing and radially maximal graph with given radius r, $3 \leq r < \infty$, and degree $k \geq 2n+3$. In section 4 it is shown that radially maximal graphs with radius three may have at most two cut-nodes and those with radius $r \geq 4$ may have an arbitrary number of cut-nodes. In section 5 some open problems are posed.

Let G be a graph. By $V(G)$ is denoted the node set of G, $E(G)$ the edge set of G, $\overline{G}$ the complement of G, $deg_G(u)$ the degree of a node u in G. By $d_G(u,v)$ is denoted the distance between the nodes $u, v \in V(G)$, i.e. the number of edges of a shortest u - v path in G. The eccentricity of a node $u \in V(G)$ is denoted by $e_G(u) = \max_{v \in V(G)} d_G(u,v)$. The radius and the diameter of G are denoted by $r(G)$ and $diam(G)$, respectively, i.e. $r(G) = \min_{u \in V(G)} e_G(u)$ and $diam(G) = \max_{u \in V(G)} e_G(u)$.

The subgraph of G induced by nodes with the minimum eccentricity is called the centre. The set of central nodes of G is denoted by $C(G)$. If $C(G) = V(G)$, G is *selfcentred* graph. Otherwise, G is a *non-selfcentric* graph.

Further, $D_i(x)$ denotes the set of nodes with distance i from a node x, thus:

$$D_i(x) = \{z \in V(G) : \ d_G(x,z) = i\} .$$

By $Rim(G)$ is denoted the set of nodes that are eccentric to some central node (see Buckley, Lewinter [2]), i.e. $Rim(G) = \cup_{c \in C(G)} D_{r(G)}(c)$.

Definitions and notations not included here can be found in [1].

From now on we use the terms maximal and decreasing graph to mean radially maximal and radially decreasing graph, respectively.

2. Extensions of maximal graphs

In this short section we deal with an operation that creates maximal graphs from smaller ones. Let G be a graph and $a \in V(G)$. Then a *node-extension* of G in a, $G[a]$, is a graph such that:

$V(G[a]) = V(G) \cup \{a'\}$

$E(G[a]) = E(G) \cup \{xa' : \ xa \in E(G)\} \cup \{aa'\}$,

where $a' \notin V(G)$. Note that $r(G[a]) = r(G)$ for $a \in V(G)$, since no eccentricity is changed and $e_{G[a]}(a') = e_{G[a]}(a) = e_G(a)$ except for the trivial case $V(G) = \{a\}$.

Lemma 5. *Let G be a maximal graph and $a \in V(G) - Rim(G)$. Then $G[a]$ is a maximal graph.*

Proof. Denote $H = G[a]$. By way of contradiction suppose that there is $h \in E(\overline{H})$ such that $r(H+h) = r(G)$. Denote $\{a'\} = V(H) - V(G)$.

First assume h is incident to neither a nor a'. Then we have $H + h = (G + h)[a]$. Since the node-extension preserves radius and G is maximal, we have $r(H + h) = r(G + h) < r(G) = r(H)$.

In the following suppose $h = ab$ (see Fig. 1).

Let $c \in C(G + h)$. Then $d_{G+h}(c, y) < r(G)$ for all $y \in V(G)$. Thus, $d_{H+h}(c, y) < r(G)$ for all $y \in V(H) - \{a'\}$. Since $r(H + h) = r(G)$, we have $d_{H+h}(c, a') = r(G)$, and therefore $d_{H+h}(c, a) = r(G) - 1$.

Let $x \in V(H)$ such that $xa \in E(H)$, $x \neq a'$. Since $d_{H+h}(c, a') = r(G)$ and $d_{H+h}(c, x) < r(G)$, we have $d_{H+h}(c, x) = r(G) - 1$. Hence, $d_G(c, x) = r(G) - 1$ for all nodes x of G adjacent to a.

Since $a \notin Rim(G)$ and $d_G(c, a) = r(G)$, we have $c \notin C(G)$. Thus, there is $z \in V(G)$ such that $d_G(c, z) > r(G)$. Since $d_{G+h}(c, z) < r(G)$, there is a path from c to z containing h with length at most $r(G) - 1$ in $G + h$. But $z \neq a$, since $d_G(c, a) = r(G)$. Thus, $d_{G+h}(c, z) > d_{G+h}(c, a) = d_{H+h}(c, a) = r(G) - 1$, a contradiction. $\square$

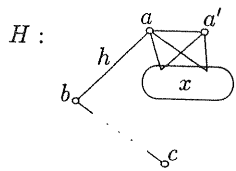

Fig. 1

3. Induced subgraphs

As shown in the introduction, maximal graphs with radii ∞, 0, 1, and 2 are characterized and they are very simple. In this section we prove that any graph G can be an induced subgraph of some regular maximal graph H with given radius $r(H)$, $3 \leq r(H) < \infty$. Further, we show a relation between maximal and decreasing graphs.

From Theorem 8 in [8] it follows that there exists an infinite number of maximal graphs with radius three. Using a construction from Theorem 4 in [5] we obtain a more general existence theorem:

Theorem 6. *Let G be a nonempty graph with n nodes, and let r and k be given integers, $r \geq 3$ and $k \geq 2n + 3$. Then there exists a regular graph of degree k with radius r that is decreasing and maximal and contains G as an induced subgraph.*

Proof. Let H be an arbitrary graph and $u \in V(H)$. We say that u has a property (*) if and only if:

$$(*) \qquad \forall a \in D_1(u)\ \exists b \in D_1(u) \text{ such that } ab \notin E(H).$$

Let G be a nonempty graph, and $l \geq 1$. In the following we construct a graph H_l from G:

(1) Let us construct a graph G_1 from G by successively verifying (*) for the nodes of G, and by joining one new node to every node not fulfilling (*). Clearly $|V(G_1)| \leq 2n$.

(2) Let a graph G_2 arise from G_1 by adding l isolated nodes, and let a graph G_3 arise from G_2 by adding one new node w joined to every node of G_2. Since $l \geq 1$, the node w satisfies (*) in G_3.

(3) Let G_3' be a copy of G_3 and let u' be a node of G_3' corresponding to the node u of G_3. Let $V(H_l) = V(G_3) \cup V(G_3')$, and let the edge set of H_l consist of $E(G_3)$ and $E(G_3')$, and moreover for all $x, y \in V(G_3)$ we have:

$$(**) \qquad xy', yx' \in E(H_l) \Longleftrightarrow xy \notin E(G_3)$$

except the case $x = y$, where $xx' \notin E(H_l)$.

Clearly, all the nodes of H_l satisfy (*). Since a mapping φ, such that $\varphi(u) = u'$ and $\varphi(u') = u$ for all $u \in V(G_3)$ and $u' \in V(G_3')$, is an automorphism of H_l, it is enough to prove the following statements for the nodes of G_3.

(*i*) The graph H_l is regular of degree $|V(G_1)| + l$.

Clearly, w has the required degree, and for all $a \in V(G_2)$, (**) implies $deg_{H_l}(a) = deg_{H_l}(w)$.

(*ii*) Let $a \in V(G_3)$. Then $d_{H_l}(a, a') = 3$, and $d_{H_l}(a, x) \leq 2$ for all $x \in V(H_l)$, $x \neq a'$.

If $x \in V(G_3)$, we have $d_{H_l}(a, x) \leq 2$, since $aw, wx \in E(H_l)$.

Suppose that $x = b' \in V(G_3')$, $b' \neq a'$, and $ab' \notin E(H_l)$. By (**) we have $ab \in E(G_3)$. However, there is $y \in V(H_l)$ such that $ay \in E(H_l)$ and $by \notin E(H_l)$ by (*). Hence, $yb' \in E(H_l)$ by (**), and $d_{H_l}(a, b') \leq 2$.

Clearly, $aa' \notin E(H_l)$. Since the existence of $y \in V(H_l)$ such that ay, $ya' \in E(H_l)$ contradicts (**), we have $d_{H_l}(a, a') = 3$.

(*iii*) For each $a \in V(G_3)$ and $x \in V(H_l)$ we have $d_{H_l}(a, x) + d_{H_l}(x, a') = 3$.

If $ax \notin E(H_l)$ and $x \neq a'$, we have $xa' \in E(H_l)$ by (**).

If $ax \in E(H_l)$ and $d_{H_l}(x, a') \geq 3$, we have $x = a$ by (*ii*).

Let $l = k - |V(G_1)| - \delta$, where $\delta = 0$ if $r = 3$, $\delta = 1$ if $r = 4$, and $\delta = 2$ if $r \geq 5$. Since $k \geq 2|V(G)| + 3$, we have $l \geq 1$.

Let $H_{3,k} = H_l$, $H_{4,k} = H_l \times K_2$, and $H_{r,k} = H_l \times C_{2(r-3)}$ if $r \geq 5$. (By C_n is denoted a cycle on n nodes, and by $\times$ is denoted the Cartesian product.) In the following we show that $H_{r,k}$ satisfy all the conditions of Theorem 6.

Clearly, G is an induced subgraph of $H_{r,k}$, and $H_{r,k}$ is regular of degree k by (*i*). Since $r(H_l) = 3$, we have $r(H_{r,k}) = r$.

The graph H_l is decreasing by (*ii*) and Lemma 2. Since the Cartesian product of two decreasing graphs is a decreasing graph, by Lemma 2, $H_{r,k}$ is decreasing, too.

The graph H_l is maximal by (*iii*). Clearly, a statement analogous to (*iii*) holds also for $H_{r,k}$. Thus, $H_{r,k}$ is a maximal graph. $\square$

To close this section we comment on the relation between decreasing and maximal graphs.

The only decreasing graph with radius one is the graph K_2, and the only decreasing graphs with radius two are the complete multipartite graphs $K_{2,2,\ldots,2}$, by Lemma 2. These graphs are maximal, too. Thus, all decreasing graphs with radius $r \le 2$ are maximal. It is known that there are some other decreasing graphs that are maximal, e.g. cube, dodecahedron, icosahedron, and all the graphs constructed in the proof of Theorem 6.

However, for each r, $3 \le r < \infty$, there are decreasing graphs with radius r that are not maximal, as shown in Fig. 2 for the case $r = 3$ and in Fig. 3 for the case $r \ge 4$. (The graph G in Fig. 3 consists of two cycles $C_{2(r-1)}$ joined by four edges, where $d_G(y, x') = d_G(x, y') = r - 1$.)

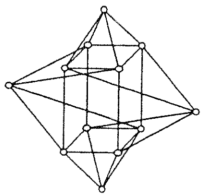

Fig. 2

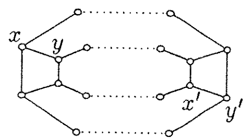

Fig. 3

On the other hand, there are maximal graphs with radius $r \ge 1$ that are not decreasing. Just take a graph with radius r and an odd number of nodes, and complete it to a maximal graph G by adding edges that do not decrease the radius. Since each decreasing graph has an even number of nodes by Lemma 2, G is not a decreasing graph.

4. Cut-nodes in maximal graphs

In this section we describe the structure of maximal graphs with cut-nodes. Moreover, we show that a maximal graph with radius three may have at most two cut-nodes, and a maximal graph with radius r, $4 \le r < \infty$, may have arbitrarily many cut-nodes.

Lemma 7. *Let G be a (radially) maximal graph with radius $r \ge 3$ containing a cut-node y. Then the graph $G - y$ has exactly two components, say A' and B'. Let A and B be the subgraphs of G induced on $V(A') \cup y$ and $V(B') \cup y$, respectively, and let $e_A(y) \ge e_B(y)$. Then:*
1) $e_A(y) + e_B(y) \le 2r - 2, \quad e_A(y) \ge r, \quad e_B(y) \le r - 2$
2) B is a diametrically maximal graph with diameter $e_B(y)$.

Proof. Since $r(G) = r$, there is a branch at y, say A, such that $e_A(y) \ge r$. Let B be the union of all other branches at y.

Suppose that there are two non-adjacent nodes v, w in $(D_i(y) \cup D_{i+1}(y)) \cap V(B)$ for some $i \ge 1$. Let $H = G + vw$. Since G is radially maximal, there is a node $z \in V(H)$ such that $e_H(z) < r$. Clearly, $z \in V(A)$. For each $u \in V(A)$ we have $d_H(z,u) = d_G(z,u)$, and for each $u \in V(B)$ we have $d_H(z,u) = d_H(z,y) + d_H(y,u) = d_G(z,u)$. Hence, $e_H(z) = e_G(z) \geq r$, a contradiction.

Thus, $G-y$ has only two components. Moreover, $B = \{y\}+K_{a_1}+\cdots+K_{a_n}$, where $n = e_B(y)$ and $+$ denotes the sequential join (see the note following Lemma 1).

Suppose $a_n > 1$ and $x \in V(K_{a_n})$. If $n = 1$, we have $r(G+xz) = r(G)$ for any $z \in D_1(y) \cap A$. If $n \geq 2$, we have $r(G+xz) = r(G)$ for any $z \in D_{n-2}(y) \cap B$. Hence, $a_n = 1$ and B is a diametrically maximal graph with diameter $e_B(y)$, by Lemma 1.

According to Lemma 4, $e_A(y) + e_B(y) \leq diam(G) \leq 2r - 2$ and combining this with $e_A(y) \geq r$ we get $e_B(y) \leq r - 2$. $\square$

Now we give the upper bound for the number of cut-nodes in maximal graphs with radius three.

Theorem 8. *Each maximal graph with radius three contains at most two cut-nodes.*

Proof. Let y be a cut-node in maximal graph G. By Lemma 7 the graph $G-y$ consists of two components, and by Lemma 4 the smaller one is a single node, say x. Clearly, $D_1(x) = \{y\}$. Since $diam(G) \leq 2r(G) - 2 = 4$, by Lemma 4, we have $D_i(x) = \emptyset$ for all $i \geq 5$.

Since $e_G(y) \geq 3$, we have $D_4(x) \neq \emptyset$. Suppose that $u, v \in D_4(x)$ and $uv \notin E(G)$. Since G is maximal, we have $r(G+uv) \leq 2$. Thus, there is $c \in D_2(x)$, such that $e_{G+uv}(c) \leq 2$. Since $e_G(c) \geq 3$, $d_G(c,u) \geq 2$, and $d_G(c,v) \geq 2$, we have $e_{G+uv}(c) \geq 3$, a contradiction. Hence, $D_4(x)$ induces a complete graph in G.

Let y' be another cut-node in G, i.e. $y' \neq y$. Denote by x' the isolated node in $G - y'$ (see Lemma 7). Suppose that $y' \in D_2(x)$. Since G is maximal, there is $c \in D_2(x)$, $c \neq y'$, such that $e_{G+xx'}(c) \leq 2$. Since $e_G(c) \geq 3$, $d_G(c,x) = 2$, and $d_G(c,x') \geq 2$, we have $e_{G+xx'}(c) \geq 3$, a contradiction.

Thus, $y' \in D_3(x)$ and $x' \in D_4(x)$. Since $D_4(x)$ induces a complete graph, we have $D_4(x) = \{x'\}$, by Lemma 7, and G has at most two cut-nodes. $\square$

Fig. 4 shows some maximal graphs with radius three containing cut-nodes. From these graphs an infinite number of maximal graphs with cut-nodes can be obtained by node-extensions (see Lemma 5). However, the rightmost graph in Fig. 4 is not a node extension of the middle one.

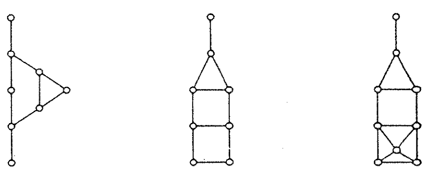

Fig. 4

We finish this section with the following theorem:

Theorem 9. *Let k and r be given integers, $r \geq 4$. Then there is an infinite number of maximal graphs with radius r containing k cut-nodes.*

Proof. At first, we construct a graph $G_{r,l}$ from a complete graph K_l, $l \geq 3$, in three steps:

(1) We attach the end of a path of length $r-3$ to every node of K_l.
(2) We insert three nodes into each edge of K_l. We use the term "superedge" for the paths of length four joining the nodes of K_l.
(3) We paste six edges to every triangle formed by three superedges according to Fig. 5.

The graph $G_{5,3}$ is shown in Fig. 5.

Verifying eccentricities of three nodes in $G_{r,l}$ and using symmetry one can see that $r(G_{r,l}) = r$. Clearly, $G_{r,l}$ has $l(r-3)$ cut-nodes. Now we show, that if we add to $G_{r,l}$ any edge, which decreases the number of cut-nodes, the radius of resulting graph will be less than r.

Denote by G^* the block of $G_{r,l}$ containing $V(K_l)$. Let $uv \notin E(G_{r,l})$, and suppose that $G_{r,l} + uv$ has fewer cut-nodes than $G_{r,l}$. Then at least one node of the edge uv lies outside G^*. Suppose that the distance between u and G^* is not smaller than the distance between v and G^*.

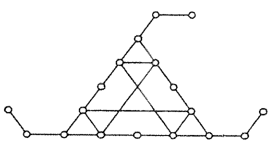

Fig. 5

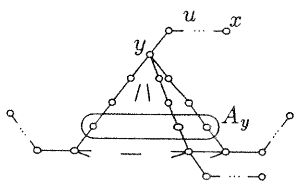

Fig. 6

Denote by x the endnode of a path containing u, and y the node of K_l at shortest distance from u, see Fig. 6. Moreover, let A_y be the set of interior nodes of superedges incident to y, that are at distance three from y. Clearly, $|A_y| = l - 1$ (see Fig. 6).

Let $a \in A_y$. In the following we show that $a \in C(G_{r,l})$ and $D_r(a) = \{x\}$.

- Since a and y have no common neighbor, we have $d_{G_{r,l}}(a,x) = r$.
- Since A_y induces a complete subgraph in $G_{r,l}$, we have $d_{G_{r,l}}(a,z) \leq 3$ for all $z \in V(G^*)$.
- Clearly, $d_{G_{r,l}}(a,z) \leq 2$ for all $z \in V(K_l)$, $z \neq y$.
- Since $r-1 \geq 3$, we have $d_{G_{r,l}}(a,z) \leq r-1$ for all $z \in V(G_{r,l}) - \{x\}$.

Thus, $D_r(a) = \{x\}$ for each $a \in A_y$. Denote $G' = G_{r,l} + uv$. In the following we show, that there exists $a \in A_y$ such that $d_{G'}(a,x) \leq r-1$. Distinguish three cases:

- Suppose that v is a node of path attached to a node $y' \in V(K_l)$.

  Clearly, $d_{G'}(a,x) \leq r-1$ for each $a \in A_y$, if $y' = y$.

  Suppose that $y' \neq y$. Let $a \in A_y$ be the node of superedge joining y and y'. Since $d_{G_{r,l}}(u,y) \geq d_{G_{r,l}}(v,y')$, we have $d_{G_{r,l}}(a,v) \leq d_{G_{r,l}}(a,u) - 2$. Hence, $d_{G'}(a,x) = d_{G_{r,l}}(a,v) + 1 + d_{G_{r,l}}(u,x) \leq r-1$.

- Suppose that v is a node of superedge incident to y.

  Clearly, $d_{G'}(a, x) \leq r - 1$ for all $a \in A_y$, if $v = y$.

  Suppose that $v \neq y$. Let $a \in A_y$ be the node of superedge containing v. Since $d_{G_{r,l}}(a, v) \leq 2 < d_{G_{r,l}}(a, y)$, we have $d_{G'}(a, x) \leq r - 1$.
- Suppose that v is an interior node of a superedge non-incident to y.

  Clearly, there is $a \in A_y$ such that $d_{G'}(v, a) \leq 2$ (see Fig. 5). Thus, $d_{G'}(a, x) \leq r - 1$.

Hence, $r(G') \leq r - 1$. Clearly, from each graph with radius r we can construct a maximal graph with radius r simply by adding edges. As shown above, if we complete the graph $G_{r,l}$ to a maximal graph in any way, we cannot decrease the number of cut-nodes.

In the following let l satisfy $l(r - 3) \geq k$. Let G be some maximal graph constructed from $G_{r,l}$ by adding edges. As noted above, G has exactly $l(r-3)$ cut-nodes and $r(G) = r$.

Clearly, $z \notin Rim(G)$ for every cut-node z of G. Thus, applying the node-extension to $l(r-3) - k$ different cut-nodes, we obtain a maximal graph with radius r and exactly k cut-nodes, by Lemma 5. To obtain an infinite family of graphs satisfying the conditions of Theorem 10, just apply the node-extensions to an interior node of superegde. □

5. Open problems

It is easy to see that $|V(G)| \geq 2r$ for any connected graph G with radius r. Thus, the cycle on $2r$ nodes is maximal graph with radius r and the minimum number of nodes. However, for non-selfcentric maximal graphs we have the following conjecture:

Conjecture 10. *Let G be a non-selfcentric maximal graph with radius $r \geq 3$ and the minimum number of nodes. Then $|V(G)| = 3r - 1$, G is planar containing a cut-node, and the maximum degree of G is three.*

For $r = 3$ there are only two non-selfcentric maximal graphs with radius three on eight nodes (see Fig. 4). For $r = 4$ just eight graphs satisfying Conjecture 10 are known, and for $r = 5$ twenty-two graphs satisfying Conjecture 10 were found by computer.

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