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Critical equimatchable graphs

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Abstract

A graph G is equimatchable if every maximal matching of G has the same cardinality. In this paper, we investigate equimatchable graphs such that the removal of any edge creates a graph that is not equimatchable, called edge-critical equimatchable graphs (ECE-graphs). We show that apart from two simple cases, namely bipartite ECE-graphs and even cliques, all ECE-graphs are 2-connected factor-critical. Accordingly, we give a characterization of factor-critical ECE-graphs with connectivity 2. Our result provides a partial answer to an open question posed by Levit and Mandrescu [Eur. J. Comb. 20 (2019), 261–272] on the characterization of well-covered graphs with no shedding vertex. We also introduce equimatchable graphs such that the removal of any vertex creates a graph that is not equimatchable, called vertex-critical equimatchable graphs (VCE-graphs). To conclude, we clarify the relationship between various subclasses of equimatchable graphs (including ECE-graphs and VCE-graphs) and discuss the properties of factor-critical ECE-graphs with connectivity at least 3.

1 Introduction

Matching theory is one of the fundamental fields that encompasses both practical and theoretical challenges [14]. Given a graph G, a *matching* is a set of edges of G having pairwise no common endvertices. It is well-known that given a graph, a

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matching of maximum size can be efficiently computed whereas finding an inclusion-wise maximal matching of minimum cardinality is an NP-complete problem even in several restricted cases [22].

A graph G is called *equimatchable* if every maximal matching of G has the same cardinality. The structure of equimatchable graphs has been widely studied in the literature (see for instance [1, 2, 4, 5, 8, 9, 11, 15, 16, 17]). The counterpart of equimatchable graphs for independent sets are the well-covered graphs: a graph is *well-covered* if all its maximal independent sets have the same size. Well-covered graphs have been first introduced in [18] and studied extensively since then. Given a graph G, the line graph $L(G)$ is the graph obtained by representing every edge of G with a vertex in $L(G)$ and making two vertices of $L(G)$ adjacent if the edges of G represented by these vertices have a common endvertex. It follows that a graph G is equimatchable if and only if its line graph $L(G)$ is well-covered. Motivated by this link and the related research on well-covered graphs, we investigate in this paper the criticality of equimatchable graphs, which has been posed as an open question on well-covered graphs in [13] and reformulated in [6] in terms of equimatchable graphs.

A graph is 1-*well-covered* if it is well-covered and remains well-covered upon removal of any vertex [19]. Recently, the stability of being equimatchable with respect to edge removals has been studied in [6]. An equimatchable graph G is called *edge-stable* if the graph obtained by the removal of any edge of G remains equimatchable. Edge-stable equimatchable graphs are denoted ESE-graphs as a shorthand. So, a graph is edge-stable equimatchable if and only if its line graph is 1-well-covered. A *shedding vertex* is a vertex x such that for every independent set I in the graph obtained by removing the neighborhood of x and the vertex x, there exists some neighbor y of x such that $I \cup \{y\}$ is independent. Shedding vertices are strongly related to the combinatorial topology of independence complexes of graphs [13, 21], and play an important role in identifying vertex decomposable graphs [3]. In [13], Levit and Mandrescu showed that all vertices of a well-covered graph G without isolated vertices are shedding if and only if G is 1-well-covered, and posed their characterization as an open problem. A partial answer has been given in [6] by showing that the characterization of edge-stable equimatchable graphs (with no component isomorphic to an edge) provides a characterization for well-covered line graphs such that all vertices are shedding. In the same paper [13], finding all well-covered graphs having no shedding vertex has been posed as an open problem. In terms of equimatchable graphs, this corresponds to the notion of criticality which is the opposite of stability. In this paper, we investigate edge-critical equimatchable graphs which correspond to well-covered line graphs with no shedding vertex; we provide their characterization in some cases and shed light to their structure from various perspectives.

For an equimatchable graph G, we say that $e \in E(G)$ is a *critical edge* if the removal of e from G makes it non-equimatchable. Note that if an equimatchable graph G is not edge-stable, then it has a critical edge. A graph G is called *edge-critical equimatchable*, denoted ECE for short, if G is equimatchable and every $e \in E(G)$ is critical. We note that ECE-graphs can be obtained from any equimatchable graph by recursively removing non-critical edges. By definition of ECE-graphs, a graph G

with no component isomorphic to an edge is ECE if and only if $L(G)$ is well-covered and has no shedding vertex. Thus, the complete characterization of ECE-graphs would clarify the structure and the recognition of well-covered line graphs with no shedding vertex.

At the expense of losing the link with 1-well-covered graphs, one can also extend the notion of criticality of equimatchable graphs to vertex removals. An equimatchable graph G is called *vertex-critical* if G looses its equimatchability by the removal of any vertex. We denote vertex-critical equimatchable graphs shortly by VCE.

We start with formal definitions and frequently used results on equimatchability in Section 2. We proceed with the characterization of VCE-graphs in Section 3. Our findings point out that apart from an easily detectable simple structure, VCE-graphs coincide with factor-critical equimatchable graphs and that they contain all factor-critical ECE-graphs. This motivates once again the study of ECE-graphs, which we start in Section 4. We first show that ECE-graphs are either 2-connected factor-critical or 2-connected bipartite or even cliques. Noting that 2-connected bipartite ECE-graphs admit a simple characterization, we focus on factor-critical ECE-graphs. We give a complete characterization of ECE-graphs with connectivity 2. In Section 5, we provide a comparison of various subclasses of equimatchable graphs in terms of inclusions and intersections; ECE-graphs, VCE-graphs, edge-stable equimatchable graphs and factor-critical equimatchable graphs are illustrated in Figure 3. We conclude in Section 6 with a discussion on factor-critical ECE-graphs with connectivity at least 3.

2 Definitions and preliminaries

Given a graph $G = (V, E)$ and a subset of vertices I, $G[I]$ denotes the subgraph of G induced by I, and $G \setminus I = G[V \setminus I]$. If I is a singleton $\{v\}$, we denote $G \setminus I$ by $G - v$. We also denote by $G \setminus e$ the graph $G(V, E \setminus \{e\})$. For a subset I of vertices, we say that I is *complete* to another subset I' of vertices (or by abuse of notation, to a subgraph H) if all vertices of I are adjacent to all vertices of I' (respectively H). K_r is a clique on r vertices. For a vertex v, the neighborhood of v in a subgraph H is denoted by $N_H(v)$. We omit the subscript H when it is clear from the context. For a subset $V' \subseteq V$, $N(V')$ is the union of the neighborhoods of the vertices in V'. The *degree* of a vertex v is the number of its neighbors, denoted by $d(v)$. For a graph G, $\Delta(G)$ denotes the maximum degree of a vertex in G. For a connected graph G, a k-cut set is a set of k vertices whose removal disconnects the graph into at least two connected components. A 1-cut set is called a *cut vertex*. The smallest k such that G has a k-cut set is called the *connectivity* of G. Also, a graph G having no $(k-1)$-cut is called *k-connected*. For simplicity, we sometimes abuse the language and use a connected component and the graph induced by this connected component interchangably.

Given a graph G, the size of a maximum matching of G is denoted by $\nu(G)$. A matching is *maximal* if no other matching properly contains it. A matching M is said to *saturate* a vertex v if v is an endvertex of some edge in M, otherwise it leaves a vertex *exposed*. If every matching M of G *extends* to a perfect matching, in other words, for every matching M (including a single edge) there is a perfect matching that contains M, then G is called *randomly matchable*. Clearly, if an equimatchable graph has a perfect matching, then it is randomly matchable. If $G - v$ has a perfect matching for every $v \in V(G)$, then G is called *factor-critical*. For short, a factor-critical equimatchable graph is denoted an EFC-*graph*. For a vertex v, a matching M is called a *matching isolating* v if $\{v\}$ is a connected component of $G \setminus V(M)$. If G is factor-critical, it follows from its definition that for every vertex v, there is a matching M_v isolating v.

The following result serves as a guideline to study the structure of equimatchable graphs.

Theorem 2.1 *[17] A 2-connected equimatchable graph is either factor-critical or bipartite or K_{2t} for some $t \geq 1$.*

In the view of Theorem 2.1, a systematic way to study various properties of (subclasses of) equimatchable graphs is to consider i) equimatchable graphs with a cut vertex, ii) 2-connected EFC-graphs, iii) 2-connected bipartite equimatchable graphs, and iv) K_{2t} for some $t \geq 1$.

The case of bipartite graphs has been settled as follows.

Lemma 2.2 *[17] A connected bipartite graph $G = (U \cup W, E)$, $|U| \leq |W|$ is equimatchable if and only if for every $u \in U$, there exists a non-empty set $S \subseteq N(u)$ such that $|N(S)| \leq |S|$.*

Lemma 2.2, together with the well-known Hall's condition implies the following more insightful characterization of connected bipartite equimatchable graphs.

Theorem 2.3 (Hall's Theorem) *[12] A bipartite graph $G = (A \cup B, E)$ has a matching saturating all vertices in A if and only if it satisfies $|N(S)| \geq |S|$ for every subset $S \subseteq A$.*

Corollary 2.4 *[6] Let $G = (U \cup W, E)$ be a connected bipartite graph with $|U| \leq |W|$. Then G is equimatchable if and only if every maximal matching of G saturates U.*

While studying equimatchable graphs with a cut vertex, the following will be useful:

Lemma 2.5 *[2] Let G be a connected equimatchable graph with a cut vertex v, then each connected component of $G - v$ is also equimatchable.*

Recall that a graph is equimatchable if and only if every connected component of it is equimatchable. Therefore, in the remainder of this paper, we assume that all graphs are simple, finite, connected.

It should be noted that in the studies of equimatchable graphs with respect to various properties in [9, 10, 11], the case of factor-critical equimatchable graphs has been the most complicated one. The following basic observations will guide us through our proofs. Since the size of any maximal matching in a factor-critical equimatchable graph is $(n-1)/2$ where n is the number of vertices of the graph, we have the following:

Lemma 2.6 *[6] Let G be a factor-critical graph. G is equimatchable if and only if there is no independent set I with three vertices such that $G \setminus I$ has a perfect matching.*

An equivalent reformulation of Lemma 2.6 is the following:

Corollary 2.7 *Let G be a factor-critical equimatchable graph. Then every maximal matching of G leaves exactly one vertex exposed.*

Another useful result on factor-critical equimatchable graphs is the following.

Lemma 2.8 *[9] Let G be a 2-connected factor-critical equimatchable graph. Let v be a vertex of G and M_v a minimal matching isolating v. Then $G \setminus (V(M_v) \cup \{v\})$ is isomorphic to K_{2n} or $K_{n,n}$ for some $n \in \mathbb{N}$.*

Lastly, equimatchable graphs with a perfect matching are precisely randomly matchable graphs whose structure is well-known:

Lemma 2.9 *[20] A connected graph is randomly matchable if and only if it is isomorphic to a K_{2n} or a $K_{n,n}$ $(n \geq 1)$.*

3 Vertex-critical equimatchable graphs

Let us first investigate vertex-critical equimatchable graphs. As suggested by Theorem 2.1, we will proceed seperately with VCE-graphs with a cut vertex, 2-connected bipartite VCE-graphs, even cliques (showing that all three of them are empty), and finally with 2-connected factor-critical VCE-graphs. As a result, we will show that VCE-graphs are almost equivalent to factor-critical equimatchable graphs. Building upon the results obtained in this section, we will show later that VCE-graphs contain factor-critical ECE-graphs. This motivates even further the study of factor-critical ECE-graphs.

Recall that a graph G is VCE if G is equimatchable and $G - v$ is non-equimatchable for every $v \in V(G)$. Let us call a vertex $v \in V(G)$ *strong* (in G) if every maximal matching of G saturates v, (or equivalently there is no maximal matching of $G - v$ saturating all neighbours of v), otherwise it is called *weak* (in G).

We have the following by noticing that every maximal matching of G saturates v if and only if the size of every maximal matching of G decreases exactly by one when v is removed from G:

Remark 3.1 Let G be an equimatchable graph. Then v is a strong vertex if and only if $\nu(G-v)=\nu(G)-1$.

Proposition 3.2 *Let G be an equimatchable graph. Then, for a vertex $v \in V(G)$, the graph $G-v$ is equimatchable if and only if one of the following holds:*

(i) *v is a strong vertex in G,*
(ii) *all vertices in $N(v)$ are strong in $G-v$.*

Proof. Let G be an equimatchable graph, and assume that $G-v$ is equimatchable for a vertex $v \in V(G)$. There are two possibilities: Either $\nu(G-v)=\nu(G)-1$ and then v is a strong vertex by Remark 3.1. Otherwise, $\nu(G-v)=\nu(G)$, i.e., v is not strong, hence it is a weak vertex. Then we claim that $N(v)$ is a set of strong vertices in $G-v$. Indeed, if $u \in N(v)$ is a weak vertex in $G-v$, then there exists a maximal matching M of $G-v$ leaving u exposed with $|M|=\nu(G-v)=\nu(G)$. Then $M \cup \{vu\}$ is a maximal matching in G, a contradiction with the equimatchability of G. Hence $N(v)$ is a set of strong vertices in $G-v$.

We now suppose the converse. Let G be an equimatchable graph, and let v be a strong vertex. Then $\nu(G-v)=\nu(G)-1$ and the size of each maximal matching decreases exactly by one. It follows that $G-v$ is equimatchable. Now, let $N(v)$ be a set of strong vertices in $G-v$. Then every maximal matching of $G-v$ saturates $N(v)$, and therefore those are also maximal matchings of G. Since G is equimatchable, they all have the same size, thus $G-v$ is also equimatchable. □

By Lemma 2.5, if an equimatchable graph has a cut vertex, then its removal from the graph leaves an equimatchable graph. Then we have the following.

Proposition 3.3 VCE-*graphs are* 2-*connected.*

Proposition 3.4 *There is no bipartite* VCE-*graph.*

Proof. Let $G=(U \cup W, E)$ be a bipartite equimatchable graph with $|U| \leq |W|$. By Corollary 2.4, every vertex of U is strong. It follows that $G-u$ is equimatchable for every $u \in U$ by Proposition 3.2. Therefore, G is not VCE. Hence, there is no bipartite VCE-graph. □

Since every complete graph is an equimatchable graph, the removal of a vertex from K_t yields an equimatchable graph. Thus we have the following.

Proposition 3.5 *K_t for some integer $t \geq 2$ is not* VCE*.*

Theorem 2.1 together with Propositions 3.3, 3.4 and 3.5 imply the following:

Corollary 3.6 *VCE-graphs are 2-connected factor-critical.*

So, the following result provides a characterization of all VCE-graphs.

Theorem 3.7 *Let G be a 2-connected graph with $2r+1$ vertices. Then G is* VCE *if and only if G is a $(K_{2r}, K_{r,r})$-free* EFC*-graph.*

Proof. Let G be a VCE-graph with $2r+1$ vertices, then it is 2-connected by Proposition 3.3. We claim that G is $(K_{2r}, K_{r,r})$-free, since otherwise there exists a vertex $v \in V(G)$ such that $G-v$ is isomorphic to a connected randomly matchable graph. In such a case $G-v$ is equimatchable, a contradiction with the vertex criticality of G.

We now suppose the converse. Let G be a $(K_{2r}, K_{r,r})$-free EFC-graph. Assume for a contradiction that there is a vertex $v \in V(G)$ such that $G-v$ is equimatchable. Since G is factor-critical, the graph $G-v$ has a perfect matching. It follows that $G-v$ is a connected randomly matchable graph which is either K_{2r} or $K_{r,r}$ by Lemma 2.9, contradicting our assumption. Therefore G is VCE. □

Having obtained a characterization of VCE-graphs as a subclass of EFC-graphs, let us now investigate the difference of EFC-graphs from VCE-graphs. This will allow us to complete the containment relationships between various subclasses of equimatchable graphs as depicted in Figure 3 of Section 5.

Theorem 3.8 *[11] G is an EFC-graph with a cut-vertex v if and only if every connected component C_i of $G-v$ is isomorphic to $K_{r,r}$ or to K_{2t} for some integers $r, t \geq 1$ and where v is adjacent to at least two adjacent vertices of each C_i.*

Theorem 3.7 together with Propositions 3.3 and Theorem 3.8 allow us to describe all EFC-graphs that are not VCE as follows:

Proposition 3.9 *Let G be an EFC-graph which is not VCE. Then there exists a vertex $v \in V(G)$ such that each connected component C_i of $G-v$ is a $K_{r,r}$ or a K_{2t} for some integers $r, t \geq 1$ and where v is adjacent to at least two adjacent vertices of each C_i.*

Proof. If G has a cut-vertex, then the result follows from Theorem 3.8. Otherwise G is a 2-connected graph with $2r+1$ vertices and contains one of K_{2r} or $K_{r,r}$ by Theorem 3.7. Since G is 2-connected, v has at least two neighbors x and y in $G-v$; moreover $xy \in E$. Indeed, if $G-v$ is K_{2r} then clearly $xy \in E$; if $G-v$ is $K_{r,r}$ then x and y belong to the same (r)-stable set of $K_{r,r}$ and v has no neighbor in the other (r)-stable set; then $G-x$ has no perfect matching, contradicting that G is factor-critical. □

It follows from the above discussion that VCE-graphs are almost equivalent to the class of factor-critical equimatchable graphs; indeed this is the most intriguing subclass of equimatchable graphs as the structure of the remaining equimatchable graphs are rather well-known [5, 17]. By Proposition 3.9, the only factor-critical equimatchable graphs that are not VCE are those graphs G admitting a vertex v such that $G-v$ leaves a graph whose connected components are $K_{r,r}$ or K_{2t} for some integers r and t and where v is adjacent to at least two adjacent vertices of each component of $G-v$.

We now start the investigation of ECE-graphs. It is worth noting that while comparing subclasses of equimatchable graphs in Section 5, Proposition 3.3 and Theorem 3.7 will allow us to derive (in Corollary 5.4) that all factor-critical ECE-graphs are VCE.

4 Edge-critical equimatchable graphs

In this section, we investigate ECE-graphs. Our preliminary results in Section 4.1 show that apart from two simple cases, namely bipartite ECE-graphs and complete graphs of even order, all ECE-graphs are (2-connected) factor-critical. Then, we characterize factor-critical ECE-graphs with connectivity 2 in Section 4.2 (Theorem 4.9). It is worth mentioning that one can easily observe that ECE-graphs consist of at least four vertices, i.e., no graphs of order at most three can be an ECE-graph.

4.1 Preliminaries on ECE-graphs

We start with a lemma that will be frequently used in our proofs.

Lemma 4.1 *Let G be an equimatchable graph except K_2. Then $uv \in E(G)$ is critical if and only if there is a matching of G containing uv and saturating $N(\{u,v\})$.*

Proof. Assume that uv is a critical edge in G. Then $G \setminus uv$ admits two maximal matchings M_1 and M_2 with $|M_1| < |M_2|$. Note that M_1 leaves both u and v exposed in $G \setminus uv$ since otherwise M_1 would be a maximal matching of G, contradicting that G is equimatchable. This implies that M_1 saturates all vertices in $N_{G\setminus uv}(\{u,v\})$. Hence, $M_1 \cup \{uv\}$ is a maximal matching of G as desired.

We now suppose the converse. If there is such a matching M, then $M \setminus \{uv\}$ is a maximal matching in $G \setminus uv$. However, there exists another vertex w which is without loss of generality a neighbour of v since $G \neq K_2$. So there is also another maximal matching M' in $G \setminus uv$ which can be obtained by extending the edge vw. Clearly, M' has size $\nu(G)$ since it is also a maximal matching of G. Then $G \setminus uv$ is not equimatchable, implying that uv is a critical-edge in G. □

By Lemma 4.1, if a graph G is factor-critical ECE, then for every $uv \in E(G)$, there exists a matching of G containing uv and saturating $N(\{u,v\})$. However such a matching does not exist if $N(\{u,v\}) = V(G)$ since $|G|$ is odd. It follows that:

Corollary 4.2 *If G is a factor-critical* ECE*-graph, then there is no edge $uv \in E(G)$ such that $N(\{u,v\}) = V(G)$.*

The following is a direct consequence of Lemma 4.1, since any randomly matchable graph has a matching containing uv and saturating $N(\{u,v\})$ for every edge uv.

Remark 4.3 Randomly matchable graphs except K_2 are edge-critical equimatchable.

In what follows, we shall prove that ECE-graphs have no cut vertex.

Lemma 4.4 ECE-*graphs are* 2-*connected.*

Proof. Assume that G is an ECE-graph, and has a cut-vertex z. Let $H_1, H_2, \ldots, H_k$ be the components of $G - z$ for $k \geq 2$. By Lemma 2.5, each H_i is equimatchable. Besides, since G is an ECE-graph, there exists a matching containing uv and saturating $N(\{u,v\})$ for every $uv \in E(G)$ by Lemma 4.1. Let us pick a vertex w_i from each component H_i such that $w_i \in N(z) \cap H_i$. Then, for the edge zw_1, there exists a maximal matching M in G containing zw_1 and saturating $N(\{z, w_1\})$. Let $M \cap E(H_i) = M_i$ for $i \in [k]$. Observe that M_1 saturates all vertices in $N_{H_1}(w_1)$. Also, for each $i \geq 2$, the matching M_i saturates all vertices in $N_{H_i}(z)$. We now consider the edge zw_2, similarly as above; there exists a maximal matching L in G containing zw_2 and saturating $N(\{z, w_2\})$. It follows that there exists a maximal matching $T = L \cap E(H_1)$ in H_1 such that T saturates all vertices in $N_{H_1}(z)$. In this manner, we obtain a maximal matching $T \cup M_2 \cup \ldots \cup M_k$ in G isolating z, and so we have $\nu(G) = |T| + |M_2| + \ldots + |M_k|$ since G is equimatchable. This also implies that $\nu(G) = \sum \nu(H_i)$. On the other hand, observe that M_1 is a maximal matching in H_1 since M_1 saturates all vertices in $N_{H_1}(w_1)$. Moreover the matchings M_1 and T are of the same size since H_1 is equimatchable. Thus $M_1 \cup M_2 \cup \ldots \cup M_k$ must be of size $\nu(G)$. However, this contradicts that we have the maximal matching $M = M_1 \cup M_2 \cup \ldots \cup M_k \cup \{zw_1\}$ in G. Hence G has no cut-vertex. $\square$

The following is an immediate consequence of Theorem 2.1 together with Lemma 4.4.

Theorem 4.5 ECE-*graphs are either factor-critical or bipartite or* K_{2t} *for some* $t \geq 2$.

In view of Theorem 4.5, we consider ECE-graphs under three disjoint categories: 2-connected factor-critical, 2-connected bipartite, and complete graphs of even order (which are randomly matchable thus ECE).

The characterization of bipartite ECE-graphs has been given in [5] as follows.

Theorem 4.6 *[5] A connected bipartite graph* $G = (U \cup V, E)$ *with* $|U| \leq |V|$ *except* K_2 *is a bipartite* ECE-*graph if and only if for every* $u \in U, |N(S)| \geq |S|$ *holds for any subset* $S \subseteq N(u)$ *and the equality holds only for* $S = N(u)$.

It remains to clarify the structure of factor-critical ECE-graphs. Recall that all factor-critical ECE-graphs are 2-connected by Lemma 4.4. In the next subsection, we provide a characterization of factor-critical ECE-graphs with connectivity 2.

4.2 Factor-critical ECE-graphs with connectivity 2

The following result on factor-critical equimatchable graphs with connectivity 2 will guide us in this subsection.

Theorem 4.7 *[11] Let G be an* EFC*-graph of order at least* 5 *and connectivity* 2 *and let $S = \{s_1, s_2\}$ be a 2-vertex-cut of G. Suppose that a_i and b_i are distinct neighbours of s_i in respectively A and B for $i = 1, 2$. Then $G \setminus S$ has precisely two components A and B such that*

- *(i) B is one of the four graphs $K_{2p+1}, K_{2p+1} \setminus b_1b_2$, $K_{p,p+1}$ or $K_{p,p+1} + b_1b_2$ and in the two last cases b_1 and b_2 belong to the $(p + 1)$-stable set of $K_{p,p+1}$.*
- *(ii) $A \setminus \{a_1, a_2\}$ is either K_{2q-2} or $K_{q-1,q-1}$, and if $|B| > 1$, then A is either K_{2q} or $K_{q,q}$*

First, we show that there is no factor-critical ECE-graph of order 5 or less.

Remark 4.8 Factor-critical ECE-graphs have at least 7 vertices.

Proof. It is clear that there is no ECE-graph on 3 or fewer vertices. Assume that there exists a connected factor-critical ECE-graph G with 5 vertices. If $\Delta(G) = 2$, then for any edge $uv \in E(G)$ with $d(u) = d(v) = 2$, there is no matching containing uv and saturating $N(\{u, v\})$. Thus, uv is not critical by Lemma 4.1. For the other case, if $\Delta(G) \geq 3$, then for a vertex u with $d(u) \geq 3$, there exists a neighbour v of u such that $N[\{u, v\}] = V(G)$, a contradiction by Corollary 4.2. □

The general structure of a factor-critical ECE-graph G of order at least 7 and connectivity 2 follows from Theorem 4.7; for a 2-cut $S = \{s_1, s_2\}$, the graph $G - S$ has exactly two components A and B as described in Theorem 4.7 and illustrated in Figure 1.

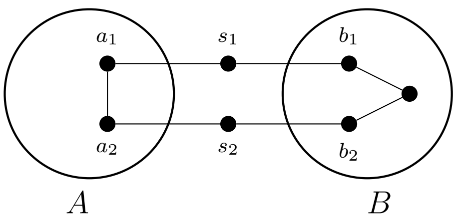

Figure 1: The structure of factor-critical ECE-graphs where $|A|$ is even and $|B|$ is odd.

We will introduce five possible configurations with respect to A and B and then show that a factor-critical ECE-graph with connectivity 2 falls into one of these five types. For a graph G, consider $A, B \subset V(G)$ each one with at least two vertices. We say that A is *partially-complete* to B if there exist a non-empty partition A_1, A_2 of A and a non-empty partition B_1, B_2 of B such that for each $i = 1, 2$, A_i is complete to B_i and A_i has no neighbour in B_{3-i}. Let $S = \{s_1, s_2\}$ be an independent set.

- **Type I**: $A \cong K_{2q}$, $B \cong K_{2p+1}$ for $p, q \geq 1$ such that S is complete to A, and S is partially-complete to B (see Figure 2(a)).
- **Type II**: $A \cong K_{q,q}$, $B \cong K_{2p+1}$ for $p, q \geq 1$ such that for $i = 1, 2$, each s_i is complete to a distinct (q)-stable set of A, and S is partially-complete to B (see Figure 2(b)).
- **Type III**: $A \cong K_{2q}$, $B \cong K_{p,p+1}$ for $p, q \geq 1$ such that S is complete to A, and S is partially-complete to the $(p+1)$-stable set of B (see Figure 2(c)).
- **Type IV**: $A \cong K_{q,q}$, $B \cong K_{p,p+1}$ for $p, q \geq 1$ such that for $i = 1, 2$, each s_i is complete to distinct (q)-stable sets of A, and S is partially-complete to the $(p+1)$-stable set of B (see Figure 2(d)).
- **Type V**: $A \setminus \{a_1, a_2\} \cong K_{2q-2}$ for $q \geq 3$, $B \cong K_1$, and there is a vertex $w \in A$ such that $\{a_1, a_2, w\}$ is a stable set, each s_i is complete to $\{b, a_i, w\}$ for $i = 1, 2$, and $\{a_1, a_2, w\}$ is complete to $A \setminus \{a_1, a_2, w\}$ (see Figure 2(e)).

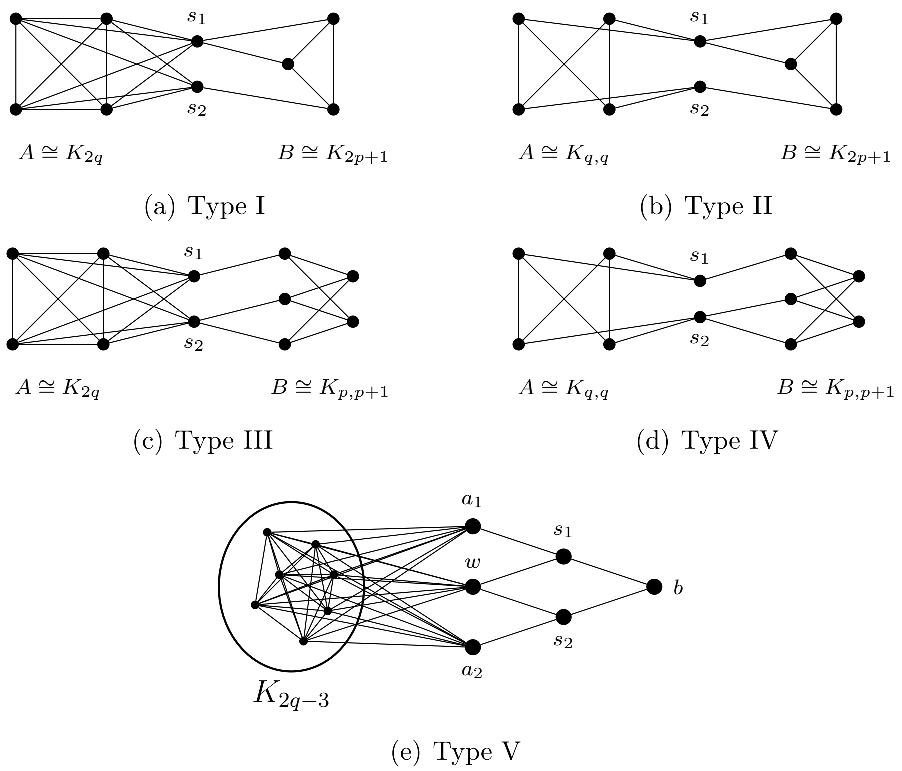

Figure 2: The family $\mathcal{F}$ of factor-critical ECE-graphs with connectivity 2.

Let $\mathcal{F}$ stand for the family of all graphs falling into one of the five types of configurations I, II, III, IV, V depicted in Figure 2. The main result of this section is the following.

Theorem 4.9 *A graph G is factor-critical* ECE *with connectivity* 2 *if and only if $G \in \mathcal{F}$ where the five types describing $\mathcal{F}$ are depicted in Figure 2.*

We will obtain the proof of Theorem 4.9 as two separate lemmas, each one proving one direction.

Lemma 4.10 *All graphs belonging to $\mathcal{F}$ are* ECE*-graphs.*

Proof. Let $G \in \mathcal{F}$. We will show that each one of the five types of configuration is an ECE-graph. Let $\mathcal{G}_i$ be the class of all graphs of Type i as described above and depicted in Figure 2.

$G \in \mathcal{G}_1$ **is equimatchable:** Since S is complete to A and partially-complete to B, and since $G \setminus S$ consists of two cliques, it follows that the independence number of G is equal to 2. Thus G is equimatchable by Lemma 2.6.

$G \in \mathcal{G}_2$ **is equimatchable:** Let A_1, A_2 be the bipartition of A with $A_1 = \{a_1, a_2, \ldots, a_q\}$ and $A_2 = \{a'_1, a'_2, \ldots, a'_q\}$ and without loss of generality suppose that for $i = 1, 2$, each s_i is complete to A_i. Consider an independent set I of size 3. It is clear that I cannot contain both s_1 and s_2 since $N(\{s_1, s_2\}) = V(G)$. Thus, there are two possibilities. Either I contains at least two vertices of A, say $a_1, a_2 \in A_1$, then $G \setminus I$ has no perfect matching since $\{a'_1, a'_2, \ldots, a'_q\}$ is an independent set in $G \setminus I$ having only $q - 1$ neighbours in $G \setminus I$. Or I contains one vertex from each one of the sets A, S and B, say $I = \{a, s, b\}$, respectively. Note that if $s = s_1$, then $a \in A_2$. In this case, $G \setminus I$ has no perfect matching since $\{a_1, a_2, \ldots, a_q\}$ is an independent set in $G \setminus I$ having only $q - 1$ neighbours in $G \setminus I$. It then follows from Lemma 2.6 that G is equimatchable.

$G \in \mathcal{G}_3$ **is equimatchable:** Let B_1, B_2 be the bipartition of B with $B_1 = \{b_1, b_2, \ldots, b_{p+1}\}$ and $B_2 = \{b'_1, b'_2, \ldots, b'_p\}$. Note that S is complete to A and partially-complete to B_1. Consider an independent set I of size 3. Suppose I contains both vertices of $S = \{s_1, s_2\}$, then its third vertex belongs to B_2. We then observe that $G \setminus I$ has no perfect matching since B_1 is an independent set of $G \setminus I$ of size $p + 1$ but it has only $p - 1$ neighbours in $G \setminus I$. So assume that I contains at most one vertex from $A \cup S$, thus at least two vertices x, y in B, necessarily both in B_1 or both in B_2. If $x, y \in B_1$, then $G \setminus I$ has no perfect matching, since B_2 is an independent set in $G \setminus I$ of size p but it has at most $p - 1$ neighbours in $G \setminus I$. Finally, if $x, y \in B_2$, then $G \setminus I$ has no perfect matching since B_1 is an independent set in $G \setminus I$ of size $p + 1$ but it has at most p neighbours in $G \setminus I$. Hence G is equimatchable by Lemma 2.6.

$G \in \mathcal{G}_4$ **is equimatchable:** Let A_1 and A_2 be the bipartition of A with $|A_1| = |A_2| = q \geq 1$ and s_i is complete to A_i for $i = 1, 2$. Consider an independent set I of size 3. If I contains both vertices of $S = \{s_1, s_2\}$ or at least two vertices of B, then we can show that G is equimatchable in a similar way as above. Thus, assume that I contains at least two vertices of A which should be clearly in the same part of A, say A_1. Then $G \setminus I$ has no perfect matching since A_2 is an independent set in $G \setminus I$ of size q but it has at most $q - 1$ neighbours in $G \setminus I$ (in $A_1 \cup \{s_2\}$). Finally, if I consists of one vertex from each one of the sets A, S and B, without loss of generality $I = \{a, s_1, b\}$ where $b \in V(B)$ and $a \in A_2$, then $G \setminus I$ has no perfect matching since A_1 is an independent set in $G \setminus I$ of size q but it has $q - 1$ neighbours in $G \setminus I$. Hence, in each case, G is equimatchable by Lemma 2.6.

$G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$ **is ECE:** We will show that for every $e = uv \in E(G)$ the graph $G \setminus e$ is not equimatchable by considering every possible type of edge. First, suppose e is incident to two vertices in A and S, say without loss of generality $u \in A$ and $v = s_1$. Then for some $b \in B \setminus N(s_1)$ there is a perfect matching M_b in $B \setminus \{b\}$, and a perfect matching M_A in $A \setminus \{u'\}$ for some u' such that $uu', s_2u' \in E$. Now, the set $M_A \cup M_b \cup \{uv, u's_2\}$ is a matching containing uv and saturating $N(\{u, v\})$. It follows from Lemma 4.1 that $G \setminus e$ is not equimatchable. Similarly, if e is incident to two vertices in B and S, say $u \in B$ and $v = s_1$, then letting M_A and M_u being perfect matchings in A and $B \setminus \{u\}$ respectively, the set $M_A \cup M_u \cup \{uv\}$ is a matching containing uv and saturating $N(\{u, v\})$. So, $G \setminus e$ is not equimatchable. Now, if e belongs to A, then there is a perfect matching M_A of A containing the edge uv such that for some $b_i \in N_B(s_i)$ for $i = 1, 2$, the set $M_A \cup \{s_1b_1, s_2b_2\}$ is a matching containing uv and saturating $N(\{u, v\})$. So, $G \setminus e$ is not equimatchable. Finally, if e belongs to B, then there exists $b \in B \setminus \{u, v\}$ due to $p \geq 1$, say $bs_i \in E(G)$ for some $i \in \{1, 2\}$ such that there exists a perfect matching M_b of $B \setminus \{b\}$ containing the edge uv, and a vertex $a \in N_A(s_{3-i})$ such that the set $M_b \cup \{s_ib, s_{3-i}a\}$ is a matching containing uv and saturating $N(\{u, v\})$. So, $G \setminus e$ is not equimatchable. Hence G is ECE.

$G \in \mathcal{G}_5$ **is equimatchable:** By definition, the set $\{a_1, a_2, w\}$ is independent, s_i is complete to $\{b, a_i, w\}$ for $i = 1, 2$, and $\{a_1, w, a_2\}$ is complete to $A' = A \setminus \{a_1, w, a_2\}$. Consider an independent set I of size 3. If I contains S or $I = \{a_1, a_2, w\}$, then $G \setminus I$ has an odd component implying that $G \setminus I$ has no perfect matching. The only remaining possibility is that I consists of the vertex b and two vertices from $\{a_1, a_2, w\}$. In this case, $\{s_1, s_2\}$ is an independent set in $G \setminus I$, but it has a unique neighbour in $G \setminus I$. So $G \setminus I$ has no perfect matching. Therefore, G is equimatchable by Lemma 2.6.

$G \in \mathcal{G}_5$ **is ECE:** Let us show that for every possible $e = uv \in E(G)$, the graph $G \setminus e$ is not equimatchable using Lemma 4.1. Let $A' = A \setminus \{a_1, w, a_2\}$. If e is incident to two vertices in B and S, say $u = b$ and $v = s_1$, then for some $a' \in A'$, the set $\{s_1b, s_2w, a_1a'\}$ is a matching containing s_1b and saturating $N(\{s_1, b\})$. Similarly, if e is incident to two vertices in $\{a_1, a_2, w\}$ and S, say without loss of generality $u \in \{a_1, w\}$ and $v = s_1$, then for $x \in \{a_1, w\} \setminus \{u\}$, there is a perfect matching M_x of the graph induced by $A' \cup \{x\}$ such that the set $M_x \cup \{s_1u, bs_2\}$ is a matching containing s_1u and saturating $N(\{s_1, u\})$. Finally, the cases where e is incident to two vertices in $\{a_1, a_2, w\}$ and A', say without loss of generality $u = a_1$ and $v \in A'$, or two vertices in A' will be handled commonly. In these cases, there is a perfect matching M_{a_1} of the graph induced by $A' \cup \{a_1\}$ that contains the edge a_1v. It follows that the set $M_{a_1} \cup \{s_1w, s_2a_2\}$ is a matching containing a_1v and saturating $N(\{a_1, v\})$. Hence G is ECE-graph by Lemma 4.1. □

Now, we will show that all factor-critical ECE-graphs with connectivity 2 belong to the family $\mathcal{F}$. To this end, we will first give an equivalent formulation for a graph to belong to the family $\mathcal{F}$ which follows directly from the definitions of Types I, II, III, IV, V forming the family $\mathcal{F}$ (as depicted in Figure 2).

Proposition 4.11 *Let G be a factor-critical graph of order at least 7 and connectivity 2. Then G is a member of $\mathcal{F}$ if and only if there exists a 2-cut $S = \{s_1, s_2\}$ such that $G \setminus S$ has exactly two components A and B, and the following hold:*

- (i) *B is isomorphic to either K_1 or K_{2p+1} or $K_{p,p+1}$ for $p \geq 1$. Moreover, for $p \geq 1$, if $B \cong K_{2p+1}$ (resp. $K_{p,p+1}$), then S is partially-complete to B (resp. $(p+1)$-stable set of B).*
- (ii) *S is an independent set.*
- (iii) *If $|B| > 1$, then $A \cup \{s_1, s_2\}$ induces either $K_{2q+2} \setminus s_1s_2$ or $K_{q+1,q+1} \setminus s_1s_2$ for $q \geq 1$. If $|B| = 1$, then we have $N(s_i) = \{b, a_i, w\}$ where $b \in B$, and $a_i, w \in A$ for $i = 1, 2$, and $A \cong K_{2q} \setminus \{a_1a_2, wa_1, wa_2\}$ for $q \geq 3$.*

Lemma 4.12 *Let G be a factor-critical graph and connectivity 2. If G is* ECE*-graph, then G is a member of $\mathcal{F}$.*

Proof. Suppose that G is a factor-critical ECE-graph with connectivity 2. By Theorem 4.7, there is a 2-vertex-cut $S = \{s_1, s_2\}$ such that $G - S$ has precisely two components A and B as described in items (i) and (ii) of Theorem 4.7. Let $s_1b_1, s_2b_2 \in E(G)$ for vertices $b_1, b_2 \in B$ (where b_1 and b_2 are distinct if $|B| > 1$, and $b_1 = b_2$ if $B = K_1$), and let $s_1a_1, s_2a_2 \in E(G)$ for distinct vertices $a_1, a_2 \in A$ (see Figure 1). We will prove that G satisfies the conditions $(i), (ii), (iii)$ in Proposition 4.11 to show that G has one of the five configurations in Figure 2.

(i) If $|B| = 1$, then $B = K_1$, and so the claim clearly holds. Thus we may assume $|B| > 1$, and so $|B| \geq 3$. First, note that A is either K_{2q} or $K_{q,q}$ by Theorem 4.7 (ii). In addition, we infer that $a_1a_2 \in E(G)$, since otherwise A would be $K_{q,q}$ where a_1 and a_2 belong to the same q-stable set. However, extending s_1a_1, s_2a_2 into a maximal matching in G leaves two vertices of A exposed, contradicting to the equimatchability of G by Corollary 2.7.

We now claim that for every $b \in N_B(\{s_1, s_2\})$, $B \setminus \{b\}$ is a randomly matchable graph. Without loss of generality, assume $bs_1 \in E(G)$. Consider a matching M_a containing bs_1, s_2a_2 and saturating all vertices but a vertex $a \in A$. This is a minimal matching isolating a. Thus, by Lemma 2.8, the graph $G \setminus (V(M_a) \cup \{a\})$ which is the graph induced by $B - b$ is randomly matchable. It then follows from Lemma 2.9 that $B \setminus \{b\}$ is isomorphic to $K_{p,p}$ or K_{2p} for every $b \in N_B(\{s_1, s_2\})$. Therefore, if B is isomorphic to $K_{p,p+1}$ or $K_{p,p+1} + b_1b_2$ as described in Theorem 4.7, then $N_B(S)$ is included in the $(p+1)$-stable set of $K_{p,p+1}$. Moreover, if B is $K_{2p+1} \setminus b_1b_2$, then S has no neighbour in B other than b_1 and b_2.

Claim 1 *If there is a vertex $w \in B \setminus N(S)$, then $B \setminus \{w\}$ has no perfect matching.*

Proof of the Claim. Assume for a contradiction that there is a vertex $w \in B$ which is not adjacent to S such that $B \setminus \{w\}$ has a perfect matching M. Clearly, M is a minimal matching isolating w, and therefore $A \cup S$ induces a connected randomly matchable graph in G by Lemma 2.8. That is, $A \cup S$ induces a graph which is isomorphic to K_{2q+2} or $K_{q+1,q+1}$ for some $q \geq 1$. It then follows that $s_1s_2 \in E(G)$ since $a_1a_2, a_1s_1, a_2s_2 \in E(G)$. In this case, however, we show that $s_1b_1 \in E(G)$ is not a critical edge. Indeed, if $B \cong K_{p,p+1}$ or $B \cong K_{p,p+1} + b_1b_2$, then $T = N(\{s_1, b_1\})$ contains the (p)-stable set of $K_{p,p+1}$. On the other hand, T contains either $A \cup \{s_2\}$ (which induces a K_{2q+1}) or the $(q + 1)$-stable set of $A \cup S$. We then deduce that every matching of G containing s_1b_1 leaves a vertex of T exposed. It can be checked that the same holds if $B \cong K_{2p+1}$ or $B \cong K_{2p+1} \setminus b_1b_2$ (recall that in this case $N_B(\{s_1, s_2\}) = \{b_1, b_2\}$). Consequently, there is no matching containing s_1b_1 and saturating $T = N(\{s_1, b_1\})$, contradicting to the criticality of s_1b_1. ♢

We have already noticed that if B is $K_{2p+1} \setminus b_1b_2$ then S has no neighbor in B other than b_1 and b_2. So, Claim 1 implies that B is not isomorphic to $K_{2p+1} \setminus b_1b_2$ for $p \geq 2$ (note that the case $p = 1$ corresponds to the graph $K_{1,2}$). We also note that B is not isomorphic to $K_{p,p+1} + b_1b_2$ neither. Indeed, S has no neighbour in (p)-stable set of $K_{p,p+1} + b_1b_2$, but for a vertex b in (p)-stable set of $K_{p,p+1} + b_1b_2$, the graph $(K_{p,p+1} + b_1b_2) - b$ has obviously a perfect matching, it contradicts Claim 1. Hence B is not isomorphic to $K_{p,p+1} + b_1b_2$. It then follows from Claim 1 that either $B \cong K_{2p+1}$ and every vertex of B is adjacent to at least one of s_1, s_2, or $B \cong K_{p,p+1}$ and every vertex of $(p + 1)$-stable set of B is adjacent to at least one of s_1, s_2.

To complete the proof, it remains to show that s_1 and s_2 have no common neighbour in B. To this end, we first clarify the links between S and A as follows:

Claim 2 *Both $A \cup \{s_1\}$ and $A \cup \{s_2\}$ induce either K_{2q+1} or $K_{q,q+1}$.*

Proof of the Claim. We first claim that none of s_1 or s_2 is complete to $B \cong K_{2p+1}$ or the $(p + 1)$-stable set of $B \cong K_{p,p+1}$. Assume for a contradiction that s_1 is complete to $B \cong K_{2p+1}$ or the $(p + 1)$-stable set of $B \cong K_{p,p+1}$, then the edge s_1a_1 is not critical. Indeed, $N_{G \setminus s_1a_1}(\{s_1, a_1\})$ contains all vertices of $B \cong K_{2p+1}$ or the $(p + 1)$-stable set of $B \cong K_{p,p+1}$, as well as $N_A(a_1)$ which is either an odd clique K_{2q-1} or the q-stable set of A. In all cases, there is no matching containing s_1a_1 and saturating $N(\{s_1, a_1\})$, contradicting that s_1a_1 is a critical edge in G by Lemma 4.1. Thus, for each $i = 1, 2$, there is a vertex $b \in B$ such that $s_ib \notin E(G)$ where b is a vertex in $B \cong K_{2p+1}$ or the $(p + 1)$-stable set of $B \cong K_{p,p+1}$. This implies that there exists a vertex $b \in B$ with $b \in N(s_2) \setminus N(s_1)$ such that $B \setminus \{b\}$ has a perfect matching P. In addition, $P \cup \{s_2a_2\}$ is a matching isolating b. It then follows from Lemma 2.8 that $G[A \cup \{s_1\}] \setminus \{a_2\}$ is a randomly matchable graph. By symmetry, $G[A \cup \{s_2\}] \setminus \{a_1\}$ is randomly matchable as well. Moreover, any vertex $a \in N_A(s_1)$ (or $a \in N_A(s_2)$) can play the role of a_1, i.e., the graph $G[A \cup \{s_2\}] \setminus \{a\}$ is randomly matchable graph for every $a \in N_A(s_1)$. Likewise, the graph $G[A \cup \{s_1\}] \setminus \{a\}$ is randomly matchable graph for every $a \in N_A(s_2)$. This implies the following: if s_1 and s_2 have a common neighbour in A then each of the sets $A \cup \{s_1\}$ and $A \cup \{s_2\}$ induce cliques of size $2q + 1$; if s_1 and s_2 have no common neighbour in A then each of the sets $A \cup \{s_1\}$ and $A \cup \{s_2\}$ induce $K_{q,q+1}$, and $s_1a_2, s_2a_1 \notin E(G)$. ♢

Let us now show that s_1 and s_2 have no common neighbour in B. Assume for a contradiction that there is a vertex $b \in B$ such that $s_1b, s_2b \in E$. In this case, we show that the edge s_1b is not critical. Indeed, $R = N_{G\setminus s_1b}(\{s_1, b\})$ contains either $B \setminus \{b\}$ (when B is an even clique) or (p)-stable set of $B \cong K_{p,p+1}$. Moreover, R contains either $A\cup\{s_2\}$ (when A is an even clique) or $(q+1)$-stable set of $A\cup\{s_2\}$. In all cases, one can easily check that there is no matching containing s_1b and saturating $N(\{s_1, b\})$, contradicting the criticality of s_1b by Lemma 4.1.

(ii) If $s_1s_2 \in E(G)$, then by the statement (i), $N(\{s_1, s_2\})$ contains B or the $(p+1)$-stable set of B when $B \cong K_{2p+1}$ or $B \cong K_{p,p+1}$, respectively. Thus, there is no matching containing s_1s_2 and saturating $N(\{s_1, s_2\})$, implying that s_1s_2 is not a critical edge by Lemma 4.1. It follows that $s_1s_2 \notin E(G)$.

(iii) We shall prove this item under two main cases with respect to the size of B.

Case 1: $|B| > 1$.

As we have already shown in Claim 2 that $G[A \cup \{s_1, s_2\}]$ is either $K_{q+1,q+1} \setminus s_1s_2$ or $K_{2q+2} \setminus s_1s_2$.

Case 2: $|B| = 1$.

Let b be the unique vertex in B. By Theorem 4.7-(ii), $A' = A \setminus \{a_1, a_2\}$ is either $K_{q-1,q-1}$ or K_{2q-2}. Recall that $q \geq 2$ by Remark 4.8. We first note that if s_1 or s_2 has only two neighbours in G, say $d_G(s_1) = 2$, then it is similar to Case 1 as we take the 2-cut $S = \{a_1, s_2\}$. Thus, $d_G(s_i) \geq 3$ for $i = 1, 2$. Similarly, if none of s_1, s_2 has a neighbour in A', then it boils down to the Case 1 as we take the 2-cut $S = \{a_1, a_2\}$. Thus there exists $w \in N_{A'}(S)$. We then deduce that if s_1 or s_2 has no neighbour in A', say this is s_1, then a_2 would be a common neighbour of s_1 and s_2 since $d(s_1) \geq 3$ and $s_1s_2 \notin E$ by the item (ii). It follows that for every pair $x, y \in \{a_1, a_2, w\}$ there exists a matching which saturates $\{s_1, s_2, x, y\}$ and isolates b; thus $A \setminus \{x, y\}$ is randomly matchable by Lemma 2.8. Hence we may exchange the roles of a_2 and w as we desired. That is, if we define $A' = A \setminus \{a_1, w\}$, then both s_1 and s_2 would have a neighbour in A'. It follows that s_1 and s_2 has a common neighbor in A' (and s_1 has no other neighbor in A'). The only remaining case is if both s_1 and s_2 have distinct neighbors in A'.

Combining the two cases, in what follows, we assume that each of $\{s_1, s_2\}$ has at least one neighbour in A'. So there exist $w_1, w_2 \in A'$ such that $w_1 \in N(s_1)$ and $w_2 \in N(s_2)$ (with possibly $w_1 = w_2$). Since $\{s_1w_1, s_2a_2\}$ is a matching isolating b, the graph $G[A' \cup \{a_1\}] - w_1$ is randomly matchable by Lemma 2.8. Similarly, $G[A' \cup \{a_2\}] - w_2$ is randomly matchable since $\{s_1a_1, s_2w_2\}$ is a matching isolating b. It then follows that if A' is isomorphic to $K_{q-1,q-1}$ with a bipartition R and T, then for each $i = 1, 2$, we deduce that both a_i and w_i are complete to either R or T. That is, all neighbours of s_i in A are complete to either R or T. Similarly, if A' is isomorphic to K_{2q-2}, then for each $i = 1, 2$, we deduce that a_i is complete to $A' - w_i$.

For the remaining cases of the proof, we distinguish all possible cases according to $A' = K_{q-1,q-1}$ and $A' = K_{2q-2}$ and vertices w_1, w_2, s_1, s_2. We will see that there is no ECE-graph where $A' = K_{q-1,q-1}$, and the only possible configuration of an ECE-graph with $A' = K_{2q-2}$ corresponds to Type V (see Figure 2(e)).

Subcase 2.1: Suppose that $A' \cong K_{q-1,q-1}$, and there exists $w_i \in N(s_i)$ for each $i = 1, 2$ such that $w_1 \neq w_2$ and $w_1w_2 \in E(G)$.

We claim that $s_1a_2, s_2a_1 \notin E(G)$. Assume for a contradiction that without loss of generality $s_1a_2 \in E(G)$. Then $\{s_1a_2, s_2w_2\}$ is a matching isolating b. By Lemma 2.8, the remaining graph $H = G[A' \cup \{a_1\}] \setminus w_2$ must be randomly matchable. We then say that a_1 and w_2 are complete to the same $(q-1)$-stable set of A'. Recall also that $a_1, w_1 \in V(H)$ are complete to the same $(q-1)$-stable set of A', and $w_1w_2 \in E(G)$. This implies that H is not randomly matchable whenever $q > 2$. If however $q = 2$, i.e, $A' = \{w_1, w_2\}$, then H being randomly matchable implies that $a_1w_1 \in E(G)$. Now, noting that $a_1w_2 \in E(G)$, the edge a_1s_1 is not critical in G since all vertices w_1, w_2, a_2, b belong to $N(\{a_1, s_1\}) \setminus \{a_1, s_1\}$ whereas there is no matching in $G \setminus \{a_1, s_1\}$ saturating $\{w_1, w_2, a_2, b\}$, a contradiction by Lemma 4.1. Thus, we conclude that $s_1a_2 \notin E(G)$. By symmetry, we also have $s_2a_1 \notin E(G)$. Moreover, a_1 is adjacent to a_2, since otherwise for a perfect matching M in $A' \setminus \{w_1, w_2\}$, the set $M \cup \{s_1w_1, s_2w_2\}$ is a perfect matching in $G \setminus \{a_1, a_2, b\}$ where $\{a_1, a_2, b\}$ is an independent set, it is contradiction to the equimatchability of G by Lemma 2.6.

Now, assume A' has a third vertex $w_2' \in N(s_2)$ with $w_2' \notin \{w_1, w_2\}$, then there is also w_1' in A since $A' \cong K_{q-1,q-1}$. Then, a_1 is adjacent to both w_2 and w_2', similarly a_2 is adjacent to both w_1 and w_1'. Now we claim that the vertex a_2 has no neighbor in the stable set R of A' containing w_2 and w_2'. Indeed, if $a_2w \in E(G)$ for some $w \in R$ (which is not necessarily a neighbor of s_2) then letting without loss of generality w_2' to be a neighbor of s_2 different from w, the matching $\{s_1a_1, s_2w_2'\}$ isolates b. However, the remaining graph $G \setminus \{b, s_1, s_2, a_1, w_2'\}$ is not randomly matchable since the vertices w_1, w, a_2, w_1' induce $K_4 \setminus w_1w_1'$ in G, a contradiction to the equimatchability of G by Lemma 2.8. Thus, $R \cup \{a_2\}$ is a stable set in G. In this case however, the edge s_1a_1 is not critical because there is no matching containing s_1a_1 and saturating $N = N(\{s_1, a_1\})$; indeed N contains the independent set $R \cup \{b, a_2\}$ of size $q+1$ while $G \setminus \{s_1, a_1\}$ has $2q+1$ vertices. We therefore conclude that for $i = 1, 2$, each s_i is adjacent to only w_i in A'.

We next claim that $w_1a_1, w_2a_2 \in E(G)$. Assume it is not true and without loss of generality, let $w_2a_2 \notin E(G)$. In this case, the edge s_1a_1 is not critical. To show this, let us first note that a_2 has no neighbor in the $(q-1)$-stable set R of A' containing w_2 since $\{s_1w_1, s_2w_2\}$ is a matching isolating b, the graph $G[A \setminus \{w_1, w_2\}]$ is randomly matchable by Lemma 2.8. Now, $N = N(\{s_1, a_1\})$ contains the stable set $R \cup \{b, a_2\}$ of size $(q+1)$ while $G \setminus \{a_1, s_1\}$ has $2q+1$ vertices. So, there is no matching containing s_1a_1 and saturating N, a contradiction by Lemma 4.1.

Lastly, we show that the obtained graph is not ECE as follows. The matching $\{w_1a_1, bs_2\}$ is a matching isolating s_1, however, the remaining graph $H = G \setminus \{w_1, a_1, b, s_2, s_1\}$ is isomorphic to $K_{q-2,q} + a_2w_2$. Clearly, H is not randomly matchable if $q > 2$ which yields a contradiction to the equimatchability of G by Lemma 2.8. In addition, if $q = 2$, then a_1s_1 is not a critical edge, since $N(\{a_1, s_1\}) \setminus \{a_1, s_1\}$ consists of the vertices w_1, w_2, a_2, b which cannot be saturated by any matching in $G \setminus a_1s_1$. We conclude that there is no such type of ECE-graph.

Subcase 2.2: Suppose that $A' \cong K_{q-1,q-1}$, and there exists $w_i \in N(s_i)$ for each $i = 1, 2$ such that $w_1 \neq w_2$ and $w_1w_2 \notin E(G)$.

Consider the graph $A' \cong K_{q-1,q-1}$ with a bipartition R and T, and let without loss of generality $w_1, w_2 \in R$ and take $w_1', w_2' \in T$. If $a_1a_2 \in E(G)$, then for a perfect matching M in $A' \setminus \{w_1, w_1', w_2, w_2'\}$, the set $M \cup \{s_1w_1, s_2w_2, a_1a_2\}$ is a perfect matching in $G \setminus \{w_1', w_2', b\}$ where $\{w_1', w_2', b\}$ is a stable set, it is a contradiction to the equimatchability of G by Lemma 2.6. Thus $a_1a_2 \notin E(G)$. Besides, we observe that if none of the edges w_1a_1 and w_2a_2 is present, then G is bipartite with bipartition $T \cup \{s_1, s_2\}$ and $R \cup \{a_1, a_2, b\}$. However, bipartite graphs are not factor-critical, a contradiction. So we conclude that one of w_1a_1, w_2a_2 appears in G. If both of them are present, then for a perfect matching M in $A' \setminus \{w_1, w_1', w_2, w_2'\}$, the set $M \cup \{w_1a_1, w_2a_2, bs_2\}$ is a perfect matching in $G \setminus \{w_1', w_2', s_1\}$ where $\{w_1', w_2', s_1\}$ is a stable set, contradicting that G is equimatchable by Lemma 2.6. Therefore precisely one of w_1a_1 or w_2a_2 appears on the graph G, without loss of generality assume $w_1a_1 \notin E(G)$ and $w_2a_2 \in E(G)$. Then, we claim that $a_1s_2 \notin E(G)$, since otherwise $\{a_1s_2, w_1s_1\}$ is a matching isolating b whereas the remain graph $G \setminus \{a_1, s_2, w_1, s_1, b\}$ is not randomly matchable since it contains the edge w_2a_2, a contradiction by Lemma 2.8. Likewise, if s_2 is adjacent to w_1, then $\{a_1s_1, w_1s_2\}$ is a matching isolating b whereas the remain graph $G \setminus \{a_1, s_1, w_1, s_2, b\}$ is not randomly matchable, a contradiction by Lemma 2.8. Therefore $\{w_1, a_1, s_2\}$ is a stable set. Then, for a perfect matching M in $A' \setminus \{w_1, w_1'\}$, the set $M \cup \{w_1'a_2, bs_1\}$ is a perfect matching in $G \setminus \{w_1, a_1, s_2\}$, contradicting that G is equimatchable by Lemma 2.6. Consequently, there is no such type of ECE-graph.

Subcase 2.3: Suppose that $A' \cong K_{q-1,q-1}$, and the vertices s_1, s_2 have a unique neighbour $w_1 = w_2 = w$ in A'.

Consider the graph $A' \cong K_{q-1,q-1}$ with a bipartition R and T, and let $w \in R$. Then, for $i = 1, 2$ each a_i is adjacent to all vertices in T. Besides, if w is adjacent to none of a_1 and a_2, then G induces a bipartite graph with a bipartition $T \cup \{s_1, s_2\}$ and $R \cup \{a_1, a_2, b\}$. However, bipartite graphs are not factor-critical, a contradiction. So there exists at least one edge between w and $\{a_1, a_2\}$, without loss of generality assume $wa_1 \in E(G)$. Then, there is no matching containing a_1w and saturating $N = N(\{a_1, w\})$ since N includes $T \cup \{s_1, s_2\}$ which is an independent set of size $q + 1$ whereas $G \setminus \{a_1, w\}$ has $2q + 1$ vertices, a contradiction to the fact that the edge wa_1 is critical by Lemma 4.1. Hence, s_1 and s_2 has no common neighbour in A'. Consequently, there is no such type of ECE-graph.

Subcase 2.4: Let $A' \cong K_{2q-2}$ for $q \geq 2$.

Recall first that for each $i = 1, 2$, a_i is complete to $A' \setminus \{w_i\}$. Let us first show that $a_1a_2 \notin E(G)$. Indeed, if $a_1a_2 \in E(G)$, then $N(\{s_1, a_1\})$ contains all vertices in $V(G) \setminus \{s_2\}$. It can be observed that there is no matching containing s_1a_1 and saturating $N(\{s_1, a_1\})$, a contradiction to the criticality of the edge s_1a_1 by Lemma 4.1. We next claim that each vertex of s_1, s_2 has a unique neighbour in A', and they are the same. Indeed, if $w_1 \neq w_2$, then $\{s_1w_1, s_2w_2\}$ would be a matching isolating b. However, $A \setminus \{w_1, w_2\}$ is not randomly matchable since $a_1a_2 \notin E(G)$, a contradiction by Lemma 2.8. Hence the vertices s_1, s_2 have a unique neighbour in A', say $w = w_1 = w_2$. If one of the edges a_1w, a_2w is present in G, say $a_1w \in E(G)$, then this edge is not critical since there is no matching containing a_1w and saturating $N(\{a_1, w\})$, a contradiction by Lemma 4.1. Therefore $\{a_1, a_2, w\}$ is a stable set. Lastly, observe that if $A' \setminus \{w\}$ consists of a single vertex, then G induces a bipartite graph. However, bipartite graphs are not factor-critical. Hence $q \geq 3$ and $A' \setminus \{w\} \cong K_{2q-3}$ (see Figure 2(e)). $\square$

By combining Lemmas 4.10 and 4.12, and noting that factor-critical ECE-graphs have at least 7 vertices by Remark 4.8, we obtain the characterization of factor-critical ECE-graphs with connectivity 2 given in Theorem 4.9.

5 An overview of subclasses of equimatchable graphs

Let us give a comparison of VCE-graphs and ECE-graphs with other subclasses of equimatchable graphs that are well-studied in the literature, namely factor-critical equimatchable (EFC) graphs and edge-stable equimatchable (ESE) graphs (see Figure 3). To this end, we define the following disjoint families of graphs:

- $\mathcal{A}$ is the class of EFC-graphs admitting a vertex v such that $G-v$ is isomorphic to K_{2r} for some integer $r \geq 2$ and $2 \leq d(v) \leq 2r-2$.
- $\mathcal{B}$ is the class of EFC-graphs admitting a vertex v such that $G-v$ is isomorphic to $K_{r,r}$ for some integer $r \geq 2$ where v is adjacent to at least two adjacent vertices in $K_{r,r}$, and v has at least one non-neighbor in each one of the (r)-stable sets of $K_{r,r}$.
- $\mathcal{C}$ is the class of graphs which are isomorphic to K_3, K_{2r+1} or $K_{2r+1} \setminus e$ for an edge $e \in E(K_{2r+1})$ for some integer $r \geq 2$.
- $\mathcal{D}$ is the class of EFC-graphs admitting a vertex v such that $G-v$ is isomorphic to $K_{r,r}$ for some integer $r \geq 2$ where v is adjacent to at least two adjacent vertices in $K_{r,r}$, and v is complete to an (r)-stable set of $K_{r,r}$.
- $\mathcal{E}$ is the class of EFC-graphs with a cut vertex.

Note that an EFC-graph must be connected since it is factor-critical. Thus, EFC-graphs consist of two parts: EFC-graphs with a cut vertex and 2-connected EFC-graphs. We observe that all graphs in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ are 2-connected.

We first investigate the relation of the graph class ESE with the above defined graph classes. It has been shown in [6] that ESE-graphs are either 2-connected factor-critical or bipartite.

Proposition 5.1 *We have* $(\mathcal{A} \cup \mathcal{B} \cup \mathcal{E}) \cap \text{ESE} = \emptyset$ *and* $(\mathcal{C} \cup \mathcal{D}) \subseteq \text{ESE}$.

Proof. Let us first show that no graph in $\mathcal{A}\cup\mathcal{B}$ is ESE. Recall that an equimatchable graph G is edge-stable if $G \setminus e$ is also equimatchable for any $e \in E(G)$. In [6], it has been proved that an EFC-graph is edge-stable if and only if there is no induced $\overline{P_3}$ in G such that $G \setminus \overline{P_3}$ has a perfect matching where $\overline{P_3}$ is the complement of a path on 3 vertices. Consider a graph $G \in \mathcal{A}\cup\mathcal{B}$ with the vertex v as described in the definitions of the families $\mathcal{A}$ and $\mathcal{B}$. Then there exist two adjacent non-neighbors x and y of v such that $\{v, x, y\}$ induces a $\overline{P_3}$ in G, and $G \setminus \overline{P_3}$ has a perfect matching (since $G \setminus \overline{P_3}$ is isomorphic to K_{2r-2} or $K_{r-1,r-1}$). Therefore, $(\mathcal{A}\cup\mathcal{B}) \cap \mathrm{ESE} = \emptyset$. Besides, ESE-graphs with a cut-vertex are bipartite [6], thus not factor-critical. It follows that $\mathcal{E} \cap \mathrm{ESE} = \emptyset$.

On the other hand, for $r \geq 2$, a clique K_{2r+1} and $K_{2r+1} \setminus xy$ are ESE by Theorem 12 in [6] as well as K_3 which is an ESE-graph. So every graph in $\mathcal{C}$ is an ESE-graph. Besides, if G is a graph in $\mathcal{D}$ with the vertex v as stated, the graph G has no $\overline{P_3}$ as an induced subgraph since v is complete to an (r)-stable set of $G - v$. Thus, every graph in $\mathcal{D}$ is an ESE-graph. It follows that $(\mathcal{C}\cup\mathcal{D}) \subseteq \mathrm{ESE}$. □

Let us now establish the links between VCE-graphs, ECE-graphs and the families $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ and $\mathcal{E}$. First of all, it is known that both VCE-graphs and ECE-graphs are 2-connected (by Proposition 3.3 and Lemma 4.4 respectively); thus $\mathrm{VCE} \cap \mathcal{E} = \emptyset$ and $\mathrm{ECE} \cap \mathcal{E} = \emptyset$. Moreover, we have clearly $\mathrm{ESE} \cap \mathrm{ECE} = \emptyset$ by definition of these classes.

We next show that the graphs in $\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}\cup\mathcal{D}$ are not ECE.

Proposition 5.2 *Let G be a connected graph with $2r + 1$ vertices for some $r \geq 1$. If G contains one of $K_{2r}, K_{r,r}$ as an induced subgraph, then G is not* ECE*.*

Proof. Let G be a connected graph with $2r+1$ vertices, and suppose that G contains one of $K_{2r}, K_{r,r}$ as an induced subgraph. Then there exists a vertex $v \in V(G)$ such that $G-v$ is isomorphic to K_{2r} or $K_{r,r}$. Since G is connected, the vertex v is adjacent to a vertex u in $V(G - v)$. Note that $G \setminus uv$ contains one of K_{2r} and $K_{r,r}$ as an induced subgraph. So every maximal matching in $G \setminus uv$ is of size r. This means that the edge uv is not critical, thus G is not ECE. □

By Proposition 5.2 and the definitions of the families $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$, we have the following:

Corollary 5.3 $(\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}\cup\mathcal{D}) \cap \mathrm{ECE} = \emptyset$.

Since VCE-graphs are equivalent to 2-connected $(K_{2r}, K_{r,r})$-free EFC-graphs on $2r + 1$ vertices (by Theorem 3.7), and ECE-graphs on $2r + 1$ vertices do not contain K_{2r} or $K_{r,r}$ (by Proposition 5.2), we have the following:

Corollary 5.4 *All factor-critical* ECE*-graphs are* VCE*.*

As we have $(\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}) \cap \mathrm{ECE} = \emptyset$ by Corollary 5.3, Corollary 5.4 implies that $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \subseteq \mathrm{EFC} \setminus \mathrm{VCE}$. It remains to show that VCE is equivalent to the class $\mathrm{EFC} \setminus (\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E})$.

Corollary 5.5 $\mathrm{EFC} \setminus \mathrm{VCE} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E}$.

Proof. Let G be a graph in $\mathrm{EFC} \setminus \mathrm{VCE}$. Then, by Proposition 3.9, G has a vertex v such that every component of $G - v$ is isomorphic to $K_{r,r}$ or K_{2t} for some $r, t \geq 1$ and where v is adjacent to at least two adjacent vertices of $G - v$. If v is a cut-vertex then $G \in \mathcal{E}$. Assume v is not a cut-vertex, then $G - v$ is a connected graph on $2r$ vertices. If $G - v$ is a $K_{r,r}$ for $r \geq 2$ then $G \in \mathcal{B} \cup \mathcal{D}$. If however $G - v$ is a K_{2r} then $G \in \mathcal{A} \cup \mathcal{C}$. It follows that the families $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ form together all 2-connected graphs in $\mathrm{EFC} \setminus \mathrm{VCE}$ as decribed in Proposition 3.9. $\square$

As we stated above, EFC-graphs with a cut vertex are equivalent to the family $\mathcal{E}$ while 2-connected EFC-graphs consist of three disjoint subclasses; $\mathcal{A} \cup \mathcal{B}$, $\mathcal{C} \cup \mathcal{D}$ and VCE-graphs. Let us also recall that ECE-graphs are either factor-critical, or bipartite, or even cliques by Theorem 4.5. This completes the full containment relationship between VCE-graphs, ECE-graphs, ESE-graphs and EFC-graphs as illustrated in Figure 3 where these classes are represented by sets VCE, ECE, ESE and EFC respectively, and FC means factor-critical.

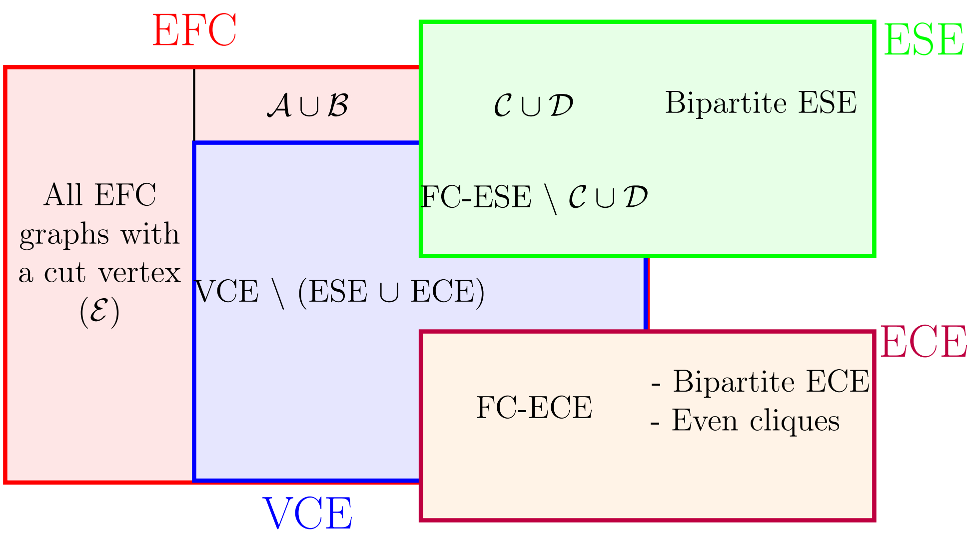

Figure 3: The world of equimatchable graphs.

6 Conclusion

In this paper, we shed light on the structure of equimatchable graphs from a new perspective, namely the criticality with respect to vertex removals and edge removals. We first showed that VCE-graphs boil down to factor-critical equimatchable graphs apart from a few simple exceptions. We also noted that factor-critical ECE-graphs form a subclass of VCE-graphs. This motivated our studies on factor-critical ECE-graphs, whose structure can be analyzed according to their connectivity [10, 11]. We gave a full characterization of factor-critical ECE-graphs with connectivity 2.

We also investigated the case of factor-critical ECE-graphs with connectivity at least 3. We obtained partial results towards their full characterization, leaving some open cases. The reader is referred to the Appendix in [7] for details.

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