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Characterising 3-polytopes of radius one with unique realisation

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Abstract

The class of graphs that are planar, 3-connected, of radius one, are exactly the 1-skeletons of polyhedra with one vertex adjacent to all others. Let F be a planar, 3-connected graph of radius one on p vertices, with a vertices of degree three. We characterise all unigraphic degree sequences for such graphs, when $a \geq 3$ and p is large enough with respect to a. The proof methods reveal the structure of this class of graphs. We also solve the case $a = 2$ for any value of p.

1 Introduction

1.1 Results

The distance $d(u, v)$ between two vertices u, v of a (finite, simple) connected graph $F = (V, E)$ is the length of a minimal uv-path. The graph radius of F is defined as

$$\mathrm{rad}(F) := \min\{\mathrm{ecc}(v) : v \in V\},$$

where

$$\mathrm{ecc}(v) := \max\{d(v, w) : w \in V\}$$

is the eccentricity of v. In this paper we will refer to the graph radius simply as the radius.

The planar, 3-connected graphs are the 1-skeletons of 3-polytopes [26]. In the literature these graphs are in fact called 3-polytopes, or sometimes polyhedra. They are the planar graphs that are *uniquely* embeddable in a sphere [28]. Their regions are bounded by cycles (polygons), and this fact is true more generally for planar, 2-connected graphs [5, Proposition 4.26]. For recent work on distance, radius, and related topics for graphs, see e.g. [23, 1, 15]. For distance and radius topics for polytopes, see e.g. [24, 25, 22].

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According to Tutte's Theorem [27, Theorem (6.1)], if F is a 3-polytope of size (i.e. number of edges) q that is not a pyramid (i.e. a wheel graph), then either F or its dual may be obtained by adding an edge to a 3-polytope of size $q-1$. This yields an algorithm to generate all 3-polytopes. For the sub-family of 3-polytopes of graph radius 1, there is the following faster algorithm to generate them all. Henceforth, expressions such as $F-v$, $F+v$, $F-\{e_1,e_2\}$ stand for removing/adding vertices/edges/sets of vertices or edges from/to a graph.

Remark 1.1. Every 3-polytope of graph radius 1 and size q that is not a pyramid may be obtained by adding an edge to a 3-polytope of radius 1 and size $q-1$. To see this, observe that a graph F is 3-polytopal of radius 1 if and only if $F-v$ is planar and has a region bounded by all remaining vertices, where v is a vertex of eccentricity 1 in F. If F is not a pyramid, then there is an edge e such that $F-v-e$ still has a region bounded by all remaining vertices. The author wishes to thank Lionel Brütsch and Niels Willems for helping to make this point explicit.

The degree of a vertex is the number of edges incident to it (vertices adjacent to it). Letting $V=\{v_1,v_2,\dots,v_p\}$ and $d_i=\deg(v_i)$ for $i=1,2,\dots,p$, we call

$$s : d_1, d_2, \dots, d_p \tag{1.1}$$

the degree sequence of F. The elements of (1.1) are usually written in non-increasing order, however, in this paper we cannot make this assumption globally. That is because several sequences that we will consider depend on one or more parameters, and which entries in the sequence are bigger sometimes depends on the values of these parameters.

Vice versa, a sequence s (1.1) is called graphic (sometimes 'graphical') if there exists an order (i.e. number of vertices) p graph F of degree sequence s. We then say that F is a realisation of s. The classical theorems of Havel [11], Hakimi [9] and Erdős-Gallai [7] determine when s is graphic. With the Havel–Hakimi algorithm [11, 9], one can construct such an F realising s. For recent work on degree sequences of graphs, see e.g. [2, 23, 15]. For recent results on degree sequences of 3-polytopes, see e.g. [20, 19].

A sequence s is called *unigraphic* if, up to isomorphism, there is exactly one graph of degree sequence s. With some abuse of terminology, we will refer to the corresponding realisation F as unigraphic. Koren [13] and Li [14] devised criteria to establish when a given s is unigraphic. Unigraphic non-2-connected graphs have been classified [12], and also those satisfying $d_2=d_{p-1}$ when the sequence (1.1) is written in non-increasing order [13, Theorem 6.1].

The problem might be more treatable if we ask which s are uniquely realisable by a graph F belonging to a given sub-class of graphs, i.e., satisfying certain properties. For instance, it is easy to explicitly list the types of unigraphic sequences of tree graphs [10, Exercise 6.11]. Then it may or may not be the case that s is unigraphic with respect to the class of all graphs, e.g. $2,2,2,1,1$ is unigraphic *with respect to the sub-class of trees*, but not unigraphic with respect to the class of all graphs.

In this paper, we consider the following question. What are the degree sequences that are realised by one and only one 3-polytopal graph? In this paper, we attempt to answer this question for the class of 3-polytopes of radius 1. Plainly $\mathrm{rad}(F) = 1$ means that there exists a vertex, v_1 say, adjacent to all others, i.e. we can take $d_1 = p - 1$. In what follows, we will denote by a the number of degree 3 vertices in F,

$$a = a(F) := \#\{v \in V : \deg(v) = 3\} \tag{1.2}$$

(this is the smallest possible degree, as F is 3-connected). It is not difficult to show that $a \geq 2$ (see Lemma 1.5 below).

Our main result characterises the families of unigraphic sequences where $a \geq 3$ and p is large enough compared to a. The notation n^m in a sequence means that the value n appears m times.

Theorem 1.2. *Let s be a sequence as in (1.1), and denote by a the number of degree three vertices. Assuming $a \geq 3$ and $p \geq 3a$, then s is unigraphic as a 3-polytope of radius 1 if and only if s is one of the following:*

$$\begin{aligned}
B1 &: p-1, (x+3)^2, p-1-2x+3, 4^{p-7}, 3^3, && p \geq 10,\ 3 \leq x \leq \lfloor (p-4)/2 \rfloor; \\
B2 &: p-1, \left(\frac{p+1}{4}+3\right)^4, 4^{p-9}, 3^4, && p \geq 15,\ p \equiv 3 \pmod 4; \\
B3 &: p-1, \left(\frac{p+3}{5}+3\right)^5, 4^{p-11}, 3^5, && p \geq 22,\ p \equiv 2 \pmod 5; \\
C &: p-1, x+3, 2(a-1)-x+3, 5^{p-a-3}, 3^a, && {\scriptstyle p\geq 8,\ 3\leq a\leq p/3,} \\
& && {\scriptstyle 1+2\lceil (a-2)/2\rceil \leq x \leq 2a-3,\ x \textit{ odd};} \\
D &: p-1, \left(\frac{p}{4}+3\right)^4, 4^{p-8}, 3^3, && p \geq 12,\ p \equiv 0 \pmod 4,
\end{aligned}$$

or the exceptional $14, 5^9, 3^5$.

The proof of Theorem 1.2 takes up the entirety of Sections 2 and 3. The proof, which is highly non-trivial, uncovers much of the structure of these graphs.

We point out that once $a \geq 3$ is fixed and p grows, *all except finitely many* unigraphic 3-polytopes of radius 1 with a vertices of degree three will be of one of the types B1, B2, B3, C, D.

On the other hand, if we fix p, it is straightforward to see that, if s is unigraphic and F is not a pyramid (i.e. wheel graph), then $2 \leq a \leq p/2$ (for the upper bound e.g. one may adapt Lemma 1.5). Then Theorem 1.2 classifies the unigraphic s for $2 \leq a \leq p/3$ but not for $p/3 < a \leq p/2$.

The unigraphic 3-polytopes corresponding to these sequences will be described once we have introduced some notation. Recall that expressions of the form $F + v$, $F - \{e_1, e_2\}$ mean adding/removing vertices or edges.

Definition 1.3. For F the graph of a 3-polytope satisfying $\deg(v_1) = p-1$, we note that $F - v_1$ is a non-empty, planar, Hamiltonian graph. It has a region containing all of its vertices. Henceforth, H will denote a Hamiltonian cycle in $F - v_1$. We will always immerse $F - v_1$ in the plane so that H bounds the external region. Note also that

$$G := F - v_1 - E(H)$$

is a non-empty planar graph, of sequence

$$s' : d_2 - 3, d_3 - 3, \ldots, d_p - 3. \tag{1.3}$$

Exactly a entries in s' are zeroes. We will denote by Z the set of isolated vertices in G.

Now we are in a position to better describe the polytopes F corresponding to the sequences in Theorem 1.2. We may do this by characterising $G - Z$. In the exceptional $14, 5^9, 3^5$, $G - Z$ is the disjoint union of three triangles. In all other cases $G - Z$ is connected. For types B1, B2, and B3, $G - Z$ is a triangle, triangulated quadrilateral (i.e., diamond graph) and triangulated pentagon respectively, together with vertices of degree one adjacent to the boundary points of the triangle, or triangulated quadrilateral or pentagon. In B1, at least two of the vertices of degree > 1 have the same degree; in B2 (resp. B3), all of the 4 (resp. 5) vertices of degree > 1 have the same degree.

For C, $G - Z$ is formed of a set of triangles intersecting at a vertex u pairwise, another (possibly empty) set of triangles intersecting at a vertex v pairwise, and a uv-path (that may be trivial, i.e. possibly $u = v$). For D, $G - Z$ is a triangle together with a fourth vertex adjacent to exactly one point on the boundary of the triangle, and with extra degree one vertices adjacent each to one of these four vertices, so that these four have the same degree in G.

Theorem 1.2 leaves out the case $a = 2$. For this case, we have the following.

Proposition 1.4. *Let s be a sequence as in* (1.1)*, with exactly* 2 *vertices of degree three. Then s is unigraphic as a* 3*-polytope of radius* 1 *if and only if s is one of the following:*

$A1 : p - 1, 4^{p-3}, 3^2,$	$p \geq 5,\ p$ *odd*;
$A2 : p - 1, p - 3, 4^{p-4}, 3^2,$	$p \geq 8$;
$A3 : p - 1, (x + 3)^{(p-5)/(x-1)}, 4^{(p-5)(x-2)/(x-1)+2}, 3^2,$	$p \geq 6,\ (p - 5)/(x - 1) \in \mathbb{N}$;
$A4 : p - 1, x + 3, p - x, 4^{p-5}, 3^2,$	$p \geq 8,\ \lfloor (p - 1)/2 \rfloor \leq x \leq p - 5.$

Proposition 1.4 will be proven in Section 4.

In all of the types A1, A2, A3, and A4, G is a forest, and either it has at most two non-trivial trees, or all non-trivial trees are copies of K_2 (type A1).

With similar methods but an increasing amount of work, one can go about proving a statement of this flavour (i.e. without the restriction $p \geq 3a$ of Theorem 1.2) for the cases $a = 3$, $a = 4$, ... These are omitted here.

Related literature. To the best of our knowledge, the problem of characterising 3-dimensional polytopes by their degree sequences, and finding those that do not share their sequence with any other 3-polytopes, was posed for the first time in this paper. At the time of writing, there was not a lot of information in this direction in the literature, to the best of our knowledge. Since then, with Delitroz [4] we have addressed the analogous problem of unigraphic degree sequences of 3-polytopes where the largest entry is $p-2$ rather than $p-1$. In the general case, this seems to be a difficult problem. The first real breakthrough came in [17], where we proved that if a 3-polytope is the only one realising its sequence, then it has no n-gonal faces for $n \geq 8$. The arguments in [17] are independent from, and consistent with, the results of the present paper.
A related problem is to determine which degree sequences of 3-polytopes have a unique realisation among the class of all graphs, rather than among the class of 3-polytopal graphs. For instance, the n-gonal pyramid is unigraphic with respect to the class of 3-polytopes, but for $n \geq 6$, not with respect to the class of all graphs. As it turns out, there are only eight solutions to this related problem [18].
Recently, we have shown that

$$x, y, 3^{x+y-4}, \qquad x \geq y \geq 3$$

are the only graph degree sequences with exactly one *self-dual* 3-polytopal realisation [16, Theorem 4]. Moreover, among sequences with exactly one 3-polytopal realisation, this realisation is self-dual only in the case of pyramids and of $4, 4, 3, 3, 3, 3$ [16, Corollary 6].

1.2 Conventions

Everywhere we fix the notation $F = (V, E)$, $|V| = p$ and $|E| = q$ for a 3-polytope of radius 1, order p, size q, and degree sequence s (1.1). We write $F_1 \simeq F_2$ when F_1, F_2 are isomorphic graphs. The notation H, G, s' will always be as in Definition 1.3.

We use K_p for the complete graph on $p \geq 1$ vertices, C_p with $p \geq 3$ for a cycle, and $\mathcal{S}_n$ for the star graph on $n \geq 2$ edges. A *caterpillar* is a tree where if we delete all degree one vertices, we are left with a central path $c_1, c_2, \ldots, c_\ell$, $\ell \geq 1$. The caterpillar depends only on ℓ and on $x_j := \deg(c_j)$, $j = 1, \ldots, \ell$, and we will denote it by $\mathcal{C}(x_1, \ldots, x_\ell)$. A special case is the star, $\mathcal{C}(x) = \mathcal{S}_x$.

A connected graph with no separating vertices is called a *block*. A *block of a graph G* is a G-subgraph that is a block, and maximal with respect to the property of being a block. Equivalently, a block of a graph is either an isolated vertex (trivial block), or a bridge, or a maximal 2-connected subgraph. A *cyclic block* is a block of a graph containing a cycle, i.e., a maximal 2-connected subgraph. Isolated vertices and bridges are *acyclic blocks*. The *circumference* of a graph is the length of its longest cycle.

1.3 Initial considerations

As mentioned in the Introduction, for each radius one 3-polytope we have $a \geq 2$ (recall (1.2)), and actually we can show rather more.

Lemma 1.5. *If F is a radius one 3-polytope, then*

$$a \geq 2 + \sum_{i \geq 3} (i-2) \cdot B_G(i), \tag{1.4}$$

where

$$B_G(i) := \#\{\textit{blocks of } G \textit{ with circumference } i\}.$$

Proof. We begin by showing that $a \geq 2$. As the graph G is non-empty, we may take $e_1 = uv$ to be any edge. The claim is, at least one vertex on each of the two uv-paths of the cycle H has degree 0 in G. To see this, fix any $w \neq u, v$ on one of the uv-paths (w exists as $e_1 = uv$ is an edge in G so it cannot be an edge in H). Either $\deg_G(w) = 0$, or there is an edge $e_2 = wx \in E(G)$. Now G is planar so that e_1, e_2 cannot cross. We then consider the wx-subpath of the initial uv-path, and reason as above to conclude that since G is finite, we indeed have $a \geq 2$.

Now let G contain exactly one block, with circumference i, say. This determines i (internally disjoint) u_1u_2-, u_2u_3-, $\dots$, u_iu_1- paths in H, hence reasoning as above $a \geq i = 2 + (i-2)$ and (1.4) is proven in the case of one block.

Next, call J the subgraph of G induced by vertices lying on cyclic blocks of G. We claim that there exists at least one block B of J with vertices all lying on the same path $u_1, u_2, \dots, u_n$ of H, such that $u_2, \dots, u_{n-1}$ belong to no other cyclic block of J. We start by checking if a block B_1 satisfies this property. If not, we can find an edge $e = w_1w_m$ on the boundary of B_1, such that $w_1, w_2, \dots, w_m$ are consecutive on H, no vertices from $w_2, \dots, w_{m-1}$ belong to B_1, and moreover by planarity of J there is at least one cyclic block B_2 of J with vertices a subset of the $w_1, w_2, \dots, w_m$. We repeat the above procedure with B_2 in place of B_1, and by finiteness of J we eventually find at least a cyclic block B with vertices all lying on the same path $u_1, u_2, \dots, u_n$ of H, such that $u_2, \dots, u_{n-1}$ belong to no other cycle of J.

Let i be the circumference of B. We denote the vertices of a cycle in B of longest length by

$$u_{k_1}, u_{k_2}, \dots, u_{k_i},$$

in order along $u_1, u_2, \dots, u_n$. Then there is a degree 0 vertex of G along $u_1, u_2, \dots, u_n$ between u_{k_l} and $u_{k_{l+1}}$ for each $1 \leq l \leq i-1$. We finally remove B from J so that we can argue that any such block added to J increases a by at least $(i-1)-1 = i-2$, and the proof of the present lemma is complete. □

1.4 Overview of the proofs

Sections 2 and 3 are entirely dedicated to proving Theorem 1.2. In Section 2, we will show that apart from the exceptional case $14, 5^9, 3^5$, if $p \geq 3a$ then $G - Z$ is connected. In the second part of the proof (Section 3), we will analyse several cases for connected $G - Z$, and assuming s is unigraphic, either determine F or bound p with respect to a. The first case is when $G-Z-Y$ is 2-connected, where Y is the set of degree 1 vertices in G (Section 3.1). The second and third cases are for $G-Z-Y$ not 2-connected, distinguishing between when no block of $G-Z-Y$ contains all vertices that are separating in G (Section 3.2), and when such a block exists (Section 3.3). A proposition summarising results closes each section. Combining these propositions, we will prove Theorem 1.2 (Section 3.4).

One general idea is that, outside of the types of sequence listed in Theorem 1.2, the number of vertices of G not lying on a cyclic block of G is *bounded.* This fact allows us to obtain an upper bound for p depending on a.

The proof of Proposition 1.4 (the case $a = 2$) may be found in Section 4. Finally, we have collected some data on the enumeration of radius one and unigraphic 3-polytopes. This is relegated to appendix A.

2 Proof of Theorem 1.2: first part

Henceforth we assume that the number of degree three vertices in F is $a \geq 3$. Recall Definition 1.3 for v_1, H, G, s', Z. All figures in this paper are sketches for the graph $F - v_1$. Its Hamiltonian cycle H is the external cycle.

Lemma 2.1. *If s is unigraphic, $p \geq 7$, and $a \geq 3$, then G has at least one cycle.*

Proof. Suppose by contradiction that G is a forest. Firstly, we will show that every tree degree sequence admits a realisation as a caterpillar. Indeed, if

$$x_1, x_2, \ldots, x_\ell, 1^m, \qquad x_1, \ldots, x_\ell \geq 2, \tag{2.1}$$

is the degree sequence of a $(\overline{p}, \overline{q})$ tree graph, then

$$\sum_{i=1}^{\ell} x_i + m = 2\overline{q} = 2(\overline{p} - 1) = 2(\ell + m - 1),$$

hence

$$m = \sum_{i=1}^{\ell} x_i - 2\ell + 2 = \sum_{i=1}^{\ell} (x_i - 2) + 2.$$

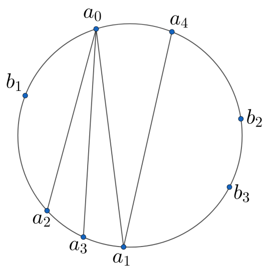
The caterpillar $\mathcal{C}(x_1, \ldots, x_\ell)$ has exactly

$$(x_1 - 1) + (x_\ell - 1) + \sum_{i=2}^{\ell-1} (x_i - 2) = m$$

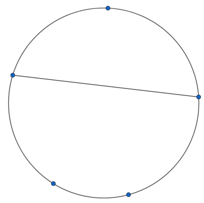
vertices of degree 1, hence it is a realisation of (2.1).

Therefore, if s is unigraphic, and G a forest, then every tree of G is a caterpillar. It follows that the sequence obtained by s on removing all vertices of degree three (i.e. the isolated vertices of G) save two of them has a realisation as a 3-polytope of radius one (cf. Section 4).

The main idea is that, for $a \geq 3$ and G a disjoint union of caterpillars, we may relocate one of the isolated vertices of G around H, obtaining another non-isomorphic realisation of s, except for the case $5, 4^2, 3^3$ of Figure 1b. Indeed, since F is not a pyramid, G has at least two non-isolated vertices. If it has exactly two of them, then s is not unigraphic as soon as there are four or more isolated vertices in G, i.e., as soon as $p \geq 7$; if it has three or more of them, then s is not unigraphic either – refer to Figure 1. The proof of this lemma is complete. □



(a) Although $7, 6, 5, 4^3, 3^2$ is unigraphic ($a = 2$, type A4, $p = 8$, $x = 4$), the depicted $8, 6, 5, 4^3, 3^3$ is not. For instance, moving b_3 to the shortest a_0a_4-path yields a non-isomorphic realisation.



(b) The unigraphic $5, 4^2, 3^3$.

Figure 1: If s is unigraphic, $p \geq 7$, and $a \geq 3$, then G is not a forest.

Next, we consider the number k of cyclic connected components in G.

Lemma 2.2. *If s is unigraphic, $p \geq 7$, and $a \geq 3$, then G has one, two or three cyclic connected components, and if three, then they are all triangles.*

Proof. Denote by G_l, $1 \leq l \leq k$, the cyclic connected components of G. For fixed l, denote by $u_{l,j}$, $1 \leq j \leq i_l$, the $i_l \geq 3$ vertices of G_l in the order that they appear around the Hamiltonian cycle H, clockwise starting from $u_{1,1}$. Consider their following two orderings around H,

$$u_{1,1}, u_{2,1}, \dots, u_{k,1}, u_{k,2}, \dots, u_{k,i_k}, u_{k-1,2}, \dots, u_{k-1,i_{k-1}}, \dots, u_{2,2}, \dots, u_{2,i_2}, u_{1,2}, \dots, u_{1,i_1}$$

and

$$u_{1,1}, u_{2,1}, \dots, u_{2,i_2}, u_{1,2}, u_{3,1} \dots, u_{3,i_3}, \dots, u_{1,i_1}, u_{i_1+1,1}, \dots, u_{i_1+1,i_{i_1+1}}, \dots, u_{k,1} \dots, u_{k,i_k}$$

where in the second ordering $u_{l,j}$ actually appears only for $l \leq k$. In the first ordering we have four consecutive vertices belonging to four different components, namely $u_{1,1}, u_{2,1}, u_{3,1}, u_{4,1}$, *unless* $k \leq 3$, whereas it is easy to see that this cannot happen in the second ordering (recall that the G_l are cyclic, hence $i_l \geq 3$ for every $1 \leq l \leq k$). Thereby, if s is unigraphic, then necessarily $k \leq 3$.

Now suppose that $k = 3$. The second of the above two orderings reads

$$u_{1,1}, u_{2,1}, \ldots, u_{2,i_2}, u_{1,2}, u_{3,1} \ldots, u_{3,i_3}, u_{1,3}, u_{1,4}, \ldots, u_{1,i_1}. \tag{2.2}$$

We also have the feasible ordering

$$u_{1,1}, u_{2,1}, \ldots, u_{2,i_2}, u_{1,2}, u_{1,3}, u_{3,1} \ldots, u_{3,i_3}, u_{1,4}, \ldots, u_{1,i_1}. \tag{2.3}$$

In (2.2), the vertex $u_{1,2}$ from G_1 does not follow or precede any other vertex from G_1. In (2.3), there is no vertex from any of G_1, G_2, G_3 that does not follow or precede any other vertex from the same component, *unless* $i_1 = 3$ (where $u_{1,1}$ indeed has such property). Therefore, if s is unigraphic and $k = 3$, then G_1 has exactly three vertices and is cyclic, i.e. it is a triangle. We may change the roles of G_1, G_2, G_3 in (2.2) and (2.3) to show that G_2, G_3 are triangles as well. □

Next, we show that under the same assumptions, there can be no non-trivial tree components.

Lemma 2.3. *If G is unigraphic, $p \geq 7$, and $a \geq 3$, then G has no non-trivial tree components.*

Proof. By Lemma 2.2, there is at least one cyclic component of G. If there are two or more, and if by contradiction there is at least one non-trivial tree component, then there are three non-trivial components that are not all isomorphic. A slight generalisation of the scenario in $a = 2$ of two copies of K_2 and a star that is not K_2—refer to Section 4—tells us that this is impossible for s unigraphic. By the way, it follows that in the case $k = 3$ if s is unigraphic then s is simply $14, 5^9, 3^5$ (i.e. G is the disjoint union of three triangles and five isolated vertices). The same argument excludes the case of exactly one cyclic component and two or more non-trivial trees.

It remains to analyse what happens for exactly one cyclic component G_1 and one non-trivial tree T. We have already seen that this tree is a caterpillar $T = \mathcal{C}(x_1, \ldots, x_\ell)$. Let $u_1, \ldots, u_i$ be the vertices of G_1 in order around H. We can choose any among u_1, u_2, or u_2, u_3, …, or u_{i_1}, u_1 to be the two closest vertices on H to the elements of $V(T)$, and moreover we can choose to order $u_1, u_2, \ldots, u_i$ clockwise or counter-clockwise around H.

Let u_1u_j, $3 \leq j \leq i-1$ be an edge. Then by planarity $u_2u_{j+1} \notin E(G)$. Reordering $u_1, \ldots, u_i$ around H, we see that G is not unigraphic—Figure 2. Therefore, $G_1 = C_i$ is just an i-gon.

It follows that

$$s' : x_1, \ldots, x_\ell, 2^i, 1^b, 0^a$$

for some $b \geq 2$. Then there is another realisation of G as the caterpillar

$$\mathcal{C}(x_1, \ldots, x_\ell, \underbrace{2, \ldots, 2}_{i})$$

together with a isolated vertices, contradiction. □

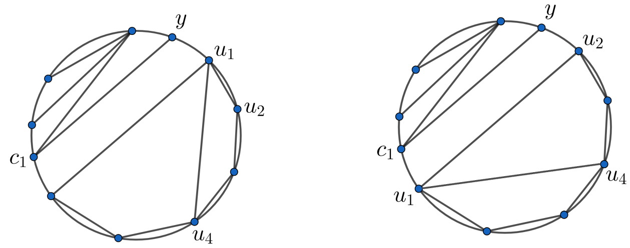


Figure 2: In this example, u_i, u_1 are the closest vertices of G_1 to $V(T)$ around H. Letting $u_1u_4 \in E(G)$, by planarity it follows that $u_2u_5 \notin E(G)$, hence G is not unigraphic.

Now assume that G has exactly two cyclic components G_1, G_2 (and thanks to Lemma 2.3 there are no non-trivial tree components). Our goal for the rest of this section is to show that in this case $p \leq 3a - 1$. The argument starts similarly to the case of G_1, T of Lemma 2.3: to not contradict unigraphicity or planarity, at least one of G_1, G_2 is just an i-gon, say $G_1 = C_i$.

We claim that G_2 cannot contain acyclic blocks: by contradiction, let $w, w' \in V(G_2)$ be the endpoints of an acyclic block ww'. Then we contradict unigraphicity by writing

$$G - ww' - u_1u_i + wu_1 + u_iw',$$

where $u_1, \ldots, u_i$ are the vertices of $G_1 = C_i$ in order around the cycle.

Thanks to Lemma 1.5,

$$a \geq i + \sum_{j \geq 3} (j-2) \cdot B_G(j),$$

where $B_G(j)$ counts blocks of G_2 of circumference j. As all blocks of G_2 are cyclic, we may rewrite

$$a \geq |V(G_1)| + |V(G_2)| - \#\{\text{blocks of } G_2\} \geq |V(G_1)| + \frac{|V(G_2)| - 1}{2},$$

so that we have the bound

$$p = 1 + a + |V(G_1)| + |V(G_2)| \leq 3a + 2 - |V(G_1)| \leq 3a - 1,$$

as claimed.

The arguments of this section imply the following.

Proposition 2.4. *If s is unigraphic, $a \geq 3$, and $p \geq 3a$, then either s is $14, 5^9, 3^5$, or G (Definition 1.3) has exactly one non-trivial connected component, and this component contains a cycle.*

3 Proof of Theorem 1.2: second part

Thanks to Proposition 2.4, to prove Theorem 1.2 it remains to inspect the scenario when G (of Definition 1.3) has exactly one non-trivial connected component, and this component contains a cycle. Recall the notation $-$ and $+$ for removing/adding vertices/edges/sets of vertices/edges from/to a graph, and also recall that

$$\begin{aligned} Z &:= \{\text{vertices of degree 0 in } G\} = \{z_1, \dots, z_a\}, \\ Y &:= \{\text{vertices of degree 1 in } G\}, \\ B &:= G - Z - Y. \end{aligned} \tag{3.1}$$

3.1 B is 2-connected

In this section we suppose that B in (3.1) is 2-connected. Say that the vertices $b_1, \dots, b_i$ of B are ordered clockwise around the Hamiltonian cycle H. If $y \in Y$, $yb_j \in E(G)$ for some $1 \leq j \leq i$, then by planarity y lies on either the $b_{j-1}b_j$- or b_jb_{j+1}-path in H not containing any of the other vertices of B. These two possibilities mean that there is more than one realisation of s, unless possibly when either $|Y| \leq 1$, or when every b_j is adjacent to at least one element of Y.

In the former case, we write $p = 1 + a + |Y| + i$. Now, b_ib_1 and b_jb_{j+1} for all $1 \leq j \leq i - 1$ are edges of B, so that they cannot be edges of H. We deduce that there is at least one element of Z between every pair of consecutive vertices of B along H, thus $i \leq a$. Therefore, we have the admissible bound on the order of the graph

$$p \leq 1 + a + |Y| + a \leq 2a + 2.$$

We are left with the case of every b_j being adjacent to at least one element of Y.

- If B is just a cycle of length $i \geq 3$, then s' reads

  $$\deg(b_1), \dots, \deg(b_i), 1^b, 0^d, \qquad b = \sum_{j=1}^{i} \deg(b_j) - 2i.$$

  For $i \geq 4$ we may alter G as follows. Let $y \in Y$ be adjacent to b_1. We take

  $$G - yb_1 - b_2b_3 + yb_2 + b_1b_3.$$

  Here and in what follows, when we apply such a transformation, if need be we also relocate vertices around H (in this case, namely y) so as to preserve planarity. Then s is not unigraphic for $i \geq 4$. On the other hand, when $i = 3$, one can check that s is unigraphic if and only if $a = 3$ and at least two of $\deg(b_1), \deg(b_2), \deg(b_3)$ are equal: s is of type B1.

- Now let B be a triangulated i-gon, $i \geq 4$. For $i \geq 6$, we may alter G as follows. Take any triangulated hexagon in the triangulation of B, delete one diagonal $b_{j_1}b_{j_2}$ of the hexagon and add another diagonal $b_{j_3}b_{j_4}$, with j_1, j_2, j_3, j_4 distinct. This has the effect of decreasing by 1 the values $\deg(b_{j_1}), \deg(b_{j_2})$ and increasing by 1 the values $\deg(b_{j_3}), \deg(b_{j_4})$. We then take two vertices in Y adjacent one each to b_{j_3}, b_{j_4}, and make them adjacent (one each) to b_{j_1}, b_{j_2} instead (this may be done without altering the value of a). Then s is not unigraphic for $i \geq 6$.

Now let $i = 4, 5$ (and here there is only one way to triangulate a quadrilateral or pentagon). It is straightforward to check that s is unigraphic if and only if all vertices of B have the same degree in G, and moreover $a = i$. We get the types B2 and B3 for $i = 4, 5$ respectively.

- It remains to inspect the case where B is neither a cycle nor a triangulated polygon. Then there exist two adjacent regions R_1 and R_2 in B, of respective boundary lengths i_1, i_2, such that $i_1 \geq i_2$ and $i_1 \geq 4$. Similarly to the case where B is a cycle, we delete the edge $b_{j_1}b_{j_2}$ between R_1, R_2, add the edge $b_{j_1}b_{j_3}$, where $b_{j_3} \neq b_{j_1}$ is adjacent to b_{j_2} on the boundary of R_1, then take a vertex in Y adjacent to b_{j_3}, and make it adjacent to b_{j_2} instead (again we do not alter a). This is another realisation of s, and if $i_1 \neq i_2 + 1$ it is clearly not isomorphic to the initial one, as i_1 has decreased by 1 and i_2 increased by 1. If $i_1 = i_2 + 1$ and $i_2 \geq 4$, we perform the transformation above but exchanging the roles of R_1, R_2 to reach the same conclusion.

  Finally, if $i_1 = 4$ and $i_2 = 3$, then R_1, R_2 form a pentagon $b_{j_1}, b_{j_2}, b_{j_3}, b_{j_4}, b_{j_5}$ with a diagonal b_{j_1}, b_{j_4}, say. We perform the transformation

  $$G - b_{j_3}b_{j_4} + b_{j_1}b_{j_3} - yb_{j_1} + yb_{j_4}$$

  with $y \in Y$, to conclude that s is not unigraphic in this case.

We summarise the results of this section as follows.

Proposition 3.1. *Assume that s is unigraphic, G (Definition 1.3) has exactly one cyclic component, and B in* (3.1) *is* 2*-connected. Then either s is of type B1, B2, or B3, or $p \leq 2a + 2$.*

3.2 B is not 2-connected, case 1

In this section we suppose that B in (3.1) is not 2-connected. For s unigraphic, in a cyclic block of G either one, or all vertices are separating in G: the idea is similar to the first argument in Section 3.1.

We assume in this section that

$$\boxed{\text{each cyclic block of } G \text{ contains exactly one vertex that is separating in } G.} \tag{3.2}$$

We deduce that, if we delete from G all cyclic blocks, we are left with one tree, that is a caterpillar $T = \mathcal{C}(x_1, \ldots, x_\ell)$ due to the arguments in Section 2 (possibly the trivial caterpillar). If T is trivial, then G has exactly one separating vertex u, contained in every non-trivial block of G.

Let us focus for now on the case that T is non-trivial. Here our first observation is that the vertices

$$c_1, c_2, \ldots, c_\ell$$

on the central path of T all have the same degree, both in T and in G, save possibly when $\ell = 2$. Indeed, otherwise, by reordering them along the path, it would be

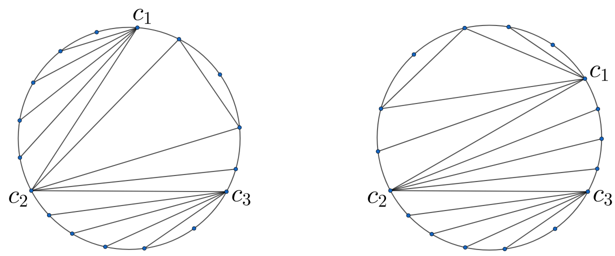


Figure 3: By unigraphicity, if $\ell \neq 2$, then $c_1, c_2, \ldots, c_\ell$ have the same degree in T and in G.

possible to construct two non-isomorphic graphs G sharing the same sequence, e.g. as in Figure 3.

Next, we claim that no vertex on the central path of T may lie on a cyclic block of G. By contradiction, assume that there exists c_j, $1 \leq j \leq \ell$, lying on a cyclic block of G. By reordering the vertices on the central path of the caterpillar as above, we may take $j = 1$. Considering the transformation in Figure 4, we see that s is not unigraphic, contradiction.

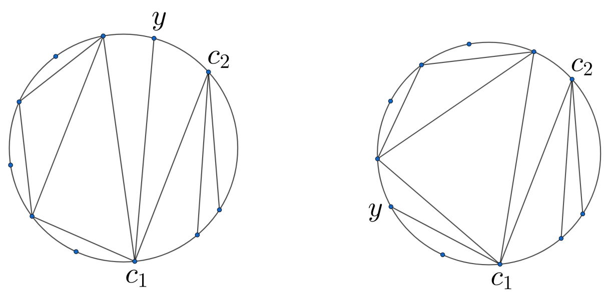


Figure 4: No vertex on the central path of T may lie on a cyclic block of G.

Hence only endvertices of T may lie on cyclic blocks of G. Next, let u be a vertex adjacent to c_1 in T, and lying on a cyclic block B_1 of G. Then we claim that

$$\deg_T(c_1) = 2.$$

By contradiction, writing $V(B_1) = \{u_1, \ldots, u_{i-1}, u = u_i\}$, we have in order around H

$$c_1, u_1, \ldots, u_{i-1}, \{\text{vertices of other cyclic blocks containing } u\}, u, y, A,$$

with $c_1 y \in E(G)$, $\deg_G(y) = 1$, and A a non-empty set, since $\deg(c_1) \geq 3$ (e.g. Figure 5, left). We take

$$G - yc_1 + c_1u_1 - u_1u_2 + u_2y$$

moving y to the u_2u_3-path in H, and moving the isolated vertices of G lying between u_1, u_2 to the c_1u_1-path in the new graph (Figure 5, right). This new graph is not isomorphic to G, as removing all cyclic blocks now leaves two non-trivial trees (it is essential that A is non-empty, i.e. $\deg(c_1) \geq 3$). Hence s is not unigraphic, contradiction.

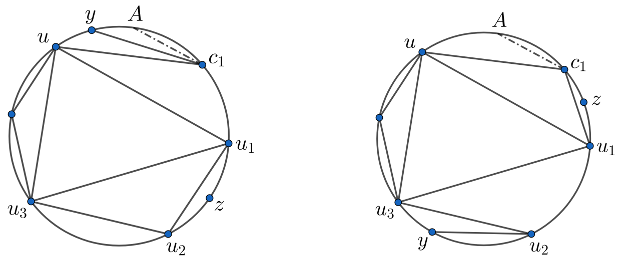


Figure 5: In this example, $z \in Z$, and the dashed-dotted line represents edges incident to a non-empty graph A. We transform the graph G on the left to $G - yc_1 + c_1u_1 - u_1u_2 + u_2y$ on the right.

We now combine the above observations to see that there cannot exist an endvertex u' of T, adjacent to c_j, $2 \leq j \leq \ell - 1$, lying on a cyclic block of G. By contradiction, we can reorder the vertices on the central path of the caterpillar, so that there also exists an endvertex u of T, adjacent to either c_1 or c_ℓ (say c_1), and lying on a cyclic block of G. As seen a moment ago, this implies $\deg_T(c_1) = 2$. Since $c_1, c_2, \ldots, c_\ell$ all have the same degree in T, this in turn implies $\deg_T(c_j) = 2$. On the other hand, since u' is adjacent to c_j, we also have $\deg_T(c_j) \geq 3$, contradiction.

To summarise, if the caterpillar T is non-trivial, then it is a simple path

$$T = \mathcal{C}(2, \ldots, 2),$$

and moreover, if u, v denote the endvertices of T adjacent to c_1, c_ℓ respectively, there is a non-empty set $\mathcal{B}$ of cyclic blocks of G containing u, and there is a possibly empty set $\mathcal{B}'$ of cyclic blocks of G containing v. Further, no other vertex of T lies on a cyclic block of G.

Altogether, if (3.2) holds, then G consists of: a path

$$u := c_0, c_1, c_2, \ldots, c_\ell, v := c_{\ell+1}, \quad \ell + 1 \geq 0,$$

a non-empty set $\mathcal{B}$ of cyclic blocks containing u, and a (possibly empty) set $\mathcal{B}'$ of cyclic blocks containing v.

Next, we will show that if $\ell \geq 2$, then each element of $\mathcal{B}, \mathcal{B}'$ is a triangle. Let $B_1 \in \mathcal{B}$. We perform the transformation

$$G - c_2c_3 + c_2u + c_3w - wu,$$

where $w \in V(B_1)$ is the vertex of B_1 closest to u along H. We then reorder the vertices of B_1 and T around H (e.g. Figure 6) so that the minimum value of a for G has not increased. This is possible as deleting wu removes a region, and we added the triangular region of boundary uc_1c_2. We may move the isolated vertices of G lying between u and w to the c_1c_2-path in the transformed graph. This transformed graph is not isomorphic to G unless B_1 is a triangle, as claimed.

Thereby, if $\ell \geq 2$, then s' is given by

$$x, y, 2^{(x-1)+(y-1)+\ell}, 0^{2+(x-1)/2+(y-1)/2}$$

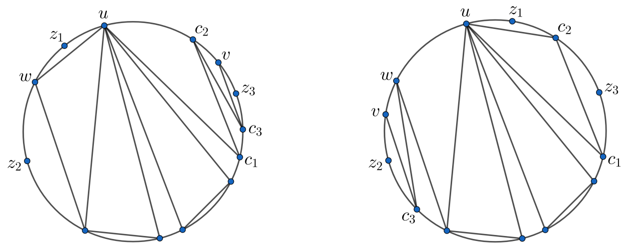


Figure 6: In the depicted example, $z_1, z_2, z_3 \in Z$, and w is the vertex of B_1 closest to u along H. We transform the graph on the left to the one on the right, contradicting unigraphicity since B_1 is not a triangle.

i.e., s is of type C.

In the rest of this section, we will assume that $\ell \leq 1$. Here we give an upper bound for the order of F. The 3-polytope contains the vertex of eccentricity one v_1, a many of degree 0 in G, at most three on the uv-path, and $\#V(B_j) - 1$ more for each $B_j \in \mathcal{B}, \mathcal{B}'$:

$$p \leq 1 + a + 3 + \sum_{j=1}^{k} (\#V(B_j) - 1),$$

where $k = \#\mathcal{B} + \#\mathcal{B}'$. We invoke Lemma 1.5,

$$a \geq 2 + \sum_{j=1}^{k} (\#V(B_j) - 2), \tag{3.3}$$

to obtain

$$p \leq 2a + 2 + k. \tag{3.4}$$

On the other hand, each cyclic block of G is of order at least three, and at least one block is of order at least four, otherwise we would be in the case where all cyclic blocks are triangles. Therefore, (3.3) also yields

$$a \geq 2 + (3-2)(k-1) + (4-2) = 3 + k. \tag{3.5}$$

We substitute (3.5) into (3.4) to see that $p \leq 3a - 1$ in this scenario.

The arguments of this section imply the following.

Proposition 3.2. *Assume that s is unigraphic, G (Definition 1.3) has exactly one cyclic component, and B in* (3.1) *is not* 2*-connected. Suppose further that there is no cyclic block of G that contains only vertices that are separating in G. Then either G is of type C, or $p \leq 3a - 1$.*

3.3 B is not 2-connected, case 2

In this section we suppose that B in (3.1) is not 2-connected, and moreover that there exists a cyclic block B_1 of G that contains only vertices that are separating in G.

We claim that there exists a cyclic endblock $B_j \neq B_1$. There are always at least two endblocks, and by contradiction, assume that both are acyclic, i.e. copies of K_2

$$a, b \quad \text{and} \quad c, d,$$

where a, c are separating vertices in B, and b, d vertices of degree 1 in B. Since $B = G - Z - Y$, the vertices b, d cannot belong to Y, i.e.

$$\deg_G(b), \deg_G(d) \geq 2.$$

This implies that there exist $y, y' \in Y$ such that $by, dy' \in E(G)$. Hence there are in G two disjoint paths of three vertices each,

$$a, b, y \quad \text{and} \quad c, d, y'.$$

We take

$$G - cd - dy' + cy' - by + bd + dy,$$

contradicting unigraphicity (Figure 7). Hence B_j exists.

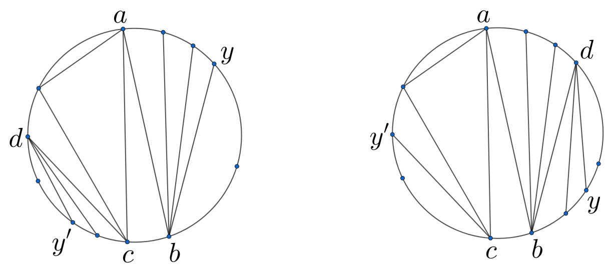


Figure 7: There cannot be two acyclic endblocks in B.

Planar, cyclic blocks with a region containing all of the vertices (i.e. polygons possibly with some diagonals) have at least two vertices of degree 2. Let w be a vertex of B_j of degree 2 in B_j, non-separating in B, and u a vertex of degree 2 in B_1 such that in $G - u$ there is still a path between B_1, B_j (u always exists, as B_1 has at least two vertices of degree 2 in B_1) – refer to Figure 8. We may transform G by moving the adjacencies of u not in $V(B_1)$ to w instead, and vice versa the adjacencies of w not in $V(B_j)$ to u (this does not affect s). By unigraphicity, we conclude that this operation produces an isomorphic graph. Now u is separating in G by definition, and by construction any neighbour of w (save for the two in $V(B_j)$) belongs to Y of (3.1) (it has degree 1 in G). It follows that u, w are adjacent to the same number $\alpha - 2 \geq 1$ of vertices in Y, where $\deg_G(u) = \deg_G(w) = \alpha$.

The arguments of Section 3.1 now imply that *all* vertices of a cyclic endblock B_j that are non-separating in B are adjacent to one or more elements of Y. Note that all vertices of B_j are separating in G, and exactly one, w_0 say, is separating in B.

Via an argument similar to previous sections, we now show that actually an endblock of G cannot be a copy of K_2, except possibly if there are only the two blocks

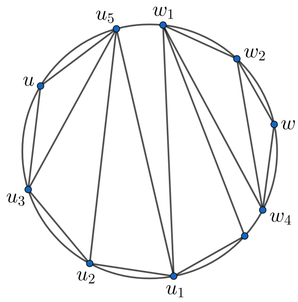


Figure 8: In this example, we assume that B_j is an endblock, $V(B_j) = \{w_1, w_2, w, w_4\}$, and $V(B_1) = \{u_1, u_2, u_3, u, u_5\}$. Moreover, all vertices of B_1 are separating in G, w_1 is separating in B, and w_2, w, w_4 are separating in G but not in B (only a subgraph of G is depicted).

B_1 and K_2 in G. Indeed, by contradiction call $V(K_2) = \{w'_0, w'_1\}$, w'_0 separating in B and w'_1 non-separating in B. By construction, w'_1 is adjacent to $y'_1, \dots, y'_i \in Y$, $i \geq 1$, and we have seen above that there exists $y \in Y$, $w'_1 y \notin E$. We perform

$$G - w'_1 y'_1 - \cdots - w'_1 y'_i + y y'_1 + \cdots + y y'_i \tag{3.6}$$

and obtain a new graph, that is non-isomorphic to G as soon as there are two or more cyclic blocks in G. We reach a contradiction unless there are only two blocks B_1 and K_2, and moreover B_1 must be a cycle in this case.

Still by the arguments of Section 3.1, a cyclic endblock B_j is either a triangle or a triangulated quadrilateral or pentagon: the arguments for the only block B of G in Section 3.1 apply here to B_j, since all vertices of B_j are separating in G, and exactly one is separating in B.

If the cyclic endblocks of G are all triangles, then we may possibly have $\alpha = 3$; if one of them is a triangulated quadrilateral or pentagon, $\alpha \geq 4$. Let us see that actually there cannot be two endblocks $B_j, B_{j'}$ that are both triangles. Indeed, in this scenario, $V(B_j) = \{w_0, w_1, w_2\}$, w_0 separating in B, $w_1 y_1, w_2 y_2 \in E$, $y_1, y_2 \in Y$, and likewise $V(B_{j'}) = \{w'_0, w'_1, w'_2\}$, w'_0 separating in B, $w'_1 y'_1, w'_2 y'_2 \in E$, $y'_1, y'_2 \in Y$. We consider the transformation

$$G - w_1 y_1 - w_2 y_2 + w_1 w'_2 + w_2 w'_2 - w'_0 w'_2 - w'_1 w'_2 + w'_1 y_1 + w'_0 y_2$$

that alters F but not s to reach a contradiction and rule out this scenario.

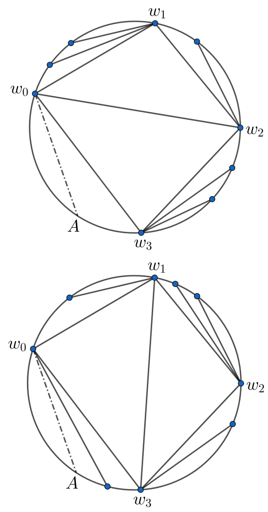
To summarise, either there are exactly two blocks in G, a cycle and a copy of K_2, or at least one endblock is a triangulated quadrilateral or pentagon, and $\alpha \geq 4$. Suppose for the moment that we are in the latter case. Let

$$V(B_j) = \{w_0, w_1, \dots, w_l\},$$

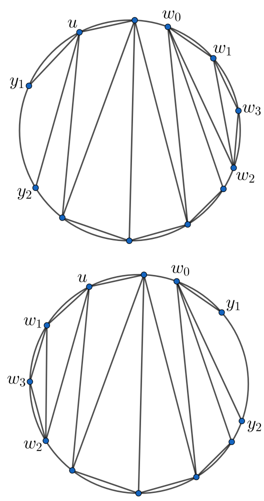
with w_0 separating in B, and $2 \leq l \leq 4$. Since $w_1, \dots, w_l$ are adjacent to one or more elements of Y, it must hold that $\deg_{B_j}(w_0) = 2$ by unigraphicity, with $w_0 w_1, w_0 w_2 \in E$, say – refer to Figure 9a. We then transform G by

$$G - u y_1 - u y_2 + u w_1 + u w_2 - w_0 w_1 - w_0 w_2 + w_0 y_1 + w_0 y_2, \tag{3.7}$$

where u is adjacent to at least 2 elements $y_1, \ldots, y_{\alpha-2} \in Y$, and in $G - u$ there is still a path between B_j and a vertex in the same block as u – refer to Figure 9b.



(a) The endblock B_j is a diamond graph. The dashed-dotted line represents edges incident to a non-empty graph A. By unigraphicity, the degrees of w_1, w_2, w_3 in G must be the same value $\alpha \geq 4$ (in this example $\alpha = 4$.) The transformation from the first graph to the second graph is applicable if and only if $\deg_{B_j}(w_0) \geq 3$.



(b) An application of (3.7) transforms the first graph into the second one. Here $\alpha = 4$, and the endblock B_j is a diamond graph ($l = 3$). Only a subgraph of G is depicted.

Figure 9: The case $\alpha \geq 4$.

By unigraphicity, (3.7) must not change F, that is to say, all possible u must belong to a block intersecting with B_j at the vertex w_0. There are thus exactly two cyclic blocks in G, one of them being a triangulated quadrilateral or pentagon, and the other a triangle or triangulated quadrilateral or pentagon. It is straightforward to see that

$$s' : k^b, 1^c, 0^a,$$

with $6 \leq b \leq 10$, $k \geq 5$, c depending only on k, and $5 \leq a \leq 8$. One checks all possibilities for b to rule out this option entirely.

Therefore, we finally see that B has exactly two blocks, one of them being a copy of K_2 on the vertices w_0, w_1' say, and the other (i.e. B_1) a cycle. We have seen that B_1 must also be a triangle or triangulated quadrilateral or pentagon, hence it is a triangle, and we write $V(B_1) = \{w_0, w_1, w_2\}$. One quickly sees that for s unigraphic, the degrees in G of w_0, w_1, w_2, w_1' must all be equal. We have obtained a graphic sequence s of type D. We summarise the work of this section as follows.

Proposition 3.3. *Assume that s is unigraphic, G (Definition 1.3) has exactly one cyclic component, and B in* (3.1) *is not* 2*-connected. Suppose further that there is a cyclic block of G that contains only vertices that are separating in G. Then s is of type D.*

3.4 Concluding the proof of Theorem 1.2

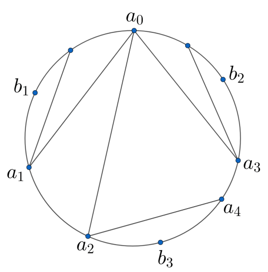
Gathering the results of Propositions 2.4, 3.1, 3.2, and 3.3, we deduce that if s is unigraphic as a 3-polytope of radius one satisfying $a \geq 3$, then either s is one of B1, B2, B3, C, D, or $p \leq 3a - 1$, or s is $14, 5^9, 3^5$. The proof of Theorem 1.2 is thus complete.

4 Proof of Proposition 1.4

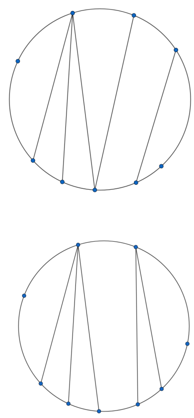
In this section we assume that the number of degree three vertices in F is $a = 2$.

Lemma 4.1. *The graph G is a forest, and every connected component of G is a caterpillar.*

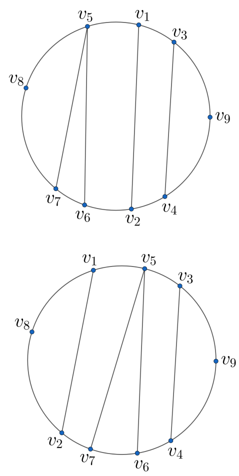
Proof. The first statement is trivial in light of Lemma 1.5: the presence of a cycle would yield $a \geq 3$. For the second statement, again by contradiction, any non-caterpillar tree contains the subgraph A depicted in Figure 10a. With the labelling as in Figure 10a, reasoning as in Lemma 1.5 there are degree 0 vertices of G along the internally disjoint u_0u_1- and u_0u_3-paths in H. Moreover, by planarity u_4 lies on one of the internally disjoint a_1a_2- and a_2a_3-paths, w.l.o.g. say the a_2a_3-path. Then there is a third degree 0 vertex of G along the a_2a_4-subpath. □



(a) Union of Hamiltonian cycle H and non-caterpillar A. There are at least three degree 0 vertices in G, namely b_1, b_2, b_3.



(b) Two realisations of the sequence $3, 2, 1^5, 0^2$.



(c) Two realisations of the sequence $2, 1^6, 0^2$. Here $p = 10$ and $t = 4$.

Figure 10: $a = 2$.

We are reduced to characterising the sequences s' with unique realisation as a disjoint union of caterpillars. Since G is a forest, its sequence s' has at least $2k$ 1's, where k is the number of non-trivial components (these are non-trivial trees). Now if s' is graphic, then it definitely has a realisation where $k - 1$ of these components are just copies of K_2. Denote by T the remaining component. For s unigraphic, we certainly have the possibility that G is just the union of $k \geq 1$ copies of K_2 and two isolated vertices, i.e. s is of type A1, with $k = (p-1-2)/2$, i.e. $p \geq 5$ and odd.

If $k \geq 2$ but $T \not\simeq K_2$, then we claim that $T \simeq \mathcal{S}_n$ is a star on $n \geq 2$ edges. To see this, suppose by contradiction that there are in s' at least two values $x, y \geq 2$. Then we would have the two realisations $\mathcal{C}(x,y) \cup K_2 \cup G'$ and $\mathcal{S}_x \cup \mathcal{S}_y \cup G'$, where G' is a forest on $k-2$ non-trivial trees, and $\mathcal{S}$, $\mathcal{C}$ denote stars and caterpillars – refer to Figure 10b. Thus indeed $T \simeq \mathcal{S}_n$, $n \geq 2$. Now we observe that in this case necessarily $k = 2$: suppose not for a contradiction. We label

$$v_1v_2, v_3v_4, \ldots, v_{t-1}v_t$$

the copies of K_2, v_{t+1} the centre of the star, $v_{t+2}, \ldots, v_{p-3}$ the remaining vertices of the star, and v_{p-2}, v_{p-1} the degree 0 vertices, where t is even and satisfies $4 \leq t \leq p-6$. Then we would have two non-isomorphic realisations of F, one with the vertices

$$v_{p-2}, v_{t+1}, v_1, v_3, \ldots, v_{t-1}, v_{p-1}, v_t, \ldots, v_4, v_2, v_{t+2}, \ldots, v_{p-3}$$

in order around H, and another with the order

$$v_{p-2}, v_1, v_{t+1}, v_3, \ldots, v_{t-1}, v_{p-1}, v_t, \ldots, v_4, v_{t+2}, \ldots, v_{p-3}, v_2$$

(Figure 10c). To summarise, G is the disjoint union of two isolated vertices, a copy of K_2, and a star $\mathcal{S}_n$, $n \geq 2$: this is possibility A2.

We are left with the case $k = 1$, i.e. G itself is a caterpillar together with two isolated vertices. Now any caterpillar $\mathcal{C}(x_1, \ldots, x_\ell)$ is determined by its length $\ell \geq 1$ and degrees $x_i \geq 2$, $1 \leq i \leq \ell$, of vertices along its path. If $x_1 \neq x_j$ for any $2 \leq j \leq \ell$, we only need to exchange the order of the corresponding vertices along the path to obtain another realisation of s, *except if* $\ell = 2$. Thereby, either the x_i's are all equal – type A3, or $\ell = 2$ – type A4.

It is straightforward to check that, on the other hand, the sequences A1, A2, A3, and A4 are indeed unigraphic. This concludes the proof of Proposition 1.4.

Remark 4.2. Let Q_i be the set of degree one vertices adjacent to u_i of degree x_i in the caterpillar $\mathcal{C}(x_1, \ldots, x_\ell)$ of G. To ensure that $a = 2$, the vertices of $\mathcal{C}(x_1, \ldots, x_\ell)$ are placed around H in the order

$$Q_1, x_1, Q_2, x_3, Q_4, x_5, \ldots, Q_{\ell-2}, x_{\ell-1}, Q_\ell, x_\ell, Q_{\ell-1}, x_{\ell-2}, \ldots, x_4, Q_3, x_2$$

if ℓ is even, and

$$Q_1, x_1, Q_2, x_3, Q_4, x_5, \ldots, Q_{\ell-1}, x_\ell, Q_\ell, x_{\ell-1}, Q_{\ell-2}, x_{\ell-3}, \ldots, x_4, Q_3, x_2$$

if ℓ is odd.

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Data availability statement

The code to produce the data of Tables 1, 2, and 3 is available on request.

A Data

We have gathered the data of Tables 1, 2, and 3 by implementing the simplified faster version of Tutte's algorithm for the case of radius one (Remark 1.1), with the help of Scientific IT & Application Support (SCITAS) High Performance Computing (HPC) for the EPFL community. In particular, we found all 3-polytopes of radius 1 with $p \leq 17$, and also those with $q \leq 41$, and among these the unigraphic ones. The unigraphic 3-polytopes found with this algorithm are consistent with the results of Theorem 1.2 and Proposition 1.4. See [8, 3, 21] for catalogues of 3-polytopes in general.

edges	3-polytopes	3-polytopes of radius 1	rad. 1 sequences	unigraphic rad. 1 seq.
6	1	1	1	1
7	0	0	0	0
8	1	1	1	1
9	2	1	1	1
10	2	1	1	1
11	4	1	1	1
12	12	2	2	2
13	22	2	1	0
14	58	4	3	2
15	158	5	4	3
16	448	7	3	1
17	1342	10	5	2
18	4199	16	7	5
19	13384	27	6	1
20	43708	42	10	3
21	144810	67	15	6
22	485704	116	11	2
23	1645576	187	18	2
24	5623571	329	28	11
25	19358410	570	21	1
26	67078828	970	35	4
27	?	1723	52	12
28		3021	38	1
29		5338	61	3
30		9563	90	15
31		16981	67	1
32		30517	103	3
33		54913	158	18
34		98847	112	2
35		179119	178	3
36		324333	258	20
37		589059	191	1
38		1072997	287	3
39		1955207	425	24
40		3573129	?	1
41		6538088	?	1

Table 1: Number of 3-polytopes or radius 1, degree sequences and unigraphic sequences up to 41 edges. For the total number of 3-polytopes on q edges see e.g. Dillencourt [6].

q \ p	4	5	6	7	8	9	10	11	12	13	14
6	1(1)										
7	–										
8		1(1)									
9		1(1)									
10			1(1)								
11			1(1)								
12			1(1)	1(1)							
13				2							
14				3(1)	1(1)						
15				3(3)	2						
16					6	1(1)					
17					7(2)	3					
18					4(4)	11	1(1)				
19						24(1)	3				
20						24(2)	17	1(1)			
21						12(6)	51	4			
22							89(1)	26	1(1)		
23							74(2)	109	4		
24							27(9)	265(1)	36	1(1)	
25								371(1)	194	5	
26								259(3)	660	50	1(1)
27								82(11)	1291(1)	345	5
28									1478	1477	65
29									891(2)	3891(1)	550
30									228(14)	6249	3000
31										6044(1)	10061
32										3176(2)	21524
33										733(18)	29133
34											24302
35											11326(3)
36											2282(19)

Table 2: Number of 3-polytopes or radius 1, sorted by order p and size q (continues in Table 3). Numbers in brackets indicate unigraphic 3-polytopes.

q \ p	15	16	17	18	19	20	21
28	1(1)						
29	6						
30	85	1(1)					
31	870	6					
32	5710	106	1(1)				
33	23747	1293	7				
34	64183(1)	10228	133	1(1)			
35	114541	51349	1896	7			
36	133464	170904	17521	161	1(1)		
37	98000(1)	384035	104349	2667	8		
38	40942(2)	586696	416385	28777	196	1(1)	
39	7528(23)	599516	1144304(1)	200137	3714	8	
40		392528	2192206	942417	45745	232	1(1)
41		148646(1)	2923018	3094421	366982	5012	9
42		24834(23)	2656742	...	...	...	...
43			1570490	...	...	...	...
44			543515(2)	...	...	...	...
45			83898(28)	...	...	...	...

Table 3: Number of 3-polytopes or radius 1, sorted by order p and size q (continued from Table 2). Numbers in brackets indicate unigraphic 3-polytopes.

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