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On the semigroup of graph gonality sequences

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Abstract

The rth gonality of a graph is the smallest degree of a divisor on the graph with rank r. The gonality sequence of a graph is a tropical analogue of the gonality sequence of an algebraic curve. We show that the set of truncated gonality sequences of graphs forms a semigroup under addition. Using this, we study which triples (x, y, z) can be the first three terms of a graph gonality sequence. We show that nearly every such triple with $z \geq \frac{3}{2}x + 2$ is the first three terms of a graph gonality sequence, and also exhibit triples where the ratio $\frac{z}{x}$ is an arbitrary rational number between 1 and 3. In the final section, we study algebraic curves whose rth and $(r+1)$st gonality differ by 1, and posit several questions about graphs with this property.

1 Introduction

The theory of divisors on graphs, developed by Baker and Norine in [2, 3], mirrors that of divisors on curves. Two important invariants of a divisor D, on either a graph or a curve, are its degree $\deg(D)$ and its rank $\mathrm{rk}(D)$. For $r \geq 1$, the rth *gonality* of a graph is the smallest degree of a divisor of rank r:

$$\mathrm{gon}_r(G) := \min_{D \in \mathrm{Div}(G)} \{\deg(D) \mid \mathrm{rk}(D) \geq r\}.$$

The *gonality sequence* of a graph G is the sequence:

$$\mathrm{gon}_1(G), \mathrm{gon}_2(G), \mathrm{gon}_3(G), \ldots$$

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In [1], the authors ask which integer sequences are the gonality sequence of some graph.

In this paper, we approach this problem by studying the first r terms of the gonality sequence. Let

$$\mathcal{G}_r := \{\vec{x} \in \mathbb{N}^r \mid \exists \text{ a graph } G \text{ with } \operatorname{gon}_k(G) = x_k \text{ for all } k \leq r\}.$$

Our first main observation is that $\mathcal{G}_r$ is a *semigroup* — that is, it is closed under addition. We say that an element $\vec{x} \in \mathcal{G}_r$ is *reducible* if it can be written as the sum of two elements in $\mathcal{G}_r$.

Theorem 1.1. *The set $\mathcal{G}_r$ is closed under addition. Moreover, if $\vec{x} \in \mathcal{G}_r$ is reducible, then for all g sufficiently large, there exists a graph G of genus g such that $\operatorname{gon}_k(G) = x_k$ for all $k \leq r$.*

The set $\mathcal{G}_r$ is always contained in the cone:

$$\mathcal{C}_r := \{\vec{x} \in \mathbb{N}^r \mid x_i < x_{i+1} \text{ and } x_{i+j} \leq x_i + x_j \text{ for all } i, j \leq r\}$$

(See Lemmas 2.1 and 2.2). Using Theorem 1.1, we give a short proof of [1, Theorem 1.5].

Theorem 1.2. *[1, Theorem 1.5] We have*

$$\mathcal{G}_2 = \mathcal{C}_2 = \{(x, y) \in \mathbb{N}^3 \mid x + 1 \leq y \leq 2x\}.$$

Moreover, if $x + 2 \leq y \leq 2x$, then for all sufficiently large g, there exists a graph G of genus g such that $\operatorname{gon}_1(G) = x$ and $\operatorname{gon}_2(G) = y$.

As noted in Section 4 of [1], Theorem 1.2 demonstrates that there are graphs whose gonality sequence cannot be the gonality sequence of an algebraic curve. For example, if C is a curve whose 2nd gonality $\operatorname{gon}_2(C) = p$ is prime, then C maps generically 1–to–1 onto a plane curve of degree p. It follows that the genus of C is at most $\binom{p-1}{2}$. On the other hand, if $p \geq 5$, then by Theorem 1.2 there exists a graph G of genus g with $\operatorname{gon}_1(G) = p-2$ and $\operatorname{gon}_2(G) = p$ for all g sufficiently large. Since the genus of a graph is determined by its gonality sequence, we see that the gonality sequence of G does not agree with that of any algebraic curve.

On the other hand, if $\operatorname{gon}_2(G) = \operatorname{gon}_1(G) + 1$, then $(\operatorname{gon}_1(G), \operatorname{gon}_2(G))$ is an irreducible element of $\mathcal{G}_2$. We know of two infinite families of graphs such that the 2nd gonality is 1 greater than the 1st gonality – the complete graph K_{x+1} and the generalized banana graph $B^*_{x,x}$ from [1]. Interestingly, both graphs have genus $\binom{x}{2}$ and 3rd gonality $\operatorname{gon}_3 = 2x$. This is exactly the genus and 3rd gonality of an algebraic curve C satisfying $\operatorname{gon}_2(C) = \operatorname{gon}_1(C) + 1 = x + 1$ (see Lemma 7.2). We ask whether this holds more generally.

Question 1.3. *Let G be a graph with the property that $\operatorname{gon}_2(G) = \operatorname{gon}_1(G) + 1$.*

(1) *Is the genus of G necessarily $g = \binom{\operatorname{gon}_1(G)}{2}$?*

(2) *For $r < g$, do we have*

$$\mathrm{gon}_r(G) = k \cdot \mathrm{gon}_2(G) - h,$$

where k and h are the uniquely determined integers with $1 \leq k \leq \mathrm{gon}_2(G) - 3$, $0 \leq h \leq k$, such that $r = \frac{k(k+3)}{2} - h$?

(3) *In particular, if $\mathrm{gon}_1(G) \geq 2$, does it follow that $\mathrm{gon}_3(G) = 2 \cdot \mathrm{gon}_1(G)$?*

Much of this paper is dedicated to studying $\mathcal{G}_3$. Unlike $\mathcal{G}_2$, we are unable to provide a complete description of $\mathcal{G}_3$. However, we have the following partial result.

Theorem 1.4. *Let $(x, y, z) \in \mathcal{C}_3$ with $z \geq 2x$. Suppose further that:*

- *if $y = x + 1$, then $z = 2x$, and*
- *if $z = x + y$, then $y = 2x$.*

Then $(x, y, z) \in \mathcal{G}_3$.

We suspect that Theorem 1.4 classifies all realizable triples $(x, y, z) \in \mathcal{G}_3$ such that $z \geq 2x$. Indeed, by Lemmas 2.1 and 2.2, we have $\mathcal{G}_3 \subseteq \mathcal{C}_3$. If $y = x+1$, then an affirmative answer to Question 1.3 would show that $z = 2x$. Similarly, if $z = x + y$, then an affirmative answer to [1, Question 4.5] would show that $y = 2x$. The goal of the rest of this paper is to study triples $(x, y, z) \in \mathcal{G}_3$ with $z < 2x$. In Section 6, we prove the following.

Theorem 1.5. *Let $(x, y, z) \in \mathcal{C}_3$ with $x + 2 \leq y \leq z - 2$ and $z \geq \frac{3}{2}x + 2$. Then $(x, y, z) \in \mathcal{G}_3$.*

Theorems 1.4 and 1.5 give a possibly complete description of triples $(x, y, z) \in \mathcal{G}_3$ with $z \geq \frac{3}{2}x + 2$. However, there exist triples $(x, y, z) \in \mathcal{G}_3$ such that $z < \frac{3}{2}x + 2$. Indeed, we have the following.

Lemma 1.6. *Let q be a rational number in the range $1 < q \leq 3$. Then there exists $(x, y, z) \in \mathcal{G}_3$ such that $\frac{z}{x} = q$.*

Unfortunately, it is difficult to write down a simple, closed-form expression for the semigroup generated by these triples. It seems likely that the techniques of this paper could be used to study $\mathcal{G}_r$ for $r \geq 4$, or to produce analogues of Theorem 1.5 where the ratio $\frac{z}{x}$ is bounded below by a constant that is smaller than $\frac{3}{2}$.

Remark 1.7. A *metric graph* is a compact, connected metric space obtained from a graph by identifying the edges with line segments. One can also define the gonality sequence of a metric graph, and it is natural to ask which integer sequences are the gonality sequence of some metric graph. By [2, Corollary 1.5], if D is a divisor on a graph G, and Γ is the corresponding metric graph in which every edge has length 1, then $\mathrm{rk}_G(D) = \mathrm{rk}_\Gamma(D)$. It follows that the rth gonality of Γ is less than or equal to that of G for all r. In some cases, the rth gonality of Γ is equal to that of G, for example, if G is a complete graph [7] or complete bipartite graph [6]. However, this is not always the case [15], and we do not know whether this holds for some of the graphs appearing in this paper. For this reason, the analogs of our results remain open for metric graphs

The paper is organized as follows. In Section 2, we present background on the divisor theory of graphs. In Section 3 we introduce graphs with known 1st, 2nd, and 3rd gonalities. In Section 4, we prove all of the main results except for Theorem 1.4, which is proved in Sections 5 and 6. Finally, in Section 7, we study the gonality sequences of certain algebraic curves, and ask several questions about graphs with the same gonality sequences.

2 Preliminaries

In this section we will introduce the notion of gonality on graphs, along with important terms and concepts. Throughout, we allow graphs to have parallel edges, but no loops.

A *divisor* on a graph G is a formal $\mathbb{Z}$-linear combination of the vertices in G. A divisor D can be expressed as

$$D = \sum_{v \in V(G)} D(v) \cdot v,$$

where each $D(v)$ is an integer. The *degree* of a divisor D, denoted $\deg(D)$, is the sum of the coefficients of D. The *support* of a divisor, denoted $\text{Supp}(D)$, is defined as

$$\text{Supp}(D) = \{v \in V(G) | D(v) > 0\}.$$

It is standard to think about divisors on graphs in terms of chip configurations. In a chip configuration, the coefficient of a vertex v is reinterpreted as the number of chips sitting on v. So, in a divisor D, v has $D(v)$ chips sitting on it. A vertex with a negative number of chips is said to be "in debt." A divisor is *effective* if, for every $w \in V(G)$, we have $D(w) \geq 0$. In other words, a divisor is effective if there are no vertices in debt. A divisor is *effective away from* v if, for every $w \in V(G) \setminus \{v\}$, we have $D(w) \geq 0$.

From this interpretation we can define a chip firing move. Firing a vertex v causes v to redistribute some of its chips by passing one chip across each of the edges incident to it. We say that two divisors D and D' are *equivalent* if D' can be obtained from D via a sequence of chip firing moves. The *rank* of a divisor D, denoted $\text{rk}(D)$, is the largest integer r such that $D - E$ is equivalent to an effective divisor for every effective divisor E of degree r. By convention, if D is not equivalent to an effective divisor, we define it to have rank -1.

The rank of a divisor is often framed as a chip firing game. Given a starting divisor we allow the "opponent" of the game to remove r chips from anywhere on the graph. A divisor has rank r if, for every choice of chips by the opponent, there is a sequence of chip firing moves that eliminates all debt on the graph.

For a graph G and a vertex $v \in V(G)$ we say that a divisor D is *v-reduced* if the following conditions are satisfied:

(1) D is effective away from v, and

(2) for any subset $A \subseteq V(G) \setminus \{v\}$, the divisor D' obtained by firing all the vertices in A is not effective.

Given a divisor D and a vertex v, by [4, Corollary 4.13], there exists a unique divisor equivalent to D that is v-reduced. Dhar's Burning Algorithm is a procedure that produces this unique representative. A good introduction to this algorithm can be found in Section 5.1 of [4]. The algorithm proceeds as follows:

(1) Replace D with a divisor that is effective away from v.
(2) Start a fire by burning vertex v.
(3) Burn every edge that is incident to a burnt vertex.
(4) Let U be the set of unburnt vertices. For each $w \in U$ we burn w if the number of burnt edges incident to w is strictly greater than $D(w)$. If no new vertices in U were burnt proceed to step (5). Otherwise return to step (3).
(5) Let U be the set of unburnt vertices. If U is empty, then D is v-reduced and the algorithm terminates. Otherwise, replace D with the equivalent divisor D' obtained by firing all vertices in U and return to step (2).

Note that a divisor is v-reduced if and only if starting a fire at v results in the entire graph being burnt. Dhar's burning algorithm is useful for determining the rank of a divisor. Specifically, we show that if D is v-reduced and $D(v) \leq 0$, then D cannot have positive rank. To see this, note that if D' is an effective divisor, then the divisor obtained by firing all unburnt vertices is also effective. It follows that the v-reduced divisor equivalent to D' is effective as well. Since D is v-reduced, we have that $D - v$ is v-reduced as well. Since $D - v$ is v-reduced and not effective, it is not equivalent to an effective divisor, and thus D does not have positive rank.

The rth *gonality* of a graph is the minimum degree over all divisors of rank r. We recall some basic facts about the rth gonality from [1].

Lemma 2.1. *[1, Lemma 3.1] Let G be a graph. For all r, we have $\mathrm{gon}_r(G) < \mathrm{gon}_{r+1}(G)$.*

Lemma 2.2. *[1, Lemma 3.2] Let G be a graph. For all r and s, we have $\mathrm{gon}_{r+s}(G) \leq \mathrm{gon}_r(G) + \mathrm{gon}_s(G)$.*

3 Dramatis Personae

This section surveys graphs for which the first few terms of the gonality sequence are known. The first of these graphs is the complete graph K_n, which has genus $g = \binom{n-1}{2}$.

Lemma 3.1. *[7, Theorem 1] For $r < g$, the rth gonality of the complete graph K_n is $\mathrm{gon}_r(K_n) = kn - h$, where k and h are the uniquely determined integers with*

$1 \le k \le n-3$, $0 \le h \le k$, such that $r = \frac{k(k+3)}{2} - h$. In particular, if $n \ge 3$, then

$$\begin{aligned}\mathrm{gon}_1(K_n) &= n-1\\ \mathrm{gon}_2(K_n) &= n\\ \mathrm{gon}_3(K_n) &= 2n-2.\end{aligned}$$

Next, we have the complete bipartite graph $K_{m,n}$, which has genus $g = (m-1)(n-1)$. Let

$$I_r = \{(a,b,h) \in \mathbb{N}^3 \mid a \le m-1, b \le n-1, \text{ and } r = (a+1)(b+1)-1-h\},$$

and let

$$\delta_r(m,n) = \min\{an + bm - h \mid (a,b,h) \in I_r\}.$$

Lemma 3.2. *[6, Theorem 4] For $r < g$, the rth gonality of the complete bipartite graph $K_{m,n}$ is $\mathrm{gon}_r(K_{m,n}) = \delta_r(m,n)$. In particular, if $2 \le m \le n$, then*

$$\begin{aligned}\mathrm{gon}_1(K_{m,n}) &= m\\ \mathrm{gon}_2(K_{m,n}) &= \min\{2m, m+n-1\}\\ \mathrm{gon}_3(K_{m,n}) &= \min\{3m, m+n\}.\end{aligned}$$

The *banana graph* B_n is the graph consisting of 2 vertices with n edges connecting them. A *generalized banana graph* is a graph with vertex set $\{v_1, \ldots, v_n\}$ such that for each $1 \le i < n$, there is at least one edge between v_i and v_{i+1} and no edges elsewhere.

In [1], the authors study the gonalty sequences of different families of generalized banana graphs. The generalized banana graph $B_{n,e}$ is the graph with vertex set $\{v_1, \ldots v_n\}$ and where there are e edges between v_i and v_{i+1} for $1 \le i \le n-1$. The generalized banana graph $B^*_{a,b}$ is the graph with vertex set $\{v_1, \ldots, v_a\}$ and with $b-a+i+1$ edges between v_i and v_{i+1} for $1 \le i \le a-1$. The generalized banana graphs $B_{4,3}$ and $B^*_{4,5}$ are depicted in Figure 1.

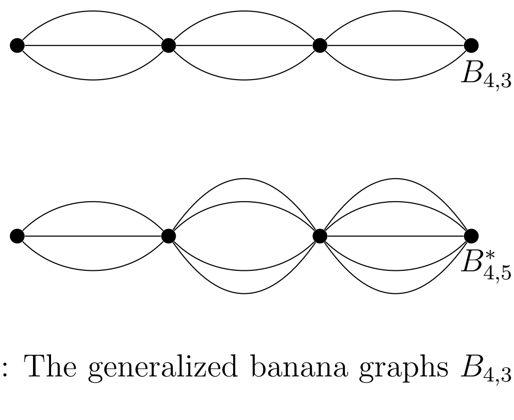

Figure 1: The generalized banana graphs $B_{4,3}$ and $B^*_{4,5}$.

Lemma 3.3. *[1, Lemmas 5.2-5.4] We have*

$$\begin{aligned}\mathrm{gon}_1(B_{n,e}) &= \min\{n, e\}\\ \mathrm{gon}_2(B_{n,e}) &= \min\{2n, 2e, n+e-1\}.\end{aligned}$$

Lemma 3.4. *[1, Lemmas 5.5 and 5.6] If* $2 \leq a \leq b \leq 2a-1$, *we have*

$$\begin{aligned} \mathrm{gon}_1(B^*_{a,b}) &= a \\ \mathrm{gon}_2(B^*_{a,b}) &= b+1. \end{aligned}$$

The 2-*dimensional* n *by* m *rook graph* is the Cartesian product of the complete graphs K_n and K_m. The vertices can be thought of as the squares of an $n \times m$ chessboard, in which 2 vertices are adjacent if a rook can move from one to the other. By convention, we assume throughout that $m \geq n$. In [14], Speeter computes the first 3 gonalities of these rook graphs.

Lemma 3.5. *[14] If* $2 \leq n \leq m$, *then*

$$\begin{aligned} \mathrm{gon}_1(K_n \square K_m) &= (n-1)m \\ \mathrm{gon}_2(K_n \square K_m) &= nm-1 \\ \mathrm{gon}_3(K_n \square K_m) &= nm. \end{aligned}$$

4 Proofs of Theorems 1.1-1.4

In this section, we prove many of the main theorems. The central construction is the following. Given two graphs G_1 and G_2, and vertices $v_1 \in V(G_1)$, $v_2 \in (G_2)$, we "put them together" by connecting v_1 to v_2 with a number of parallel edges, as in Figure 2. This construction is very similar to that of vertex gluing, as seen for instance in [12, 5, 13], the only difference being that the vertices v_1 and v_2 are connected by multiple parallel edges, rather than a single edge.

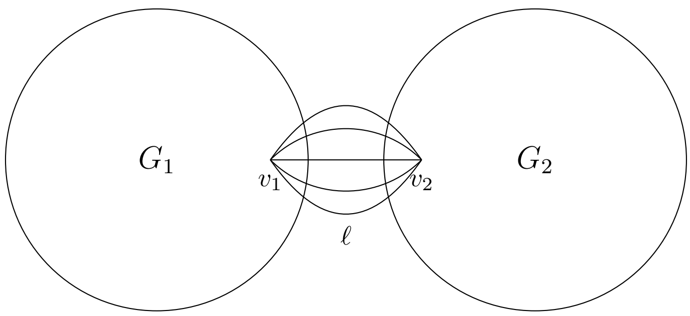

Figure 2: Gluing two graphs together

Lemma 4.1. *Let* G_1 *and* G_2 *be graphs, let* $v_1 \in V(G_1)$, $v_2 \in V(G_2)$, *and let* G *be the graph obtained by connecting* v_1 *to* v_2 *with* ℓ *edges, as in Figure 2. If* D *is a* v_1*-reduced divisor on* G*, then* $\mathrm{rk}(D|_{G_1}) \geq \mathrm{rk}(D)$.

Proof. If D has rank -1, there is nothing to prove, so we assume throughout that D has nonnegative rank. Since D is v_1-reduced, it follows that D is effective.

Now, let E be an effective divisor of degree $\mathrm{rk}(D)$ on G_1. By definition, $D - E$ is equivalent to an effective divisor. Therefore, there exists a sequence of subsets:

$$U_1 \subseteq U_2 \subseteq \cdots \subseteq U_k \subset V(G)$$

and a sequence of effective divisors $D_0, \ldots, D_k$ such that:

(1) $D_0 = D$,
(2) $D_k - E$ is effective, and
(3) D_i is obtained from D_{i-1} by firing U_i.

Since D_0 is v_1-reduced and D_1 is effective, we must have $v_1 \in U_1$. Thus, $v_1 \in U_i$ for all i.

Now, consider the sequence of subsets $W_i = U_i \cup V(G_2)$. Let $D'_0 = D$ and let D'_i be the divisor obtained from D'_{i-1} by firing W_i. Since $V(G_2) \cup \{v_1\} \subseteq W_i$ for all i, we have $D'_i(v) = D(v)$ for all $v \in V(G_2)$. Since D is v_1-reduced, it follows that $D'_k(v) \geq 0$ for all $v \in V(G_2)$. Note also that $D'_i(v) = D_i(v)$ for all $v \in V(G_1) \smallsetminus \{v_1\}$, and $D'_i(v_1) \geq D_i(v_1)$. It follows that $D'_k - E$ is effective.

Finally, note that firing W_i passes no chips from G_1 to G_2 or from G_2 to G_1. Thus, $D|_{G_1}$ is equivalent to $D'_k|_{G_1}$ on G_1. In this way, $D|_{G_1} - E$ is equivalent on G_1 to an effective divisor. Since E was arbitrary, we see that $\mathrm{rk}(D|_{G_1}) \geq \mathrm{rk}(D)$. □

If the number of parallel edges between v_1 and v_2 is large enough, then the rth gonality of the graph G is the sum of the rth gonalities of the graphs G_1 and G_2.

Proposition 4.2. *Let G_1 and G_2 be graphs, let $v_1 \in V(G_1)$, $v_2 \in V(G_2)$, and let G be the graph obtained by connecting v_1 to v_2 with ℓ edges. If $\ell \geq \mathrm{gon}_r(G_1)+\mathrm{gon}_r(G_2)$, then*

$$\mathrm{gon}_k(G) = \mathrm{gon}_k(G_1) + \mathrm{gon}_k(G_2) \text{ for all } k \leq r.$$

Proof. Let $k \leq r$, let D_1 be a divisor of rank k and degree $\mathrm{gon}_k(G_1)$ on G_1, and let D_2 be a divisor of rank k and degree $\mathrm{gon}_k(G_2)$ on G_2. Then the divisor $D_1 + D_2$ has rank at least k on G, so $\mathrm{gon}_k(G) \leq \mathrm{gon}_k(G_1) + \mathrm{gon}_k(G_2)$.

For the reverse inequality, let D be a divisor of rank at least k on G. We must show that $\deg(D) \geq \mathrm{gon}_k(G_1) + \mathrm{gon}_k(G_2)$. If $\deg(D|_{G_i}) \geq \mathrm{gon}_k(G_i)$ for $i = 1, 2$, then $\deg(D) \geq \mathrm{gon}_k(G_1) + \mathrm{gon}_k(G_2)$. On the other hand, suppose without loss of generality that $\deg(D|_{G_1}) < \mathrm{gon}_k(G_1)$. Then $D|_{G_1}$ has rank less than k, so we must be able to pass chips from G_2 to G_1. Since there are ℓ edges between G_1 and G_2, it follows that $\deg(D) \geq \ell \geq \mathrm{gon}_k(G_1) + \mathrm{gon}_k(G_2)$. □

Theorem 1.1 is a direct corollary.

Proof of Theorem 1.1. Let $\vec{x}, \vec{y} \in \mathcal{G}_r$. By definition, there exist graphs G_1 and G_2 such that $\mathrm{gon}_k(G_1) = x_k$ and $\mathrm{gon}_k(G_2) = y_k$ for all $k \leq r$. By Proposition 4.2, there exists a graph G with $\mathrm{gon}_k(G) = x_k + y_k$ for all $k \leq r$. Moreover, if G_1 has genus g_1 and G_2 has genus g_2, then by Proposition 4.2, for any $\ell \geq \mathrm{gon}_r(G_1) + \mathrm{gon}_r(G_2)$, there exists such a graph G of genus $g = g_1 + g_2 + \ell$. □

Using the fact that $\mathcal{G}_2$ is closed under addition, we provide a short proof of Theorem 1.2.

Proof of Theorem 1.2. Let G be a graph. By Lemma 2.1, we have $\text{gon}_2(G) \geq \text{gon}_1(G) + 1$. By Lemma 2.2, $\text{gon}_2(G) \leq 2 \cdot \text{gon}_1(G)$. In other words, $\mathcal{G}_2 \subseteq \mathcal{C}_2$.

We now show the reverse containment. In other words, we show that if $x + 1 \leq y \leq 2x$, then there exists a graph G with $\text{gon}_1(G) = x$ and $\text{gon}_2(G) = y$. We proceed by induction on $y - x$. For the base case, when $y = x + 1$, by Lemma 3.1, the complete graph on $x + 1$ vertices K_{x+1} satisfies

$$x = \text{gon}_1(K_{x+1}) = \text{gon}_2(K_{x+1}) - 1.$$

For the inductive step, if $y \geq x+2$, then since $(y-2)-(x-1) = y-x-1$, by induction $(x - 1, y - 2) \in \mathcal{G}_2$. If T is a tree, then $\text{gon}_r(T) = r$ for all r, so $(1, 2) \in \mathcal{G}_2$. By Theorem 1.1, therefore, (x, y) is a reducible element of $\mathcal{G}_2$, and the result follows. $\square$

A similar strategy allows us to construct triples $(x, y, z) \in \mathcal{G}_3$ where z is large relative to x.

Proof of Theorem 1.4. Let $(x, y, z) \in \mathbb{N}^3$ satisfy $x < y < z$, $y \leq 2x$, $z \leq x + y$. Suppose further that if $y = x + 1$, then $z = 2x$, and if $z = x + y$, then $y = 2x$. We will address these two edge cases first. Suppose that $y = x + 1$ and $z = 2x$. By Lemma 3.1, the first 3 terms of the gonality sequence of the complete graph K_y are $(x, x + 1, 2x)$, so $(x, x + 1, 2x) \in \mathcal{G}_3$. Similarly, suppose that $z = x + y$ and $y = 2x$. The first three terms of the gonality sequence of a tree are $(1, 2, 3)$. By Theorem 1.1, therefore, we have $(x, 2x, 3x) \in \mathcal{G}_3$. For the remainder of the proof, we assume that $y \geq x + 2$ and $z \leq x + y - 1$.

We now show that, if $z = 2x$, then $(x, y, z) \in \mathcal{G}_3$. If $y = 2x - 1$, then by Lemma 3.2, the first 3 terms of the gonality sequence of the complete bipartite graph $K_{x,x}$ are $(x, 2x - 1, 2x)$, so $(x, 2x - 1, 2x) \in \mathcal{G}_3$. Otherwise, if $x + 2 \leq y \leq 2x - 2$, then the first 3 terms of the gonality sequence of the complete graph K_{2x-y+1} are $(2x - y, 2x - y + 1, 4x - 2y)$ and the first 3 terms of the gonality sequence of the complete bipartitie graph $K_{y-x,y-x}$ are $(y-x, 2y-2x-1, 2y-2x)$. By Theorem 1.1, therefore, we have

$$(2x - y, 2x - y + 1, 4x - 2y) + (y - x, 2y - 2x - 1, 2y - 2x) = (x, y, 2x) \in \mathcal{G}_3.$$

We now consider cases where $2x < z \leq x + y - 1$. If $y = 2x$, then the first 3 terms of the gonality sequence of the complete bipartitie graph $K_{x,z-x}$ are $(x, 2x, z)$, so $(x, 2x, z) \in \mathcal{G}_3$. If $y = x + 2$, then by assumption, $z = 2x + 1$. As above, the first 3 terms of the gonality sequence of the complete graph K_x are $(x - 1, x, 2x - 2)$ and the first 3 terms of the gonality sequence of a tree are $(1, 2, 3)$. By Theorem 1.1, therefore, we have

$$(x - 1, x, 2x - 2) + (1, 2, 3) = (x, x + 2, 2x + 1) \in \mathcal{G}_3.$$

It remains to consider the cases where $x + 3 \leq y \leq 2x - 1$. Similar to the above, the first 3 terms of the gonality sequence of the complete graph K_{2x-y+2} are $(2x-y+1, 2x-y+2, 4x-2y+2)$. Since $z > 2x$, we have $z+y-3x-1 > y-x-1$, and since $z \leq x+y-1$, we have $z+y-3x-1 \leq 2(y-x-1)$. It follows that the first 3 terms of the gonality sequence of the complete bipartitie graph $K_{y-x-1,z+y-3x-1}$ are $(y-x-1, 2y-2x-2, 2y-4x-2+z)$. By Theorem 1.1, therefore, we have

$$(2x-y+1, 2x-y+2, 4x-2y+2)+(y-x-1, 2y-2x-2, 2y-4x-2+z) = (x, y, z) \in \mathcal{G}_3.$$

□

Proof of Lemma 1.6. If $q \geq 2$, the conclusion follows from Theorem 1.4. If $1 < q < 2$, then there exists an integer $n \geq 2$ such that $\frac{n+1}{n} \leq q < \frac{n}{n-1}$. If $q = \frac{n+1}{n}$, then by Lemma 3.5, for all $m \geq n+1$, we have $(nm, (n+1)m - 1, (n+1)m) \in \mathcal{G}_3$, and the conclusion follows.

If $q > \frac{n+1}{n}$, let $\epsilon_1 = q - \frac{n+1}{n}$, let $\epsilon_2 = \frac{n-(n-1)q}{n+1}$, and let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. By assumption, $\epsilon > 0$. We can therefore write $q = \frac{z}{x}$, where $x \geq \frac{1}{\epsilon}$. Finally, let $m = nz - (n+1)x$ and let $m' = nx - (n-1)z$. By construction, $m \geq n$ and $m' \geq n+1$. By Lemma 3.5 and Theorem 1.1, we have

$$((n-1)m, nm-1, nm) + (nm', (n+1)m' - 1, (n+1)m') = (x, z-2, z) \in \mathcal{G}_3.$$

□

5 Third Gonality of the Graphs $B^*_{a,b}$

The 1st and 2nd gonalities of the graph $B^*_{a,b}$ are computed in [1]. In this section, we compute the 3rd gonalities of these graphs. We first consider divisors on $B^*_{a,b}$ of rank 3 with a large number of chips on v_a.

Lemma 5.1. *Let D be a divisor of at least rank 3 on the graph $B^*_{a,b}$. If $D(v_a) \geq b+1$, then* $\deg(D) \geq a+b$.

Proof. We proceed by induction on a. For the base case consider the banana graph $B^*_{2,b}$. We will proceed by cases.

(1) If $D(v_2) = b+1$, then since D has rank at least 3, the divisor $D - 2\cdot(v_2) - (v_1)$ must be equivalent to an effective divisor. This divisor has $b-1$ chips on v_2, so we must have $D(v_1) \geq 1$. Hence, $\deg(D) \geq a+b$.
(2) If $D(v_2) \geq b+2$, then $\deg(D) \geq b+2 = a+b$.

Now for the induction step assume that the theorem holds for the banana graph $B^*_{a-1,b-1}$. If $D(v_a) \geq a+b$, we are done. If $D(v_a) = b+1$, then let $D' = D - 2v_a$. Without loss of generality, assume that D' is v_{a-1}-reduced. Then, by Lemma 4.1, we have

$$\mathrm{rk}(D|_{B^*_{a-1,b-1}}) \geq \mathrm{rk}(D'|_{B^*_{a-1,b-1}}) \geq \mathrm{rk}(D') \geq \mathrm{rk}(D) - 2 \geq 1.$$

By Lemma 3.4, it follows that $\deg(D|_{B^*_{a-1,b-1}}) \geq a$, hence $\deg(D) \geq a+b+1$. Finally, if $b+2 \leq D(v_a) < a+b$, consider the equivalent divisor obtained by firing v_a. This divisor has at least 2 chips on v_a. Note that v_a can only be fired once since $a < b$. By Lemma 4.1, the restriction of this divisor to the subgraph $B^*_{a-1,b-1}$ must have rank at least 3 and there are at least b chips on v_{a-1}, so by the inductive hypothesis there are least $a+b-2$ chips on this subgraph. So $\deg(D) \geq a+b$. □

We also consider divisors on $B^*_{a,b}$ of rank 3 with a small number of chips on v_a.

Lemma 5.2. *Let D be a divisor on $B^*_{a,b}$ of rank at least 3. If $D(v_a) \leq 2$, then $\deg(D) \geq a+b$.*

Proof. Assume without loss of generality that D is v_{a-1}-reduced. We proceed by cases.

(1) If $D(v_a) = 0$, then consider the divisor $D - v_a$. The resulting divisor has a debt on v_a, so we must have $D(v_{a-1}) \geq b$. After moving b of these chips in D to v_a and subtracting one from v_a, by Lemma 4.1, the remaining divisor must have rank at least 2 on $B^*_{a-1,,b-1}$. Hence, by Lemma 3.4, there must be at least b more chips on the rest of the graph. So $\deg(D) \geq 2b \geq a+b$.

(2) If $D(v_a) = 1$, then consider the divisor $D - 2 \cdot (v_a)$. The resulting divisor has a debt on v_a, so $D(v_{a-1}) \geq b$. After moving b of these chips in D to v_a and subtracting two from v_a, by Lemma 4.1 the remaining divisor must have rank at least 1 on $B^*_{a-1,b-1}$. Hence, by Lemma 3.4, there must be at least $a-1$ chips on the rest of the graph. Therefore, $\deg(D) \geq a+b$.

(3) If $D(v_a) = 2$, then consider the divisor $D - 3 \cdot v_a$. There is now a debt of -1 on v_a, so again there must be at least b chips on v_{a-1}. By Lemma 5.1, there must be at least $a+b-2$ chips on the subgraph induced by the vertices $\{v_1, \ldots v_{a-1}\}$. Thus, $\deg(D) \geq (a+b-2)+2 = a+b$.

□

Our computation of the 3rd gonality of $B^*_{a,b}$ will proceed by induction on a. The following lemma establishes the base case, when $a = 2$.

Lemma 5.3. *If $b \geq 4$, then* $\text{gon}_3(B^*_{2,b}) = 6$.

Proof. As with any graph $\text{gon}_3(B^*_{2,b}) \leq 3|V(B^*_{2,b})| = 6$. Assume there is a divisor D with $\deg(D) < 6$. By symmetry, we may assume that $D(v_2) \leq 2$. We proceed by cases.

(1) If $D(v_2) = 0$, then $D(v_1) \leq 5$. Consider the divisor $D - 2 \cdot (v_1) - (v_2)$. This divisor has a debt of -1 on v_2 but at most 3 chips on v_1, so D cannot be rank at least 3.

(2) If $D(v_2) = 1$, then $D(v_1) \leq 4$. Consider the divisor $D - (v_1) - 2 \cdot (v_2)$. This divisor has a debt of -1 on v_2 but at most 3 chips on v_1, so D cannot be rank at least 3.

(3) If $D(v_2) = 2$, then $D(v_1) \leq 3$. Consider the divisor $D - 3 \cdot (v_2)$. This divisor has a debt of -1 on v_2 but at most 3 chips on v_1, so D cannot be rank at least 3.

We conclude that $\text{gon}_3(B^*_{2,b}) = 6$. $\square$

Lemma 5.4. *If $b \geq 2a$, then $\text{gon}_3(B^*_{a,b}) = 3a$.*

Proof. As with any graph, $\text{gon}_3(B^*_{a,b}) \leq 3|V(B^*_{a,b,n})| = 3a$. Now, let D be a divisor of rank 3, and assume without loss of generality that D is v_{a-1}-reduced. If $\deg(D) \geq a + b$, we are done. If not, by Lemma 5.2, $D(v_a) \geq 3$. We will proceed by induction on a. The base case is Lemma 5.3. Now assume that $\text{gon}_3(B^*_{a-1,b-1}) = 3(a-1)$. By Lemma 4.1, the restriction of D to $B^*_{a-1,b-1}$ must have rank at least 3, so there are at least $3(a-1)$ chips on that subgraph. It follows that $\deg(D) \geq 3a$. $\square$

Theorem 5.5. *If $a \leq b \leq 2a - 1$ then $\text{gon}_3(B^*_{a,b}) = a + b$.*

Proof. First note that the divisor

$$(b+1) \cdot (v_a) + \sum_{i=1}^{a-1} v_i$$

has rank at least 3 and degree $a + b$, so $\text{gon}_3(B^*_{a,b}) \leq a + b$. Now, let D be a divisor of rank at least 3. If $\deg(D) \geq a + b$, we are done. If not, by Lemma 5.2, we have $D(v_a) \geq 3$. We proceed by induction on a. For the base cases, we have $\text{gon}_3(B^*_{2,2}) = 4$ and $\text{gon}_3(B^*_{2,3}) = 5$, by the Riemann-Roch Theorem for graphs [3, Theorem 1.12], and $\text{gon}_3(B^*_{a,2a}) = 3a$ by Lemma 5.4. For the induction step assume that $\text{gon}_3(B^*_{a-1,b-1}) = a + b - 2$. Since $D(v_a) \geq 3$, therefore, we have $\deg(D) > a + b$. $\square$

6 Symmetric Generalized Banana Graphs

Our goal now is to find a large family of graphs G such that $\text{gon}_3(G) < 2 \cdot \text{gon}_1(G)$. In this section, we consider a family of generalized banana graphs. Let $v_1, \ldots, v_a$ be the vertices in $L = B^*_{a,b}$ and let $w_1, \ldots, w_a$ be the vertices in $R = B^*_{a,b}$. Let $B^{0,0}_{a,b,k}$ be the graph obtained by connecting v_a to w_a with k edges, as in Figure 3.

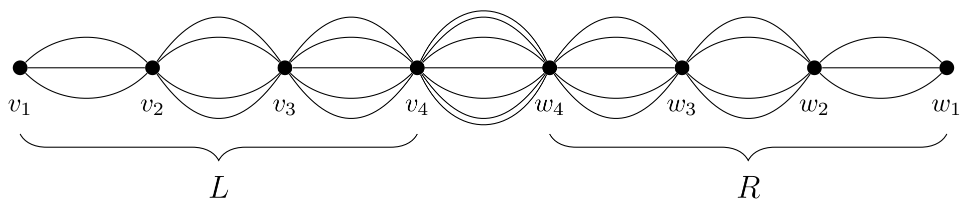

Figure 3: The symmetric generalized banana graph $B^{0,0}_{4,5,7}$.

As we will see in Lemma 6.1 and Corollary 6.2.1 below, the 1st and 2nd gonality of $B^{0,0}_{a,b,k}$ are both even. To obtain more gonality sequences, we also consider generalized banana graphs that are "almost" symmetric. Let $B^{0,1}_{a,b,k}$ be the graph obtained by connecting the vertex v_a in $L = B^*_{a,b-1}$ to the vertex w_a in $R = B^*_{a,b}$ by k edges. Let $B^{1,0}_{a,b,k}$ be the graph obtained by connecting the vertex v_{a-1} in $L = B^*_{a-1,b}$ to the vertex w_a in $R = B^*_{a,b}$ by k edges. Finally, let $B^{1,1}_{a,b,k}$ be the graph obtained by connecting the vertex v_{a-1} in $L = B^*_{a-1,b-1}$ to the vertex w_a in $R = B^*_{a,b}$ by k edges.

We begin by computing the first gonalities of these graphs.

Lemma 6.1. *If* $2 \leq a \leq b \leq 2a-1$ *and* $k \geq 2a$, *then*

$$\mathrm{gon}_1(B^{0,0}_{a,b,k}) = \mathrm{gon}_1(B^{0,1}_{a,b,k}) = 2a$$
$$\mathrm{gon}_1(B^{1,0}_{a,b,k}) = \mathrm{gon}_1(B^{1,1}_{a,b,k}) = 2a-1.$$

Proof. This follows directly from Lemma 3.4 and Proposition 4.2. □

To compute the 2nd gonality of these graphs, we will need the following refinement of Proposition 4.2.

Proposition 6.2. *Let* G_1 *and* G_2 *be graphs, let* $v_1 \in V(G_1)$, $v_2 \in V(G_2)$, *and let* G *be the graph obtained by connecting* v_1 *to* v_2 *with* ℓ *edges. If*

$$\ell \geq \mathrm{gon}_2(G_1) + \mathrm{gon}_2(G_2) - \min\{\mathrm{gon}_1(G_1), \mathrm{gon}_1(G_2)\},$$

then

$$\mathrm{gon}_2(G) = \mathrm{gon}_2(G_1) + \mathrm{gon}_2(G_2).$$

Proof. Let D_1 be a divisor of rank 2 and degree $\mathrm{gon}_2(G_1)$ on G_1, and let D_2 be a divisor of rank 2 and degree $\mathrm{gon}_2(G_2)$ on G_2. Then the divisor $D_1 + D_2$ has rank at least 2 on G, so $\mathrm{gon}_2(G) \leq \mathrm{gon}_2(G_1) + \mathrm{gon}_2(G_2)$.

For the reverse inequality, let D be a divisor of rank at least 2 on G. We must show that $\deg(D) \geq \mathrm{gon}_2(G_1) + \mathrm{gon}_2(G_2)$. Without loss of generality, assume that D is v_1-reduced. Since D has rank at least 2, $\deg(D|_{G_1}) \geq \mathrm{gon}_2(G_1)$ by Lemma 4.1. We proceed by cases. First, if $\deg(D|_{G_2}) \geq \mathrm{gon}_2(G_2)$, then $\deg(D) \geq \mathrm{gon}_2(G_1) + \mathrm{gon}_2(G_2)$.

Second, if $\mathrm{gon}_1(G_2) \leq \deg(D|_{G_2}) < \mathrm{gon}_2(G_2)$, then $(D|_{G_2})$ does not have rank at least 2, so we must be able to pass chip across the ℓ edges. Thus, $\deg(D) \geq \ell + \mathrm{gon}_1(G_2) \geq \mathrm{gon}_2(G_1) + \mathrm{gon}_2(G_2)$.

Finally, if $\deg(D|_{G_2}) < \mathrm{gon}_1(G_2)$, then $D|_{G_2}$ does not have positive rank. Again, we must be able to pass chips across the ℓ edges. Let D' be the divisor obtained by firing the subset of vertices $V(G_1)$. If E is the sum of a vertex of G_1 and a vertex of G_2, then $D - E$ is equivalent to an effective divisor. It follows that $D'|_{G_1}$ must have positive rank. Thus, $\deg(D'|_{G_1}) \geq \mathrm{gon}_1(G_1)$, so $\deg(D') \geq \ell + \mathrm{gon}_1(G_1) \geq \mathrm{gon}_2(G_1) + \mathrm{gon}_2(G_2)$. □

Corollary 6.2.1. *If $2 \leq a \leq b \leq 2a-1$ and $k \geq 2b-a+3$, then*

$$\mathrm{gon}_2(B^{0,0}_{a,b,k}) = \mathrm{gon}_2(B^{1,0}_{a,b,k}) = 2b+2$$
$$\mathrm{gon}_2(B^{0,1}_{a,b,k}) = \mathrm{gon}_2(B^{1,1}_{a,b,k}) = 2b+1.$$

Proof. This follows directly from Lemma 3.4 and Proposition 6.2. □

We now compute the 3rd gonalities of these graphs.

Theorem 6.3. *Let $2 \leq a \leq b \leq 2a-1$ and let $2b \leq k$. We have the following:*

(1) *if $k \leq 2a+b-1$, then $\mathrm{gon}_3(B^{0,0}_{a,b,k}) = k+b+1$,*
(2) *if $k \leq 2a+b-1$, then $\mathrm{gon}_3(B^{0,1}_{a,b,k}) = k+b$,*
(3) *if $k \leq 2a+b-2$, then $\mathrm{gon}_3(B^{1,0}_{a,b,k}) = k+b+1$, and*
(4) *if $k \leq 2a+b-2$, then $\mathrm{gon}_3(B^{1,1}_{a,b,k}) = k+b$.*

Proof. We prove this in the case of $B^{0,0}_{a,b,k}$. The other graphs are similar. First note that the divisor $k \cdot (v_a) + (b+1) \cdot (w_a)$ has rank at least 3. This shows that $\mathrm{gon}_3(B^{0,0}_{a,b,k}) \leq b+k+1$. For the reverse inequality, let D be a divisor of rank at least 3 on $B^{0,0}_{a,b,k}$, and assume that D is v_a-reduced. By Lemma 4.1 and Theorem 5.5, we have $\deg(D|_L) \geq a+b$. We proceed by cases.

First, if $\deg(D|_R) \geq a+b$, then $\deg(D) \geq 2a+2b \geq k+b+1$.

Next, if $b+1 \leq \deg(D|_R) < a+b$, then $D|_R$ has rank at most 2, so we must be able to pass chips across the k edges. It follows that $\deg(D|_L) \geq k$, so $\deg(D) \geq k+b+1$.

Third, if $a \leq \deg(D|_R) < b+1$, then $D|_R$ has rank at most 1, so we must able to pass chips across the k edges. Let D' be the divisor obtained from D by firing the subset of vertices $V(L)$. If E is an effective divisor with $\deg(E|_L) = 1$ and $\deg(E|_R) = 2$, then $D - E$ is equivalent to an effective divisor. It follows that $D'|_L$ must have rank at least 1. Thus, $\deg(D) \geq 2a+k \geq k+b+1$.

Finally, if $\deg(D|_R) < a$, then $D|_R$ does not have positive rank, so we must be able to pass chips across the k edges. Again, let D' be the divisor obtained from D by firing the subset of vertices $V(L)$. If E is an effective divisor with $\deg(E|_L) = 2$ and $\deg(E|_R) = 1$, then $D - E$ is equivalent to an effective divisor. It follows that $D'|_L$ must have rank at least 2. Thus, $\deg(D) \geq k+b+1$. □

We can use these graphs to identify a large collection of sequences in $\mathcal{G}_3$.

Theorem 6.4. *If $(x,y,z) \in \mathcal{C}_3$ satisfies $y \geq x+2$, $\frac{3}{2}y \leq z+2 \leq x+y$, then $(x,y,z) \in \mathcal{G}_3$.*

Proof. If x and y are both even, consider the graph $B^{0,0}_{a,b,k}$ with $a = \frac{1}{2}x$, $b = \frac{1}{2}(y-2)$, and $k = z - \frac{1}{2}y$. Since $y \geq x+2$, we have $a \leq b$. Since $z \leq x+y-2$, we have $k \leq 2a+b-1$, and since $z \geq \frac{3}{2}y-2$, we have $2b \leq k$. By Theorem 6.3, the first 3 terms of the gonality sequence of $B^{0,0}_{a,b,k}$ are $(2a, 2b+2, k+b+1) = (x,y,z)$.

Similarly, if x is even and y is odd, consider the graph $B^{0,1}_{a,b,k}$ with $a = \frac{1}{2}x$, $b = \frac{1}{2}(y-1)$, and $k = z - \frac{1}{2}(y-1)$. If x is odd and y is even, consider the graph $B^{1,0}_{a,b,k}$ with $a = \frac{1}{2}(x+1)$, $b = \frac{1}{2}(y-2)$, and $k = z - \frac{1}{2}y$. Finally, if x and y are both odd, consider the graph $B^{1,1}_{a,b,k}$ with $a = \frac{1}{2}(x+1)$, $b = \frac{1}{2}(y-1)$, and $k = z - \frac{1}{2}(y-1)$. □

Corollary 6.4.1. *If* $2a+2 \leq b \leq 3a-1$, $b \neq 2a+3$, *then* $(2a, b, 3a+1) \in \mathcal{G}_3$.

Proof. If $b = 2a+2$, then $(2a, 2a+2, 3a+1) \in \mathcal{G}_3$ by Theorem 6.4. If $2a+4 \leq b$ and $m = b-2a-1$, then $m \geq 3$. By Lemma 3.5, the first 3 terms of the gonality sequence of the rook graph $K_3 \square K_m$ are $(2m, 3m-1, 3m) \in \mathcal{G}_3$. If $n = 3a-b+1$, then $n \geq 2$. By Theorem 6.4, we have $(2n, 2n+2, 3n+1) \in \mathcal{G}_3$. Thus, by Theorem 1.1, we have

$$\begin{aligned}(2m, 3m-1, 3m) + (2n, 2n+2, 3n+1) &= (2(n+m), 3m+2n+1, 3(m+n)+1) \\ &= (2a, b, 3a+1) \in \mathcal{G}_3.\end{aligned}$$

□

We now prove Theorem 1.5.

Proof of Theorem 1.5. If $z \geq 2x$, then this follows from Theorem 1.4. For the remainder of the proof, we therefore assume that $z < 2x$.

Next, consider the case where $y = x+2$. By Theorem 6.4, if $\frac{3}{2}x + 1 \leq z \leq 2x$, then $(x, x+2, z) \in \mathcal{G}_3$. For the remainder of the proof, we assume that $y \geq x+3$.

Next, consider the cases where $z \geq 2x-2$. If $z = 2x-1$, then since $\frac{3}{2}x + 2 \leq z$, we have $x \geq 6$, and if $z = 2x-2$, then then since $\frac{3}{2}x+2 \leq z$, we have $x \geq 8$. For $x \leq 7$, the possibilities are: $(x, y, z) = (6, 8, 11), (6, 9, 11), (7, 9, 13), (7, 10, 13), (7, 11, 13)$. All of these except for $(7, 11, 13)$ are in $\mathcal{G}_3$ by Theorem 6.4. To see that $(7, 11, 13) \in \mathcal{G}_3$, note that $(3, 5, 6) \in \mathcal{G}_3$ by Theorem 1.4, and $(4, 6, 7) \in \mathcal{G}_3$ by the third graph in the right column of [1, Table 4.1]. By Theorem 1.1, $(3, 5, 6) + (4, 6, 7) = (7, 11, 13) \in \mathcal{G}_3$. For $8 \leq x \leq y-3$, by Theorem 1.4, we have $(x-6, y-8, 2x-11), (x-6, y-8, 2x-10) \in \mathcal{G}_3$, and by Lemma 3.5, we have $(6, 8, 9) \in \mathcal{G}_3$. Thus, by Theorem 1.1, $(x, y, 2x-2), (x, y, 2x-1) \in \mathcal{G}_3$ as well. For the remainder of the proof, we assume that $z < 2x-2$.

We now consider the cases where $3x \leq y+z$. Let $a = 2z-3x$, $b = 2x-z$, and $c = y+3z-6x+1$. Since $z \geq \frac{3}{2}x+2$, we have $a \geq 2$. Since $y \leq z-2$, we have $c \leq 2a-1$, and since $3x \leq y+z$, we have $c \geq a+1$. It follows from Theorem 1.4 that $(a, c, 2a) \in \mathcal{G}_3$. Similarly, since $z \leq 2x-3$, we have $b \geq 3$. By Lemma 3.5, the first 3 terms of the gonality sequence of the rook graph $K_3 \square K_b$ are $(2b, 3b-1, 3b) \in \mathcal{G}_3$. Thus, by Theorem 1.1, we have

$$\begin{aligned}(a, c, 2a) + (2b, 3b-1, 3b) &= (a+2b, 3b+c-1, 2a+3b) \\ &= (x, y, z) \in \mathcal{G}_3.\end{aligned}$$

We now consider the remaining cases. Since $y \geq x+3$ and $3x \geq y+z+1$, we see that $z \leq 2x-4$. Similarly, since $z \geq \frac{3}{2}x+2$, we have $y \leq 3x-z-1 \leq \frac{3}{2}x-3 \leq z-5$. If $a = 2z-3x-2$, then $a \geq 2$, so by Theorem 1.4, we have $(a, c, 2a) \in \mathcal{G}_3$ for all c in the range $a+1 \le c \le 2a-1$. If $b = 2x-z+1$, then since $z \le 2x-4$, we have $b \ge 5$. Thus, by Corollary 6.4.1, $(2b, d, 3b+1) \in \mathcal{G}_3$ for all d in the range $2b+2 \le d \le 3b-1$, $d \ne 2b+3$. If $a > 2$, we can choose c and d so that $c+d$ can take any integer value in the range

$$x+3 = (a+1)+(2b+2) \le c+d \le (2a-1)+(3b-1) = z-3.$$

If $a = 2$, then c must be 3, and we cannot choose d so that $c+d = 2b+3$. However, in this case we have $y = x+4$, and the sequence $(x, x+4, z)$ is in $\mathcal{G}_3$ by Theorem 1.4. Otherwise, since $x+3 \le y \le z-5$, we may choose c and d so that $c+d = y$. Thus, by Theorem 1.1, we have

$$\begin{aligned}(a, c, 2a) + (2b, d, 3b+1) &= (a+2b, c+d, 2a+3b+1)\\ &= (x, y, z) \in \mathcal{G}_3.\end{aligned}$$

□

7 Gonality Sequences of Algebraic Curves

By Theorem 1.2, the semigroup $\mathcal{G}_r$ is not finitely generated for any $r \ge 2$. Indeed, if $\vec{x} \in \mathcal{G}_r$ and $x_{i+1} = x_i + 1$ for some i, then $\vec{x}$ is irreducible. As we have seen in Theorem 1.1, if $\vec{x} \in \mathcal{G}_r$ is reducible, then there exists graphs of arbitrarily large genus with gonality sequence $\vec{x}$. Irreducible elements of $\mathcal{G}_r$ are more mysterious. In this final section, we study the gonality sequences of algebraic curves C such that $\mathrm{gon}_r(C) = \mathrm{gon}_{r-1}(C) + 1$ for some r. These curves have interesting properties, and we ask whether graphs with the same gonality sequence exhibt the same properties.

Lemma 7.1. *Let C be a smooth curve and let r be a positive integer. If $\mathrm{gon}_r(C) = \mathrm{gon}_{r-1}(C) + 1$, then C is isomorphic to a smooth curve of degree $\mathrm{gon}_r(C)$ in $\mathbb{P}^r$.*

Proof. Let $\mathcal{L}$ be a line bundle on C of rank r and degree $\mathrm{gon}_r(C)$. Let $\varphi_{\mathcal{L}} \colon C \to \mathbb{P}^r$ be the map given by the complete linear series of $\mathcal{L}$, let $B = \varphi_{\mathcal{L}}(C)$ be the image, let $\nu \colon \widetilde{B} \to B$ be the normalization of B, and let $\varphi \colon C \to \widetilde{B}$ be the induced map.

We first show that the map φ has degree 1, and is therefore an isomorphism. For any point $p \in \widetilde{B}$, the line bundle $\nu^*\mathcal{O}_B(1)(-p)$ has rank at least $r-1$ on $\widetilde{B}$. Thus, $\varphi^*\nu^*\mathcal{O}_B(1)(-p)$ has rank at least $r-1$ on C. Note that $\mathcal{L} = \varphi^*\nu^*\mathcal{O}_B(1)$, and $\deg \varphi^*(p) = \deg(\varphi)$, hence

$$\deg(\varphi^*\nu^*\mathcal{O}_B(1)(-p)) = \deg(\mathcal{L}) - \deg(\varphi) = \mathrm{gon}_r(C) - \deg(\varphi).$$

Since $\mathrm{gon}_{r-1}(C) = \mathrm{gon}_r(C) - 1$, it follows that $\deg(\varphi) = 1$.

We now show that the map ν is an isomorphism. If not, then B is singular, and projection from a singular point yields a nondegenerate map to $\mathbb{P}^{r-1}$ of degree at most $\mathrm{gon}_r(C) - 2$. Since $\mathrm{gon}_{r-1}(C) = \mathrm{gon}_r(C) - 1$, this is again impossible. It follows that the map $\varphi_{\mathcal{L}}$ is an isomoprhism onto its image. □

Lemma 7.1 has several consequences.

Lemma 7.2. *Let C be a curve with the property that* $\operatorname{gon}_2(C) = \operatorname{gon}_1(C) + 1$*. Then the genus of C is* $g = \binom{\operatorname{gon}_1(C)}{2}$ *and, for any $r < g$, we have*

$$\operatorname{gon}_r(C) = k \cdot \operatorname{gon}_2(C) - h,$$

where k and h are the uniquely determined integers with $1 \leq k \leq \operatorname{gon}_2(C) - 3$*,* $0 \leq h \leq k$*, such that* $r = \frac{k(k+3)}{2} - h$*.*

In particular, if $\operatorname{gon}_1(C) \geq 2$*, then* $\operatorname{gon}_3(C) = 2 \cdot \operatorname{gon}_1(C)$*.*

Proof. By Lemma 7.1, C is isomorphic to a smooth plane curve of degree $\operatorname{gon}_2(C)$. The genus of such a curve is $\binom{\operatorname{gon}_1(C)}{2}$, and its gonality sequence is computed in [11, 9]. □

Lemma 7.3. *Let C be a curve with the property that* $\operatorname{gon}_3(C) = \operatorname{gon}_2(C) + 1$*, and let* $m = \lceil \frac{1}{2} \operatorname{gon}_2(C) \rceil$*. Then the genus of C is at most* $m \cdot \operatorname{gon}_3(C) - m(m+2)$*. Moreover, if equality holds, then*

$$\operatorname{gon}_1(C) = \left\lceil \frac{1}{2}(\operatorname{gon}_3(C) - 1) \right\rceil.$$

Proof. By Lemma 7.1, C is isomorphic to a smooth space curve of degree $\operatorname{gon}_3(C)$. By [8, Theorem IV.6.7], the genus of C is at most $m \cdot \operatorname{gon}_3(C) - m(m+2)$, and if equality holds, then C is contained in a quadric surface. A tangent plane to the quadric meets it in two lines, which meet the curve C in $\operatorname{gon}_3(C)$ points. It follows that one of these two lines must meet C in at least $\frac{1}{2} \operatorname{gon}_3(C)$ points, and projection from this line yields a nondegenerate map to $\mathbb{P}^1$ of degree at most $\frac{1}{2} \operatorname{gon}_3(C)$. Thus,

$$\operatorname{gon}_1(C) \leq \frac{1}{2} \operatorname{gon}_3(C).$$

On the other hand, we have

$$\operatorname{gon}_1(C) \geq \frac{1}{2} \operatorname{gon}_2(C) = \frac{1}{2}(\operatorname{gon}_3(C) - 1),$$

and the result follows. □

Question 7.4. *Let G be a graph with the property that* $\operatorname{gon}_3(G) = \operatorname{gon}_2(G) + 1$*, and let* $m = \lceil \frac{1}{2} \operatorname{gon}_2(G) \rceil$*.*

(1) *Must the genus of G be at most* $m \cdot \operatorname{gon}_3(G) - m(m+2)$*?*
(2) *If equality holds, is it true that*

$$\operatorname{gon}_1(C) = \left\lceil \frac{1}{2}(\operatorname{gon}_3(C) - 1) \right\rceil?$$

Lemma 7.5. *Let C be a curve. If* $\operatorname{gon}_3(C) \leq \operatorname{gon}_1(C) + 3$*, then* $\operatorname{gon}_1(C) \leq 6$ *and* $\operatorname{gon}_1(C) \neq 5$*.*

Proof. Suppose that $\mathrm{gon}_3(C) \leq \mathrm{gon}_1(C)+3$. Then either $\mathrm{gon}_2(C) = \mathrm{gon}_1(C)+1$ or $\mathrm{gon}_3(C) = \mathrm{gon}_2(C)+1$. If $\mathrm{gon}_2(C) = \mathrm{gon}_1(C)+1$, then by Lemma 7.2,

$$2\,\mathrm{gon}_1(C) = \mathrm{gon}_3(C) \leq \mathrm{gon}_1(C)+3,$$

hence $\mathrm{gon}_1(C) \leq 3$.

If $\mathrm{gon}_3(C) = \mathrm{gon}_2(C)+1$, then by Lemma 7.1, C is isomorphic to a smooth space curve of degree $\mathrm{gon}_3(C)$. By [10, Proposition 4.1], if $\mathrm{gon}_3(C) \geq 10$, then $\mathrm{gon}_3(C) \geq \mathrm{gon}_1(C)+4$, hence we must have $\mathrm{gon}_3(C) \leq 9$.

It remains to show that, if $\mathrm{gon}_3(C) = 8$, then $\mathrm{gon}_1(C) \leq 4$. Since every curve of genus 6 or less has gonality at most 4, we may assume that C has genus at least 7. By Lemma 7.3, if $\mathrm{gon}_3(C) = 8$, then C has genus at most 9, and if it is equal to 9, then $\mathrm{gon}_1(C) \leq 4$. If C has genus 8, then $\mathcal{O}_C(2)$ has degree $16 > 2 \cdot 8 - 2$, hence $h^0(C, \mathcal{O}_C(2)) = 9$. It follows that C is contained in a quadric surface, and again, $\mathrm{gon}_1(C) \leq \frac{1}{2}\mathrm{gon}_3(C) = 4$. Finally, if C has genus 7, then by Riemann-Roch, $K_C \otimes \mathcal{O}_C(-1)$ has degree 4 and rank 1, hence $\mathrm{gon}_1(C) \leq 4$. ☐

Question 7.6. *Let G be a graph. If $\mathrm{gon}_1(G) = 5$ or $\mathrm{gon}_1(G) \geq 7$, does it follow that $\mathrm{gon}_3(G) \geq \mathrm{gon}_1(G)+4$?*

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