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On prism-hamiltonian bipartite graphs

Simon Špacapan

University of Maribor
Maribor, Slovenia
simon.spacapan@um.si

Peter Horak

University of Washington
Tacoma, WA, U.S.A.
horak@uw.edu

Abstract

A graph G is prism-hamiltonian if the prism over G, the Cartesian product of G with the complete graph K_2, is hamiltonian. In this article a characterization of prism-hamiltonian graphs is provided. Kaiser et al. conjectured that every graph with sufficiently high toughness is prism-hamiltonian. We prove a special case of this conjecture, namely that every 1-tough bipartite graph which has no adjacent vertices of degree at least four is prism-hamiltonian.

1 Introduction

A graph G is hamiltonian if there exists a spanning cycle in G. Hamiltonicity of graphs is one of the oldest and most intensively studied areas in graph theory. As a result there are literally thousands of papers on the topic. At the same time there are famous long-standing conjectures that have attracted the interest of many scholars. In this paper we focus on an extension of Chvátal's conjecture.

It is well known that the problem of whether a graph is hamiltonian is NP-complete. Therefore, it is of high interest at least to show that a graph is "close" to being hamiltonian. There are several ways to measure distance of a graph from being hamiltonian. One of them has been provided by Jackson and Wormald [7]. In their infinite hierarchy the graphs closest to hamiltonian are those which have hamiltonian paths, followed by graphs with hamiltonian prisms. Formally, a graph G is called prism-hamiltonian if its prism, the Cartesian product of G with the complete graph K_2, is hamiltonian. A typical example when it was first shown that a graph is close to being hamiltonian is the class of Middle-Levels-Graphs. In 1983 Havel raised a famous conjecture that these graphs are hamiltonian; in 2005, Horak

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et al. [6] proved that they are prism-hamiltonian, and finally, in 2016, Mütze [11] proved Havel's conjecture.

In [12] Paulraja gave a characterization of prism-hamiltonian graphs. The characterization is rather complex and uses certain edge colorings of graphs. As one of two main results of this paper we provide a simpler description of prism-hamiltonian graphs, Theorem 2.1, which we later apply in the proof of Proposition 2.2.

The notion of toughness of a graph was introduced by Chvátal, who used the concept to raise one of the central conjectures in the area of hamiltonian graphs. We focus on an extension of Chvatal's conjecture to prism-hamiltonian graphs.

The *toughness* of G, denoted $\tau(G)$, is given by

$$\tau(G) = \min\{|S|/c(S)\},$$

where $c(S)$ denotes number of connected components of $G - S$ and the minimum runs over all vertex cuts S of G.

Conjecture 1.1 *[4] There exists a $k \in \mathbb{R}$, such that every graph G with $\tau(G) > k$ is hamiltonian.*

It was shown that $k \geq 9/4$, as for every $\epsilon > 0$ there exists a non-hamiltonian graph with $\tau(G) = 9/4 - \epsilon$. Toughness of graphs in relation to their hamiltonicity properties is discussed in detail in the survey paper [1].

Kaiser et al. raised an extension of Chvátal's conjecture:

Conjecture 1.2 *[8] There exists a $k \in \mathbb{R}$, such that every graph G with $\tau(G) > k$ is prism-hamiltonian.*

It was shown that $k \geq 9/8$ in the relaxed conjecture, see [8]. More precisely, for every $\epsilon > 0$ there exists a non-prism-hamiltonian graph G with $\tau(G) = 9/8 - \epsilon$.

In this paper the focus will be restricted to prism-hamiltonicity of bipartite graphs. Let G be a bipartite graph with bipartition $A, B, |A| \leq |B|$ of its vertex set. Then $G - A$ consists of $|B|$ isolated vertices; hence, for the toughness of G, we get

$$0 \leq \tau(G) \leq 1.$$

Since $\tau(G) < \frac{1}{2}$ implies $\tau(G \square K_2) < 1$, it follows that $\tau(G) \geq \frac{1}{2}$ is a necessary condition for a (bipartite) graph G to be prism-hamiltonian. We will prove that in fact it has to be $\tau(G) > 1/2$.

On a positive note, we believe that the following restriction of Conjecture 1.2 is true.

Conjecture 1.3 *Every 1-tough bipartite graph is prism-hamiltonian.*

As a support for the conjecture we will prove the following:

Theorem 1.4 *Every 1-tough bipartite graph which has no adjacent vertices of degree at least 4 is prism-hamiltonian.*

A result of a similar nature has been proved by Paulraja [12], and separately by Čada et al. [3]. Namely, they proved, in terms of good cacti (see the definition of a good cactus below), that every 2-connected bipartite graph of maximum degree at most 3 is prism-hamiltonian. This implies that more than 1/2-tough bipartite graphs of maximum degree at most 3 are prism-hamiltonian.

A *good cactus* is a connected graph G such that every block of G is either a K_2 or an even cycle such that every vertex is contained in at most two blocks of G. The theorem below was proved in [5] (see also [13]) but has been implicitly applied in many earlier papers. It appears to be the main method by which prism-hamiltonicity of graphs is established.

Theorem 1.5 *Every graph that has a good cactus as a spanning subgraph is prism-hamiltonian.*

This theorem is (implicitly) applied in [12] and [3] to prove prism-hamiltonicity of 3-connected cubic graphs. It was also applied to the class of 3-connected planar graphs of minimum degree 4 in [13], to 1/2-tough P_4-free graphs in [5], and to bipartite 3-connected planar graphs in [2].

One might be tempted to think that all prism-hamiltonian graphs have a spanning good cactus as a subgraph. In [10] the author shows that this is not true by constructing an infinite family of 3-connected planar prism-hamiltonian graphs with no spanning good cactus (however all graphs given in [10] are non-biparitite). In this paper we also give an example of bipartite prism-hamiltonian graph with no spanning good cactus (see Proposition 2.2).

2 Results

Let $G = (V(G), E(G))$ be a graph and M a set of edges. We define the following graphs $G - M = (V(G), E(G) \setminus M)$ and $G \cup M = (V(G), E(G) \cup M)$. Let K_2 be the complete graph on two vertices and $V(K_2) = \{b, w\}$. We denote the Cartesian product of graphs G and H by $G \square H$; in particular $G \square K_2$ is the *prism* over G. The edge connecting (x, w) and (x, b) in $G \square K_2$ is called the *vertical edge* at x.

Let $P = x_1, \ldots, x_n$ be a path. P is *even* (respectively *odd*) if n is even (respectively odd). An *alternating path* in $P \square K_2$ is a path that contains exactly one vertex in $M = \{(x_1, b), (x_1, w)\}$, exactly one vertex in $N = \{(x_n, b), (x_n, w)\}$, and all vertices of $P \square K_2 - (M \cup N)$.

In [12] the author gives the definition of a SEEP-subgraph of a graph G. The definition is rather long so we skip it here; however, the author proves that prism-hamiltonian graphs are precisely those graphs that have a SEEP-subgraph. We give another characterization of prism-hamiltonian graphs, namely that a graph G is prism-hamiltonian if and only if it has a spanning subgraph in class of graphs $\mathcal{C}$, which we define below. It follows from this that G has a SEEP-subgraph (which is by the definition a spanning subgraph of G) precisely when it has a spanning subgraph in $\mathcal{C}$, and therefore the class $\mathcal{C}$ is just another description of SEEP-subgraphs.

Let $\mathcal{C}$ be the class of graphs containing P_2, and graphs obtained from even cycles by the following construction.

1. Let C be an even cycle.
2. Color an even (possibly 0) number of vertices of C by black and white so that one half of vertices are black and the other half is white (we do not require that the coloring is proper nor that all vertices are colored) so that every path P in C whose internal vertices are uncolored and endvertices are colored by the same (by distinct) color(s) is even (odd).
3. Identify every white vertex with a black vertex (bijectively).

Note that in step 3 double edges might appear, and in this case we identify any two edges with the same set of endvertices. An example of a coloring from step 2 is given in Figure 2, where the uncolored vertices are gray, and colored vertices are black or white. As we shall see in the proof below, the uncolored vertices of C correspond to vertices x of G such that the Hamilton cycle in $G\Box K_2$ uses vertical edge in x.

Theorem 2.1 *A graph G is prism-hamiltonian if and only if it has a spanning subgraph in $\mathcal{C}$.*

Proof. We prove first that every graph in $\mathcal{C}$ is prism-hamiltonian. This is clearly true for P_2 and all even cycles. Suppose that G is a graph in class $\mathcal{C}$, and let C' be the cycle obtained after step 2 of the definition of $\mathcal{C}$. Note that C' has a coloring of a subset of its vertices as described in step 2, and we fix one such coloring and refer to it in the sequel.

We define a cycle C_0 in $C'\Box K_2$ as follows. For every path $P = u_1, \ldots, u_n$ in C' such that every internal vertex of P is uncolored and its endvertices are colored we do the following:

(i) if u_1 is white and u_n is black, let C_0 contain all edges of the alternating path from (u_1, w) to (u_n, b) in $P\Box K_2$, and

(ii) if both endvertices of P are white (respectively black) then let C_0 contain all edges of the alternating path from (u_1, w) to (u_n, w) (respectively (u_1, b) to (u_n, b)) in $P\Box K_2$.

Since such paths P are pairwise internally disjoint and they cover $V(C')$ this defines a cycle C_0 in $C'\Box K_2$. Moreover, C_0 contains all vertices of $C'\Box K_2$, except (x, w) for black x, and (x, b) for white x. When we identify black and white vertices (bijectively) in step 3 of the construction we get a Hamilton cycle in $G\Box K_2$.

Suppose now that G is a prism-hamiltonian graph, and let C be a Hamilton cycle in $G\Box K_2$. First we contract all vertical edges of C, and we obtain a cycle C' (if C is not a 4-cycle). If C is a 4-cycle, then $G = P_2$ which is a graph in class $\mathcal{C}$. So assume C' is a cycle, and note that C' is an even cycle because C has an even number of vertical edges. Now we color vertices of C' which were not identified during contractions: vertices (x, b) by black and (x, w) by white (vertices that were identified during contractions remain uncolored). If $P = u_1, \ldots, u_n$ is a path in C' such that all internal vertices of P are uncolored, and u_1, u_n received the same color in this coloring, then n is even (due to the fact that uncolored vertices of C' correspond to vertices of $G \square K_2$ in which C uses the vertical edge); otherwise if u_1 and u_n have distinct colors then n is odd. Then C' together with the coloring defined above satisfies the conditions given in step 2 of the construction. Clearly, when we identify each pair of black and white vertices of C' with equal first coordinate, we obtain a spanning subgraph of G (and if there are no black or white vertices, then C' is a cycle that spans G). $\square$

Now we focus our attention to prism-hamiltonicity of bipartite graphs. As mentioned in the introduction, every graph that has a good cactus as a spanning subgraph is prism-hamiltonian. We show that having a good cactus is not equivalent to being prism-hamiltonian, and in particular this is not equivalent for the class of bipartite graphs; cf. [10] as well.

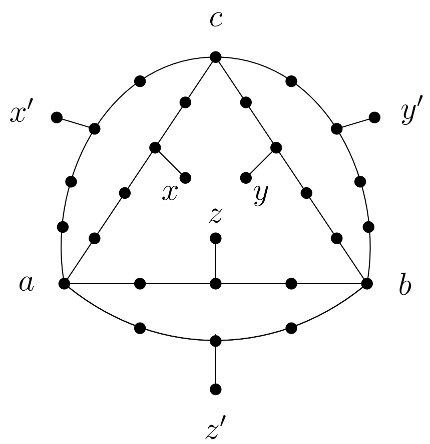

Figure 1: An example of a bipartite prism-hamiltonian graph with no spanning good cactus.

Proposition 2.2 *There exist infinitely many bipartite prism-hamiltonian graphs which have no spanning good cactus as a subgraph.*

Proof. We claim that the graph G in Figure 1 is a bipartite prism-hamiltonian graph with no spanning good cactus. Observe that G is obtained from an even cycle by identification of vertices as indicated in Figure 2 (equally named vertices are identified). Hence G is in class $\mathcal{C}$ and therefore, by Theorem 2.1, G is prism-hamiltonian.

To see that G has no spanning good cactus we observe that any spanning good cactus K in G must have a cycle. Now both cases: (1) K has a cycle containing a, b and c, and (2) K contains the cycle containing a, c and/or the cycle containing a, b, lead to a contradiction (that K spans G).

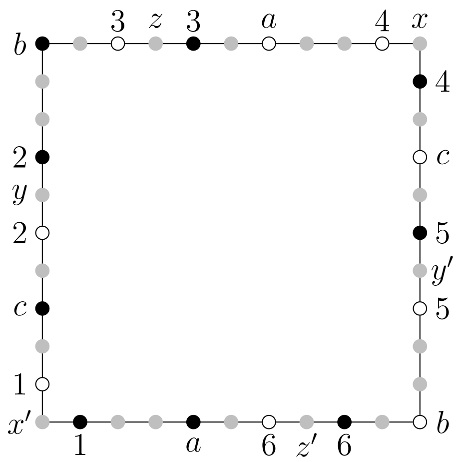

Figure 2: After identification of black and white vertices we obtain the graph in Figure 1.

Clearly, if x (or any pendant vertex) is replaced with a path, the proof would require only a trivial adjustment; this gives an infinite set of examples that prove the proposition. □

As mentioned in the introduction, $\tau(G) \geq \frac{1}{2}$ is a necessary condition for a bipartite graph G to be prism-hamiltonian, moreover we prove next that $\tau(G) \geq 1/2$ is not a sufficient condition. Consider the graph G shown in Figure 3. It is easy to see that $\tau(G) = 1/2$, and we claim that G is not prism-hamiltonian. To see this observe that prism-hamiltonicity of G implies existence of a Hamilton cycle C in the prism over $G - x'$ such that C uses the vertical edge at x. But then C uses the vertical edge at one of the neighbors of x, and therefore at least two vertices of degree 2 in the prism over $G - x'$ remain uncovered by C, which is a contradiction. It follows that G is not prism-hamiltonian, as claimed.

Now we prove Theorem 1.4. First we state the following lemma which is a key ingredient of our proof.

Lemma 2.3 *[9] Every* 1*-tough bipartite graph has 2-factor.*

Theorem 1.4 *Every* 1*-tough bipartite graph which has no adjacent vertices of degree at least* 4 *is prism-hamiltonian.*

Proof of Theorem 1.4. Let G be a 1-tough bipartite graph with no adjacent vertices of degree at least four. We shall prove that G is prism-hamiltonian. By Lemma 2.3, G has a 2-factor H. We claim that there exists a matching $M \subseteq E(G) \setminus E(H)$, such that $H \cup M$ is a connected graph (and hence G has a spanning good cactus).

Let $S = E(G) \setminus E(H)$. Since no two vertices of degree more than three are adjacent in G, $R = (V(G), S)$ is a star forest. If $e \in S$ then we denote the vertex incident to e of degree more than one in R (if any) by $v(e)$; we call it the central vertex (of the star) belonging to e. We call any star of R which is isomorphic to K_2

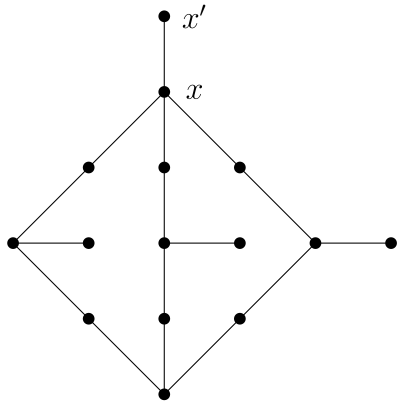

Figure 3: An example of a 1/2-tough bipartite graph which is not prism-hamiltonian.

a trivial star. Let $A \subseteq S$ be the set of edges that belong to a trivial star of R. Define $H' = H \cup A$.

An auxiliary multigraph Q is formed by contracting in G each component of H' into a single vertex and discarding loops that appear; after contractions G becomes Q and edges of R now become edges of Q, unless both endvertices of the edge are in the same component of H'. If there is more than one edge of S connecting two components of H' then there are in Q parallel edges between the two vertices representing these two components. To obtain a required matching M we have to prove that there is a spanning tree T in Q, such that T contains at most one edge of each star of R.

Note that any edge of Q belongs to a nontrivial star of R and hence it has a central vertex. Moreover, since no two vertices of degree more than 3 are adjacent in G, we find that the set of central vertices belonging to edges of Q is an independent set in G. In particular, each component of H' has a vertex which is not a central vertex of an edge of Q.

Let T be a maximal tree in Q such that T contains at most one edge of each star of R. If $V(T) = V(Q)$ we are done, so assume that $V(T) \neq V(Q)$. We will color the edges of Q in the following way: let X be the set of edges of Q belonging to a fixed star of R, then all edges of X will be colored

1. Green: if an edge of X is in $E(T)$, and an edge of X is incident to $V(Q) \setminus V(T)$.
2. Blue: if an edge of X is in $E(T)$, and no edge of X is incident to $V(Q) \setminus V(T)$.
3. Red: if no edge of X is in $E(T)$.

By maximality of T there is no red edge with one endvertex in $V(T)$ and the other in $V(Q) \setminus V(T)$. All edges of T are either green or blue, and green edges are precisely those which have an endvertex adjacent to $V(Q) \setminus V(T)$. Suppose first that there are no red edges with both endvertices in $V(T)$. We note that by this assumption, and by maximality of T, every edge of Q incident to a vertex of T is green or blue. Then removing all green and blue edges of Q results in at least $|V(T)|+1$ components in Q. Equivalently, removing the central vertex of every edge of T results in (at least) $|V(T)|+1$ components in G (recall that each component of H' has a vertex which is not a central vertex of an edge of T, hence no component of H' is entirely removed). We note that removal of central vertices results in removing all edges of G that were represented in Q by green and blue edges. Since there are $|V(T)|-1$ edges in T, there are exactly $|V(T)|-1$ central vertices. It follows that G is at most $(|V(T)|-1)/(|V(T)|+1)$-tough.

Therefore there is at least one red edge with both endvertices in $V(T)$. Let U be the set of green edges of T.

Claim 1: There is no red edge with endvertices in distinct components of $T-U$.
Proof: Suppose to the contrary, that a red edge e connects distinct components of $T-U$. Then there is a green edge f, such that the endvertices of e are contained in distinct components of $T-f$. Let T' be the tree obtained from T by deleting f and adding edges e and xy, where $x=v(f)$ and $y \in V(Q) \setminus V(T)$. Then T' is a larger tree containing at most one edge of each star of R. This contradicts the maximality of T. □

Let $N_0 = U$. For $i \in \mathbb{N}$ we inductively define M_i', M_i and N_i as follows (we only need N_{i-1} to define these sets). Let M_i' be the set of blue edges $xy \notin E(T)$, such that x and y are contained in distinct components of $T-N_{i-1}$. Let

$$M_i = \{e \in E(T) \,|\, v(e) = v(e') \text{ for some } e' \in M_i'\}$$

and define $N_i = M_i \cup N_{i-1}$. Note that M_i is the set of blue edges e in T that have an adjacent edge e', such that e and e' belong to the same star of R (i.e. $v(e)=v(e')$), and e' connects two distinct components of $T-N_{i-1}$ (i.e. $e' \in M_i'$). In the sequel we generalize Claim 1.

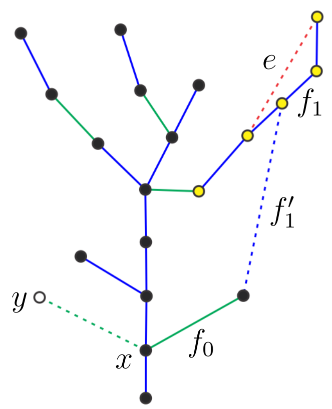

Figure 4: Edges not in T are marked by dashed lines. The only vertex not in T (the vertex y) is white. Vertices of C are yellow.

Claim 2: There is no red edge with endvertices in distinct components of $T-N_1$.
Proof: Suppose to the contrary, that a red edge e connects distinct components of

$T - N_1$. By Claim 1 both endvertices of e are contained in the same component of $T - U$, call it C (see Figure 4). Then there is a blue edge $f_1 \in N_1 \setminus U$, such that the endvertices of e are contained in distinct components of $C - f_1$. Since $f_1 \in N_1 \setminus U = M_1$, there is an edge $f_1' \in M_1'$ such that $v(f_1') = v(f_1)$. Since $f_1' \in M_1'$, there is a (green) edge $f_0 \in N_0 = U$, such that the endvertices of f_1' are contained in distinct components of $T - f_0$. Let T' be the tree obtained from T by deleting f_0, f_1 and adding edges xy, f_1', e, and vertex y, where $x = v(f_0)$ and $y \in V(Q) \setminus V(T)$. Then T' is a larger tree containing at most one edge of each star of R. This contradicts the maximality of T. □

Claim 2 has the following generalization (we skip the proof of Claim 3 since it is similar to the proof of Claim 2).

Claim 3: For every natural number k there is no red edge with endvertices in distinct components of $T - N_k$.

Let k be the minimum integer such that $N_k = N_{k+1}$. It follows from the definition of sets N_i, that then $M_{k+1} \subseteq N_k$. In other words, for every blue edge $e' \in M_{k+1}'$ with endvertices in distinct components of $T - N_k$, there is an edge $e \in N_k$ such that $v(e) = v(e')$. Moreover, by Claim 3, there are no red edges with endvertices in distinct components of $T - N_k$.

Let $Y' \subseteq E(Q)$ be the set of edges incident to a vertex in $Y = \{v(e) \,|\, e \in N_k\}$. We recall that $U \subseteq N_k$. It follows from the above discussion that Y' contains all green and blue edges between distinct components of $T - N_k$, and since there are no red edges with endvertices in distinct components of $T - N_k$, we find that $Q - Y$ has at least $|N_k| + 2$ components (there are $|N_k| + 1$ components of $T - N_k$ and at least one component contained in $V(Q) \setminus V(T)$).

Now we return back to graph G. Note that removing Y' from Q is the same as removing Y from G (in terms of connected components that appear). In each case we get at least $|N_k| + 2$ components. Hence G is at most $|N_k|/(|N_k| + 2)$-tough, a contradiction. This proves $V(T) = V(Q)$, and completes the proof of the theorem. □

We finish our paper with open problems and questions. The main problem that remains open is to solve Conjecture 1.3. The example given in Figure 3 is an example of a non-prism-hamiltonian bipartite graph which is 1/2-tough but not 2-connected. The question below asks if there exist 2-connected graphs with these properties.

Question 2.4 *Is there a 2-connected bipartite graph G with $\tau(G) \geq 1/2$ which is not prism-hamiltonian?*

The following problem is a special case of Conjecture 1.3.

Problem 2.5 *Prove that every 1-tough regular bipartite graph and every 1-tough bipartite graph of maximum degree 4 is prism-hamiltonian.*

Acknowledgements

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