AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **89(3)** (2024), Pages 400–412

Critical graphs upon multiple edge subdivision

Magda Dettlaff

Faculty of Mathematics, Physics and Informatics
University of Gdańsk, Gdańsk
Poland
magda.dettlaff@ug.edu.pl

Magdalena Lemańska

Faculty of Applied Physics and Mathematics
Gdańsk University of Technology, Gdańsk
Poland
magdalena.lemanska@pg.edu.pl

Adriana Roux*

Department of Mathematical Sciences
Stellenbosch University, Stellenbosch
South Africa
rianaroux@sun.ac.za

Abstract

A subset D of $V(G)$ is a *dominating set* of a graph G if every vertex of $V(G)-D$ has at least one neighbour in D; let the domination number $\gamma(G)$ be the minimum cardinality among all dominating sets in G. We say that a graph G is γ-q-*critical* if subdividing any q edges results in a graph with domination number greater than $\gamma(G)$ and there exists a set of $q-1$ edges such that subdividing these edges results in a graph with domination number $\gamma(G)$. In this paper we consider mainly γ-q-critical trees and give some general properties of γ-q-critical graphs; in particular, we characterize those trees T that are γ-$(n(T)-1)$-critical. We also characterize γ-2-critical trees T with $\mathrm{sd}(T)=2$ and γ-3-critical trees T with $\mathrm{sd}(T)=3$, where the domination subdivision number $\mathrm{sd}(G)$ of a graph G is the minimum number of edges which must be subdivided (where each edge can be subdivided at most once) to construct a graph with domination number greater than $\gamma(G)$.

* Corresponding author

ISSN: 2202-3518

1 Introduction

Let $G = (V, E)$ be a connected graph of order $n(G)$ and size $m(G)$.

The *open neighbourhood* $N_G(v)$ of a vertex $v \in V$ is the set of all vertices adjacent to v in G and let the *closed neighbourhood* be the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v is denoted by $\deg_G(v) = |N_G(v)|$. For a set $X \subseteq V$, the *open neighbourhood* $N_G(X)$ is the set $\bigcup_{v \in X} N_G(v)$ and the *closed neighbourhood* is the set $N_G[X] = N_G(X) \cup X$. For a set S, let $N_S[x] = N_G[x] \cap S$.

A vertex v is an *end-vertex* (or a *leaf*) of G if v has exactly one neighbour in G. The set of all end-vertices in G is denoted by $\Omega(G)$.

A vertex v is called a *support* if it is adjacent to an end-vertex. If v is adjacent to only one end-vertex, it is called a *weak support*. Otherwise, v is called a *strong support*. The set of all supports in a graph G is denoted by $S(G)$.

The *distance* between two vertices u, v is the length of a shortest $u - v$ path in a graph G and is denoted by $d_G(u, v)$. A $u - v$ path of length $d_G(u, v)$ is called a $u - v$ *geodesic*. We say that a set $A \subseteq V$ is a 2-*packing* if $d_G(x, y) > 2$ for all $x, y \in A$.

For a graph G, the *subdivision* of an edge $e = uv$ with a new vertex w (called the *subdivision vertex*) is an operation which leads to a graph G_e with $V(G_e) = V(G) \cup \{w\}$ and $E(G_e) = (E(G) \setminus \{uv\}) \cup \{uw, wv\}$. Furthermore, the graph obtained from G by subdividing all the edges in the set $F \subseteq E(G)$ is denoted by G_F.

A subset D of V is a *dominating set* of a graph G if every vertex of $V \setminus D$ has at least one neighbour in D. Let $\gamma(G)$ be the minimum cardinality among all dominating sets in G. A dominating set of cardinality $\gamma(G)$ is called a γ-*set* of G or $\gamma(G)$-*set*. For domination related concepts not defined here, consult [8].

The *domination subdivision number*, $\mathrm{sd}(G)$, of a graph G is the minimum number of edges which must be subdivided (where each edge can be subdivided at most once) in order to increase the domination number. Since the domination number of the graph K_2 does not increase when its only edge is subdivided, we therefore consider only connected graphs of order at least 3. The domination subdivision number was defined by Velammal in 1997 (see [10]) and since then it has been widely studied in graph theory papers. This parameter was studied for trees in [1] and [2]. General bounds and properties have been studied by, among others, [3], [4], [5], and [6].

In [9] Jafari Rad defined a graph to be γ_{sd}-critical if the domination number increases with the subdivision of any single edge. We generalize this concept to consider the case of the subdivision of any q edges. A graph G is γ-q-*critical* if subdividing any q edges results in a graph with domination number greater than $\gamma(G)$ and there exists a set of $q - 1$ edges such that subdividing these edges results in a graph with domination number $\gamma(G)$. The case where $q = 1$ is equivalent to the concept of γ_{sd}-critical graphs defined in [9]. Note that from the definition it follows that $\mathrm{sd}(G) \leq q$ for any γ-q-critical graph G.

In this paper we consider mainly γ-q-critical trees and give some general properties of γ-q-critical graphs; in particular, we characterize those trees T that are

γ-$(n(T)-1)$-critical. We also characterize γ-2-critical trees T with $\mathrm{sd}(T)=2$ and γ-3-critical trees T with $\mathrm{sd}(T)=3$.

2 Preliminary results

Note that the domination number of a graph cannot be decreased with the subdivision of an edge and can increase by at most one.

Proposition 2.1 [9] *For any edge e in a graph G, $\gamma(G) \le \gamma(G_e) \le \gamma(G) + 1$.*

We begin with some general remarks.

Observation 2.2 *If there is a γ-set D in G such that $V \setminus D$ contains a vertex having k neighbours in D, then G is not γ-q-critical for $q \le k-1$.*

Corollary 2.3 *If G is γ-q-critical, then for any γ-set D of G and every $v \in V \setminus D$ we have $|N_D(v)| \le q$.*

Since the subdivision of any k edges in the cycle C_n (the path P_n) leads to a graph isomorphic to C_{n+k} (P_{n+k}), we obtain the following observation.

Observation 2.4 *If a cycle C_n and a path P_n, $n \ge 3$, is γ-q-critical, then*

$$q = \mathrm{sd}(C_n) = \mathrm{sd}(P_n) = \begin{cases} 1 & \textit{if } n \equiv 0 \bmod 3, \\ 2 & \textit{if } n \equiv 2 \bmod 3, \\ 3 & \textit{if } n \equiv 1 \bmod 3. \end{cases}$$

Observation 2.5 [9] *If G contains a universal vertex, then G is γ-1-critical.*

Observation 2.6 *Let $K_{s,t}$ be a complete bipartite graph with $2 \le s \le t$. If $s = 2$, then $K_{s,t}$ is γ-$(t+1)$-critical. Otherwise $K_{s,t}$ is γ-2-critical.*

3 γ-q-critical graphs

We begin this section with some definitions.

The *corona* $G \odot H$ of two graphs G and H is defined as the graph obtained by taking $n(G)$ copies of a graph H and for each $i \le n$ adding edges between the ith vertex of G and each vertex of the ith copy of H.

A *spider* S_t is a graph obtained from the star $K_{1,t}$ for $t \ge 1$ by subdividing each edge of the star. A *d-wounded spider* $S_{t,t-d}$ is the graph formed by subdividing $t-d \le t-1$ edges of a star $K_{1,t}, t \ge 1$ (d is the number of edges that we do not subdivide; $t-1 \ge d \ge 1$). Note that $S_{t,0} = K_{1,t}$, the case where zero edges are subdivided. If $t \geq 2$ and exactly $t-1$ of the edges of a star are subdivided, i.e. $d = 1$, then the resulting graph is called a *slightly wounded spider*.

An *independent set* is a set of vertices in a graph, no two of which are adjacent. The maximum cardinality of an independent set of G is called the *independence number* of G and denoted by $\alpha(G)$. An independent set of cardinality $\alpha(G)$ is called an α*-set* of G.

In the next proposition we show that for every odd number q, there exists a γ-q-critical tree.

Proposition 3.1 *If $T = S_{t,t-k}$ is a k-wounded spider, then T is γ-q-critical for $q = n(T) - k$, where $t - 1 \geq k \geq 2$.*

Proof. Note that $\gamma(T) = t - k + 1$. Label the vertices of T as follows: label the central vertex of $K_{1,t}$ with x and its leaves with $\{v_1, \dots, v_t\}$. Subdivide the edges xv_i with vertices u_i for $i = k+1, \dots, t$.

Let $A = \{xu_i, u_iv_i \mid i = k+1, \dots, t\}$ be a set of $2(t-k) = n(T) - k - 1$ edges. If the edges in A are subdivided, then the set $D = \{x, y_{k+1}, \dots, y_t\}$ is a dominating set of cardinality $t-k+1$, where y_i is the subdivision vertex of the edge u_iv_i. Therefore, $q \geq 2(t-k) + 1 = n(T) - k$.

Let A' be a set of $n(T) - k = 2(t-k) + 1$ edges. Then A' necessarily contains at least one of the edges xv_i for $i \leq k$. Since $k \geq 2$, x is a strong support vertex and therefore the subdivision of the edges in A', producing $T_{A'}$, increases the number of support vertices of T. Hence $\gamma(T_{A'}) \geq |S(T_{A'})| > t - k + 1 = \gamma(T)$. It follows that T is γ-$(n(T) - k)$-critical. □

To show that this result also holds for even q, let T_k be the graph formed by joining the internal vertex of a path P_3 to the vertex of maximum degree of the k-wounded spider $S_{t,t-k}$. We show that T_k is γ-$(n(T_k) - k - 2)$-critical if $k \geq 2$.

Proposition 3.2 *For any even number q, the graph T_k is γ-q-critical, where $q = n(T) - k - 2$ and $k \geq 2$.*

Proof. Note that $\gamma(T_k) = t - k + 2$. In T_k let x be the vertex of maximum degree and let y be its neighbour of degree 3. Then the subdivision of all edges except the pendant edges incident to either x or y will not increase the domination number. Therefore $q \geq 2(t-k) + 2 = n(T_k) - k - 2$.

Let A' be a set of $n(T_k) - k - 2 = 2(t-k) + 1$ edges. Then A' necessarily contains at least one of the pendant edges incident to x or y. The subdivision of the edges in A', producing $(T_k)_{A'}$, increases the number of support vertices of T_k and hence $\gamma((T_k)_{A'}) \geq |S((T_k)_{A'})| > t - k + 2 = \gamma(T_k)$. It follows that T_k is γ-$(n(T_k) - k - 2)$-critical. □

Corollary 3.3 *For each $q \geq 1$ there exists a γ-q-critical tree.*

If a graph G has a strong support vertex, then $\mathrm{sd}(G) = 1$ [7]. That means both $\mathrm{sd}(S_{t,t-k}) = 1$ and $\mathrm{sd}(T_k) = 1$ if $k \geq 2$.

Corollary 3.4 *There exist γ-q-critical graphs G where the difference between q and $\mathrm{sd}(G)$ is arbitrarily large.*

Even if the graph is without leaves, we can obtain a similar result where q is odd. Construct the graph G_k for $k \geq 1$, as follows: take k 4-cycles $H_i \simeq (x_i, y_i, z_i, v_i, x_i)$ for $i = 1, \ldots, k$. Now indentify vertices v_i with one another to obtain the vertex v.

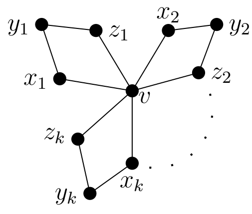

Figure 1: The graph G_k.

Proposition 3.5 *The graph G_k is γ-q-critical for $q = n(G_k) - k$, where $k \geq 1$.*

Proof. Note that $\gamma(G_k) = k+1$, any γ-set D of G_k contains v, and $|D \cap \{x_i, y_i, z_i\}| = 1$ for any $i \in \{1, \ldots, k\}$. Also, $n = n(G_k) = 3k + 1$. Now let $F = \{vx_i, vz_i \mid i = 1, \ldots, k\}$ and consider $(G_k)_F$. The set $D' = \{v\} \cup \{y_i \mid i = 1, \ldots, k\}$ is a γ-set of $(G_k)_F$ of cardinality $k + 1$ and therefore $q \geq |F| + 1 = n - k$.

On the other hand if we subdivide any set F' of $n - k$ edges, then there exists a $j \leq k$ such that $|E(H_j) \cap F'| \geq 3$. This copy becomes a cycle of length at least 7 and we need at least three vertices to dominate it. Therefore $\gamma((G_k)_{F'}) > \gamma(G_k)$ and hence G_k is γ-$(n - k)$-critical. □

It is also possible for q to be larger than n. Let A_k be the graph obtained from $K_{3,k+5}$, for $k \geq 0$, by adding a leaf to each of the vertices in the partite set V_1, where $|V_1| = 3$. Let $V_1 = \{v_1, v_2, v_3\}$, $V_2 = \{u_1, \ldots, u_{k+5}\}$ and label the leaves x_i for $i = 1, 2, 3$.

Proposition 3.6 *The graph A_k is γ-q-critical for $q = n(A_k) + k$, where $k \geq 0$.*

Proof. Note that $\gamma(A_k) = 3$ and V_1 is a γ-set of A_k. Also, $n = n(A_k) = k + 11$. Now let $F = \{v_1u_i, v_2u_i \mid i = 1, \ldots, k+5\}$. Then V_1 is a γ-set of $(A_k)_F$ and therefore $q \geq 2k + 11 = n + k$.

On the other hand consider any set F' of $n + k$ edges. If there is at least one u_i incident to three edges of F', then $\gamma((A_k)_{F'}) > \gamma(A_k)$. Otherwise, every u_i is incident to at most two edges of F' and therefore at least one pendant edge, say x_1v_1, belongs to F'. If $v_1u_j \in F'$ for some $j \leq k+5$, then clearly $\gamma((A_k)_{F'}) > \gamma(A_k)$. If $v_1u_j \notin F'$ for all $j \leq k+5$, then F' contains at least $2k+8$ edges of the form v_iu_j for $j = 2, 3$. It is easy to check that $\gamma((A_k)_{F'}) > \gamma(A_k)$. It now follows that A_k is γ-$(n+k)$-critical. □

Proposition 3.7 *Let G be a graph of order $n(G)$ and size $m(G) \geq 1$. Then $H = G \odot K_1$ is γ-$(m(G)+1+\alpha(G))$-critical.*

Proof. Of course, every minimum dominating set of $H = G \odot K_1$ has cardinality $n(G)$. Let A be an α-set of G. Then $D = V(G) \setminus A$ is a dominating set of G. Label the vertices of G as $v_1, v_2, \ldots, v_n$, where $v_1, v_2, \ldots v_{\alpha(G)} \in A$ and label the copies of K_1 in H with u_i, $i \in \{1, \ldots, n\}$.

Let $F = E(G) \cup \{v_iu_i \mid i = 1, \ldots, \alpha(G)\}$ and consider H_F. The set $D \cup \{w_1, \ldots, w_{\alpha(G)}\}$, where w_i is a subdivision vertex of v_iu_i, is a dominating set of H_F of cardinality $n(G)$, showing that the subdivision of $|F| = m(G) + \alpha(G)$ edges does not increase the domination number of H.

Now consider a set B of $m(G)+1+\alpha(G)$ edges of H and let $B' = B \cap \{v_iu_i \mid 1 \leq i \leq n\}$. Then $|B'| \geq \alpha(G) + c$ with $c \geq 1$ and there exist edges $v_ju_j, v_ku_k \in B$ such that $v_jv_k \in E(G)$, where j, k can be chosen in such a way that $v_jv_k \in B$.

Let us consider H_B and let D' be a γ-set of H_B. Then $D_1 = D' \cap \{w_i, u_i\} \neq \varnothing$ for each $v_iu_i \in B'$, where w_i subdivides v_iu_i and $D_2 = D' \cap \{v_i, u_i\} \neq \varnothing$ for $v_iu_i \notin B'$. The set $D_1 \cup D_2$ however does not dominate the subdivision vertex of the edge v_jv_k and since $|D_1 \cup D_2| \geq n(G)$, we have $|D'| > n = \gamma(H)$. This proves that H is γ-$(m(G)+1+\alpha(G))$-critical. □

Corollary 3.8 *If G is a tree, then $H = G \odot K_1$ is γ-$(n(G)+\alpha(G))$-critical.*

Since $G = K_{1,r} \odot K_1$ has $2r+2$ vertices, $n(K_{1,r}) = r+1$ and $\alpha(K_{1,r}) = r$, $r \geq 1$, it follows that $K_{1,r} \odot K_1$ is γ-$(n(G)-1)$-critical.

Jafari Rad characterized the case where $q = 1$ as follows:

Theorem 3.9 [9] *A graph G is γ-1-critical if and only if every γ-set of G is a 2-packing.*

We show that $K_{1,r} \odot K_1$, $r \geq 1$, is the only $q = (n(T)-1)$-critical tree.

Theorem 3.10 *For a γ-q-critical tree T, $q = n(T)-1$ if and only if $T = K_{1,r} \odot K_1$ for some $r \geq 1$ (i.e. T is a slightly wounded spider).*

Proof. If T is a slightly wounded spider it follows from Corollary 3.8 that T is γ-$(n(T)-1)$-critical.

Now assume that T is γ-$(n(T)-1)$-critical and let $O = V(T) \setminus (\Omega(T) \cup S(T))$. To show that T is a corona graph we show that $T = T' \odot K_1$ for some tree T' (i.e. T has only weak supports and leaves and that $O = \varnothing$).

First assume that T has a strong support vertex x with at least two neighbours $x_1, x_2 \in \Omega(T)$. Since x belongs to any γ-set of T and x_1, x_2 to no $\gamma(T)$-set, it is easy to see that subdividing the edge xx_i results in a graph with domination number greater than $\gamma(T)$, i.e. $\gamma(T_{xx_i}) > \gamma(T)$. Since any set of $n-2$ edges contains xx_1 or xx_2, $\gamma(T_F) > \gamma(T)$ for any set F of $n-2$ edges. It follows that T is not γ-$(n(T)-1)$-critical, a contradiction. Thus every support of T is a weak support.

Now assume that $O \neq \varnothing$. Since T is connected there are at least two edges between $S(T)$ and O. We consider two cases:

Case 1. If $|O| \leq 2$, then $S(T)$ is a γ-set of T and there exist two edge-disjoint paths $P_i = (z_i, x_i, y_i)$ where $y_i \in O$, $x_i \in S(T)$ and $z_i \in \Omega(T)$ for $i = 1, 2$. Note that it is possible that $y_1 = y_2$.

Let $F_i = \{z_i x_i, x_i y_i\}$ for $i = 1, 2$ and consider the graph T_{F_i}. Suppose that $x_i y_i$ are subdivided by w_i. Then w_i is not dominated by a support vertex of T_{F_i} and $\gamma(T_{F_i}) \geq |S(T_{F_i})| + 1 = |S(T)| + 1 > \gamma(T)$. Since any set of $n-2$ edges contains F_1 or F_2, $\gamma(T_F) > \gamma(T)$ for any set F of $n-2$ edges, a contradiction.

Case 2. If $|O| > 2$, there exist two non-adjacent vertices $y_1, y_2 \in O$ adjacent to two different vertices in $S(T)$, say x_1 and x_2, respectively. Let $F_i = \{z_i x_i\} \cup \{y_i v \mid v \in N(y_i)\}$, $i \in \{1, 2\}$. Suppose that the edges $z_i x_i$ and $x_i y_i$ are subdivided by $f_{i,1}$ and $f_{i,2}$, respectively, and that the remaining edges incident to y_i are subdivided by $f_{i,j}$ for $j = 3, \ldots, d_T(y_i) + 1$.

Now consider T_{F_i}. Let D_{F_i} be a γ-set of T_{F_i} with the minimum number of subdivision vertices. Since $\{z_i, f_{i,1}\} \cap D_{F_i} \neq \varnothing$, we consider two subcases:

Subcase 2.1. Let $f_{i,1} \in D_{F_i}$. If $x_i \in D_{F_i}$, then $D = (D_{F_i} - \{f_{i,j} \mid j \geq 1\}) \cup \{y_i\}$ is a dominating set of T with $|D| < |D_{F_i}|$. Hence assume that $x_i \notin D_{F_i}$. By the choice of D_{F_i} (as a dominating set containing the smallest number of subdivision vertices), $y_i \in D_{F_i}$ to dominate $f_{i,2}$ and $\{f_{i,j} \mid j \geq 2\} \cap D_{F_i} = \varnothing$. Hence, $D = (D_{F_i} - \{y_i, f_{i,1}\}) \cup \{x_i\}$ is a dominating set of T with $|D| < |D_{F_i}|$.

Otherwise, from the choice of D_{F_i} (it has the smallest number of subdivision vertices) $y_i \in D_{F_i}$ and $\{f_{i,j} \mid j \geq 2\} \cap D_{F_i} = \varnothing$. Hence, $D = (D_{F_i} - \{y_i, f_{i,1}\}) \cup \{x_i\}$ is a dominating set of T with $|D| < |D_{F_i}|$.

Subcase 2.2. Let $z_i \in D_{F_i}$; then $f_{i,1} \notin D_{F_i}$. First assume $x_i \in D_{F_i}$; then $f_{i2} \notin D_{F_i}$. To dominate y_i, either y_i or some $f_{i,j}, j \geq 3$, belongs to D_{F_i}. In this case $D = (D_{F_i} - (\{z_i\} \cup \{f_{i,j} \mid j \geq 2\})) \cup \{y_i\}$ is a dominating set of T with $|D| < |D_{F_i}|$. Hence assume $x_i \notin D_{F_i}$. To dominate $f_{i,2}$, either y_i or $f_{i,2}$ is in D_{F_i}. Thus $D = (D_{F_i} - \{f_{i,2}, y_i, z_i\}) \cup \{x_i\}$ is a dominating set of T with $|D| < |D_{F_i}|$. Since $F_1 \cap F_2 = \varnothing$, any set of $n-2$ edges contains F_1 or F_2. Therefore $\gamma(T_F) > \gamma(T)$ for any set F of $n-2$ edges, a contradiction.

It follows that $O = \varnothing$ and hence $T = T' \odot K_1$ for a tree T' and T is γ-q-critical for $q = n(T) - 1$. By Corollary 3.8, $q = n(T') + \alpha(T')$. Since $n(T) = 2n(T')$ it follows that $\alpha(T') = n(T') - 1 = m(T')$. Thus T' is a star and $T = K_{1,r} \odot K_1$ for $r \geq 1$.

If $z_i \in D_{F_i}$, then obviously $f_{i,1} \notin D_{F_i}$. Assume $x_i \in D_{F_i}$. In this case $D = (D_{F_i} - (\{z_i\} \cup \{f_{i,j} \mid j \geq 2\})) \cup \{y_i\}$ is a dominating set of T with $|D| < |D_{F_i}|$. Now let $x_i \notin D_{F_i}$. If $f_{i,2} \in D_{F_i}$, then from the choice of D_{F_i} we have $(\{y\} \cup \{f_{i,j} \mid j \geq 3\}) \cap D_{F_i} = \varnothing$. Thus $D = (D_{F_i} - \{f_{i,2}, z_i\}) \cup \{x_i\}$ is a dominating set of T with $|D| < |D_{F_i}|$. Finally, if $f_{i,2} \notin D_{F_i}$, then $y_i \in D_{F_i}$ and $\{f_{i,j} \mid j \geq 3\} \cap D_{F_i} = \varnothing$ (from the choice of D_{F_i}). In this case $D = (D_{F_i} - \{y_i, z_i\}) \cup \{x_i\}$ is a dominating set of T with $|D| < |D_{F_i}|$.

Since $F_1 \cap F_2 = \varnothing$, any set of $n - 2$ edges contains F_1 or F_2. Therefore $\gamma(T_F) > \gamma(T)$ for any set F of $n - 2$ edges, a contradiction.

It follows that $O = \varnothing$ and hence $T = T' \odot K_1$ for a tree T' and T is γ-q-critical for $q = n(T) - 1$. By Corollary 3.8, $q = n(T') + \alpha(T')$. Since $n(T) = 2n(T')$ it follows that $\alpha(T') = n(T') - 1 = m(T')$. Thus T' is a star and $T = K_{1,r} \odot K_1$ for $r \geq 1$. □

4 γ-q-critical trees with $\mathrm{sd}(T) = q$

As shown in [10], the subdivision number of any tree lies between 1 and 3. Combining the characterization of γ-1-critical graphs in [9] and the characterization of trees with $\mathrm{sd}(T) = 1$ in [2] shows which trees with $\mathrm{sd}(T) = 1$ are also γ-1-critical. We now characterize γ-2-critical trees T with $\mathrm{sd}(T) = 2$ and γ-3-critical trees T with $\mathrm{sd}(T) = 3$.

4.1 γ-2-critical trees

Theorem 4.1 *A tree T is γ-2-critical if and only if*

1. *every γ-set D of T contains at most one pair of vertices x, y such that $1 \leq d_T(x, y) \leq 2$, and if such a pair x, y exists, then each of x and y has at least two neighbours not in D, and*

2. *T has a γ-set containing exactly one such a pair of vertices x, y.*

Proof. Suppose that there is no γ-set D in T with exactly one pair of vertices $x, y \in D$ such that $d_T(x, y) \in \{1, 2\}$. Then every γ-set in T is a 2-packing or contains more than one pair of vertices at distance at most 2. In the first case it follows from Theorem 3.9 that T is γ-1-critical.

So suppose that T has a γ-set D with at least two pairs of vertices $\{x_1, y_1\}$ and $\{x_2, y_2\}$ such that $d_T(x_i, y_i) \leq 2$, for $i \in \{1, 2\}$; note that it is possible that $\{x_1, y_1\} \cap \{x_2, y_2\} \neq \varnothing$. On the $x_i - y_i$ geodesic, let v_i be the vertex adjacent to x_i (note that it is possible that $v_i = y_i$). If the edge $x_i v_i$ is subdivided with w_i, then x_i dominates w_i and y_i dominates v_i. Hence there exist two edges whose subdivision does not increase the domination number of T and hence T is not γ-2-critical.

Thus there exists a $\gamma(T)$-set D with exactly one pair of vertices $x, y \in D$ such that $d_T(x, y) \in \{1, 2\}$. We may assume that $D \cap \Omega(T) = \varnothing$, otherwise we may exchange a leaf with its support vertex. If this exchange results in a dominating set having two pairs of vertices $x, y \in D$ such that $d_T(x, y) \in \{1, 2\}$, then we obtain the case considered in the paragraph above. Hence assume this is not the case and suppose at least one of x or y, say x, has at most one neighbour not in D. Since D is a γ-set of T, x and y has at least one neighbour in $V(T) \setminus D$. Hence, x has exactly one neighbour $x' \in V(T) \setminus D$. Since x is not a leaf, $xy \in E(T)$. Subdivide the edges xx' and xy with w_1 and w_2, respectively, to form T'. Then $(D - \{x\}) \cup \{w_1\}$ is a dominating set of T' and therefore T is not γ-2-critical.

Conversely, assume that every γ-set of T has the desired property and to the contrary suppose that T is not γ-2-critical. Hence there exists a set of two edges $F = \{e_1 = x_1y_1, e_2 = x_2y_2\}$ such that $\gamma(T_F) = \gamma(T)$. Let w_1 and w_2 be the subdivision vertices of e_1 and e_2, respectively.

Let D' be a γ-set of T_F with the smallest number of subdivision vertices and consider the following cases.

Case 1. Edges e_1 and e_2 are adjacent. Without loss of generality assume that $x = x_1 = x_2$. If $x \in D'$, then there is a vertex $z_i \in N[y_i] \cap D'$ for $i = 1, 2$. From the choice of D', it follows that $w_1, w_2 \notin D'$. Therefore, D' is a γ-set of T such that $d_T(x, z_i) \leq 2$ for $i = 1, 2$, a contradiction.

Now consider the case where $x \notin D'$. If $w_1, w_2 \notin D'$, then $y_1, y_2 \in D'$ and there exists a vertex $x' \in N(x) \setminus \{w_1, w_2\}$ such that $x' \in D'$. Therefore, D' is a γ-set of T such that $d_T(x', y_i) \leq 2$ for $i = 1, 2$, a contradiction. Thus $w_i \in D'$ for at least one i and by the choice of D' exactly one, say w_1, belongs to D'. It is clear that $y_2 \in D'$. If $\deg_{T_F}(x) > 2$, then there exists $x' \in N(x) \setminus \{w_1, w_2\}$. Since $x \notin D'$, there exists $x'' \in N[x'] \cap D'$. But then $D = (D' \setminus \{w_1\}) \cup \{x\}$ is a γ-set of T such that $d_T(x'', x) \leq 2$ and $d_T(x, y_2) = 1$, a contradiction. On the other hand, if $\deg_{T_F}(x) = 2$, then $D = (D' \setminus \{w_1\}) \cup \{x\}$ is a γ-set of T such that $d(x, y_2) = 1$, but y_1 is the only neighbour of x outside of D, a contradiction.

Case 2. Edges e_1 and e_2 are not adjacent. We show that there exists a γ-set D of T such that for each edge e_i there exists a pair $u_i, v_i \in D$ such that $d_T(u_i, v_i) \leq 2$. If $w_1, w_2 \notin D'$, then at least one of x_1, y_1, say x_1, belongs to D' and at least one of x_2, y_2, say x_2, belongs to D'. Then there exists $z_i \in N[y_i] \setminus \{w_i\}$ such that $z_i \in D'$ for $i = 1, 2$. Therefore, D' is a γ-set of T such that $d_T(x_i, z_i) \leq 2$ for $i = 1, 2$, a contradiction. Thus $w_i \in D'$ for at least one i.

Subcase 2.1. $d_T(\{x_1, y_1\}, \{x_2, y_2\}) = 1$, say $d_T(x_1, x_2) = 1$.

- Suppose $w_1, w_2 \in D'$. Then by the choice of D', $x_1, x_2, y_1, y_2 \notin D'$. Thus $D = (D' \setminus \{w_1, w_2\}) \cup \{x_1, x_2\}$ is a γ-set of T.
  If $\deg_T(x_1) > 2$, then there exists a vertex $x'' \in D'$ such that $d_T(x_1, x'') \leq 2$. Since $d_T(x_1, x_2) = 1$, D is a γ-set of T containing two pairs of vertices at distance at most 2, a contradiction.

  On the other hand, if $\deg_T(x_1) = 2$, then $d_T(x_1, x_2) = 1$ and y_1 is the only neighbour of x_1 outside of D, also a contradiction.

- Assume now only one of w_1 or w_2 belongs to D', say w_1. Thus by the choice of

D', $x_1, y_1 \notin D'$. Also, $x_2 \notin D'$, otherwise $D' \setminus \{w_1\}) \cup \{y_1\}$ would be a γ-set of T_F contradicting our choice of D'. Since D' is dominating, $y_2 \in D'$ and there exists $x' \in N(x_2) \setminus \{x_1, w_2\}$ such that $x' \in D'$. Then $D = (D' \setminus \{w_1\}) \cup \{x_1\}$ is a γ-set of T and $d_T(x_1, x') = d_T(x', y_2) = 2$, a contradiction.

Subcase 2.2. $d_T(\{x_1, y_1\}, \{x_2, y_2\}) > 1$. At least one of w_1, w_2, say w_1, belongs to D'. Then $x_1, y_1 \notin D'$ and since T is connected at least one of x_1 or y_1, say x_1, has degree greater than 1. Therefore there exists a vertex x'' such that $d_T(x_1, x'') = 2$ and $x'' \in D'$.

- If $w_2 \in D'$, there similarly exists $y'' \in D'$ such that $d_T(x_2, y'') = 2$.
- If $w_2 \notin D'$, then without loss of generality $x_2 \in D'$ and there exists a vertex $z \in N[y_2] \setminus \{w_2\}$ such that $z \in D'$.

In both cases $D = (D' \setminus \{w_i\}) \cup \{x_i\}$ is a γ-set of T containing two pairs of vertices at distance at most 2, a contradiction.

Hence T is γ-2-critical. □

From Observation 2.4 we know that the paths P_{3k+2}, for $k \geq 1$, are γ-2-critical. Double stars of order more than 4, that is, trees obtained by joining the central vertices of two disjoint stars, are also γ-2-critical.

Let $\mathcal{N}(G)$ consists of those vertices which are not contained in any $\gamma(G)$-set. Benecke and Mynhardt [2] characterized all trees with domination subdivision number equal to 1 as follows:

Theorem 4.2 [2] *For a tree T of order $n \geq 3$, $\mathrm{sd}(T) = 1$ if and only if T has*

i) a leaf $u \in \mathcal{N}(T)$ or

ii) an edge xy with $x, y \in \mathcal{N}(T)$.

Note that if a tree T has a strong support vertex, the leaves adjacent to the strong support vertex belong to $\mathcal{N}(T)$ and therefore $\mathrm{sd}(T) = 1$.
We use Theorem 4.2 to characterize γ-2-critical trees T with subdivision number equal to 2.

Theorem 4.3 *The only γ-2-critical trees T with $\mathrm{sd}(T) = 2$ are the paths $T = P_{3k+2}$ for $k \geq 1$.*

Proof. Let T be a γ-2-critical tree such that $\mathrm{sd}(T) = 2$. Then $T \neq P_4$. By Theorem 4.2, T has no strong support vertices. Let D be a $\gamma(T)$-set and $L = \Omega(T) \cap D$. Then $D' = (D \setminus L) \cup N(L)$ is also a γ-set of T. If $|L| > 1$, then D' would contain more than one pair of vertices at distance at most 2, contradicting Theorem 4.1. Thus, $|L| \leq 1$ for any $\gamma(T)$-set D.

Claim 4.3.1 *If T has a γ-set D such that $|L| = 1$, then D is a 2-packing.*

Proof of claim: Let D be a $\gamma(T)$-set such that $|L| = 1$ and assume D is not a 2-packing. Then either we could find more than one pair of vertices at distance at most 2 in D' or $x \in L$ would be at distance at most 2 from another vertex in D and x would have only one neighbour in $V(T) - D$, contradicting the assumption that T is γ-2-critical.

It is easy to observe that in this case if D contains one leaf x of T, then any γ-set of $T - N[x]$ is 2-packing, so D is the unique γ-set of T such that $L = \{x\}$ and D is 2-packing. □

To show that T is a path, assume to the contrary that there exists a vertex v such that $\deg_T(v) \geq 3$. Root T at v and label the subtree rooted at v_i, where $v_i \in N(v)$, with T_i for $1 \leq i \leq \deg_T(v)$.
Since T is γ-2-critical, v is not a strong support vertex and at most one of these subtrees is the trivial graph. Assume T_j is trivial for some $1 \leq j \leq \deg_T(v)$ and let $V(T_j) = \{u\}$. Since $\mathrm{sd}(T) = 2$ it follows from Theorem 4.2 that there exists a $\gamma(T)$-set D_j such that $u \in D_j$. Since D_j is dominating, $D_j \cap N[v_i] \neq \varnothing$ for each $i \neq j$. But then $D'_j = (D_j - \{u\}) \cup \{v\}$ is a $\gamma(T)$-set with at least two vertices in D'_j at distance at most 2 from v, contradicting that T is γ-2-critical. Therefore v is not a support vertex.
Now, for each $1 \leq i \leq \deg_T(v)$, let $u_i \in \Omega(T) \cap V(T_i)$ and let $s_i \in N(u_i)$. Since $\mathrm{sd}(T) = 2$ it follows from Theorem 4.2 that there exists a $\gamma(T)$-set D_i such that $u_i \in D_i$.

Obviously, $D'_i = (D_i - \{u_i\}) \cup \{s_i\}$ is a $\gamma(T)$-set and it follows from the proof of Claim 4.3.1 that $D'_i - \{s_i\}$ is a unique 2-packing with $S(T) - \{s_i\} \subseteq D'_i$. This implies that $D_i \cap V(T_j) = D'_i \cap V(T_j) = D'_k \cap V(T_j) = D_k \cap V(T_j)$ for every $i \neq j \neq k \in \{1, \ldots, \deg_T(v)\}$ (for example $D'_1 \cap V(T_2) = D'_3 \cap V(T_2)$ and $D'_1 \cap V(T_3) = D'_2 \cap V(T_3)$ and $D'_2 \cap V(T_1) = D'_3 \cap V(T_1)$). It follows that either $v \in D_i$ for each $1 \leq i \leq \deg_T(v)$ or $v \notin D_i$ for each $1 \leq i \leq \deg_T(v)$.

If $v \in D_i$ for each $1 \leq i \leq \deg_T(v)$, then $D^* = (D_1 - V(T_2)) \cup (V(T_2) \cap D_2)$ is a $\gamma(T)$-set with more than one leaf, a contradiction. Otherwise, $v \notin D_i$ and v is dominated by v_j for some $1 \leq j \leq \deg_T(v)$. Since D'_i already has a pair of vertices at distance at most two, it follows that $\{v_i \mid i \neq j\} \cap D = \varnothing$. Then $D^* = (D_j - V(T_\ell)) \cup (V(T_\ell) \cap D_\ell)$, where $\ell \neq j$, is a $\gamma(T)$-set with more than one leaf, a contradiction.

It follows that T is a path and from Observation 2.4, $T = P_{3k+2}$ for $k \geq 1$. □

4.2 γ-3-critical trees

The following constructive characterization of the family $\mathcal{F}$ of labeled trees T with $\mathrm{sd}(T) = 3$ was given by Aram, Sheikholeslami and Favaron [1].

Let $\mathcal{F}$ be the family of labelled trees such that $\mathcal{F}$

- contains P_4 where the two leaves have status A and the two support vertices have status B; and
- is closed under the two operations $\mathcal{T}_1$ and $\mathcal{T}_2$, which extend the tree T by attaching a path to a vertex $v \in V(T)$.

Operation $\mathcal{T}_1$. Assume $sta(v) = A$. Then add a path (x, y, z) and the edge vx. Let $sta(x) = sta(y) = B$ and $sta(z) = A$.

Operation $\mathcal{T}_2$. Assume $sta(v) = B$. Then add a path (x, y) and the edge vx. Let $sta(x) = B$ and $sta(y) = A$.

If $T \in \mathcal{F}$, we let $A(T)$ and $B(T)$ be the set of vertices of status A and B, respectively, in T. It was shown in [1] that $A(T)$ is a $\gamma(T)$-set and contains all leaves of T.

Theorem 4.4 [1] *For a tree T of order $n \geq 3$,*

$$\mathrm{sd}(T) = 3 \text{ if and only if } T \in \mathcal{F}.$$

We use this result to show that that paths of order $3k + 1$, for $k \geq 1$, are the only γ-3-critical trees with $\mathrm{sd}(T) = 3$.

Theorem 4.5 *The only γ-3-critical trees T with $\mathrm{sd}(T) = 3$ are the paths $T = P_{3k+1}$ for $k \geq 1$.*

Proof. Let T be a tree with $\mathrm{sd}(T) = 3$. By Theorem 4.4, $T \in \mathcal{F}$ and there exists a $\gamma(T)$-set D containing all the leaves. Let $\Omega(T) = \{v_1, \ldots, v_\ell\}$ and let u_i be the neighbour of v_i. Now, let $F = \{u_i v_i \mid i = 1, \ldots, \ell\}$ and consider the graph T_F where the subdivision vertices are denoted by w_i, respectively. Then $(D \setminus \Omega(T)) \cup \{w_i \mid i = 1, \ldots, \ell\}$ is a γ-set of T_F. It therefore follows that if T is γ-q-critical, then $q > |\Omega(T)|$. Hence if T is γ-3-critical, $|\Omega(T)| = 2$ and T is a path. By Observation 2.4 $T = P_{3k+1}$ for $k \geq 1$. □

5 Open problems

We conclude by mentioning a number of open problems.

Problem 1 *Determine which trees T with $\mathrm{sd}(T) = 1$ are γ-2-critical or γ-3-critical.*

Problem 2 *Characterise γ-q-critical trees T for $2 \leq q \leq n(T) - 2$.*

Problem 3 *Characterise graphs G with $\mathrm{sd}(G) = q$ which are also γ-q-critical.*

In Theorem 3.10, γ-q-critical trees T for $q = n(T) - 1$ were characterised and it was shown that q is odd. Hence, if $n(T) - 1$ is even, there are no $\gamma - (n(T) - 1)$-critical trees.

Problem 4 *For which values of k do there exist $\gamma - (n(T) - k)$-critical trees, and if they exist, do they exist for all values of $n(T)$?*

References

[1] H. Aram, S. M. Sheikholeslami and O. Favaron, Domination subdivision number of trees, *Discrete Math.* **309** (2009), 622–628.

[2] S. Benecke and C. M. Mynhardt, Trees with domination subdivision number one, *Australas. J. Combin.* **42** (2008), 201–209.

[3] A. Bhattacharya and G. R. Vijayakumar, Effect of edge–subdivision on vertex–domination in a graph, *Discuss. Math. Graph Theory* **22** (2002), 335–347.

[4] O. Favaron, T. W. Haynes and S. T. Hedetniemi, Domination subdivision numbers in graphs, *Util. Math.* **66** (2004), 195–209.

[5] O. Favaron, H. Karami and S. M. Sheikholeslami, Disproof of a conjecture on the subdivision domination number of a graph, *Graphs Combin.* **24** (2008), 309–312.

[6] T. W. Haynes, S. M. Hedetniemi and S. T. Hedetniemi, Domination and independence subdivision numbers of graphs, *Discuss. Math. Graph Theory* **20** (2000), 271–280.

[7] T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi, D. P. Jacobs, J. Knisely and L. C. Van der Merwe, Domination subdivision numbers, *Discuss. Math. Graph Theory* **21** (2001), 239–253.

[8] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker Inc., New York (1998).

[9] N. Jafari Rad, Domination critical graphs upon edge subdivision, *J. Combin. Math. Combin. Comput.* **87** (2013), 101–113.

[10] S. Velammal, Studies in graph theory: covering, independence, domination and related topics, Ph.D. Thesis, Manonmaniam Sundaranar University, Tirunelveli, 1997.

(Received 16 May 2023; revised 19 Apr 2024, 7 May 2024)