AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **87(3)** (2023), Pages 403–422

The rainbow Turán number of P_5

Anastasia Halfpap

Department of Mathematics
Iowa State University
Ames, Iowa 50011, U.S.A.
ahalfpap@iastate.edu

Abstract

An edge-colored graph F is rainbow if each edge of F has a unique color. The rainbow Turán number $\mathrm{ex}^*(n, F)$ of a graph F is the maximum possible number of edges in a properly edge-colored n-vertex graph with no rainbow copy of F. The study of rainbow Turán numbers was introduced by Keevash, Mubayi, Sudakov, and Verstraëte in 2007. In this paper we focus on $\mathrm{ex}^*(n, P_5)$. While several recent papers have investigated rainbow Turán numbers for ℓ-edge paths P_ℓ, exact results have only been obtained for $\ell < 5$, and P_5 represents one of the smallest cases left open in rainbow Turán theory. In this paper, we prove that $\mathrm{ex}^*(n, P_5) \leq \frac{5n}{2}$. Combined with a lower-bound construction due to Johnston and Rombach, this result shows that $\mathrm{ex}^*(n, P_5) = \frac{5n}{2}$ when n is divisible by 16, thereby settling the question asymptotically for all n. In addition, this result strengthens the conjecture that $\mathrm{ex}^*(n, P_\ell) = \frac{\ell}{2}n + O(1)$ for all $\ell \geq 3$.

1 Introduction

An *edge-coloring* c of a graph G with edge set $E(G)$ is a function $c : E(G) \to \mathbb{N}$; for $e \in E(G)$ we call $c(e)$ the *color* of e. We say that an edge-colored graph is *properly edge-colored* if no two incident edges receive the same color, and is *rainbow* if no two edges receive the same color. Often, we build (or infer) a proper edge-coloring of G in steps, by beginning with a proper edge-coloring of a subgraph H of G and then repeatedly selecting (or deducing) the colors of edges in $E(G) \setminus E(H)$. We say that our color selections *obey coloring rules* (or are *legal*) if, for each edge $e \in E(G) \setminus E(H)$, the selected color $c(e)$ is not equal to $c(f)$ for any edge f which is incident to e and already colored. Thus, if we begin with a properly edge-colored subgraph H of G and then select colors for each edge in $E(G) \setminus E(H)$ in a manner which obeys coloring rules, we will return a proper edge-coloring of G.

Given graphs G and F, an *F-copy* in G is a (not necessarily induced) subgraph of G which is isomorphic to F; if G is edge-colored, then a *rainbow F-copy* in G is an

ISSN: 2202-3518

F-copy in G which is rainbow under the given coloring of G. An edge-colored graph is *rainbow-F-free* if it contains no rainbow F-copy.

The *rainbow Turán number* of a fixed graph F is the maximum possible number of edges in a properly edge-colored n-vertex rainbow-F-free graph G. We denote this maximum by $\mathrm{ex}^*(n, F)$, and we say that an n-vertex graph G *achieves* $\mathrm{ex}^*(n, F)$ if G has $\mathrm{ex}^*(n, F)$ edges and there exists a proper edge-coloring of G under which G is rainbow-F-free. The study of rainbow Turán numbers was introduced by Keevash, Mubayi, Sudakov, and Verstraëte [9], and has received considerable attention in the last decade. As a natural fusion of Turán theory and Ramsey-type problems, rainbow Turán problems have become interesting largely in their own right, although certain rainbow Turán problems also have the potential for application, for instance to additive number theory [9].

Observe that $\mathrm{ex}(n, F) \leq \mathrm{ex}^*(n, F)$, since any properly edge-colored F-free graph clearly contains no rainbow F-copy. In fact, it was proved in [9] that for any F,

$$\mathrm{ex}(n, F) \leq \mathrm{ex}^*(n, F) \leq \mathrm{ex}(n, F) + o(n^2).$$

However, for bipartite F, $\mathrm{ex}(n, F)$ and $\mathrm{ex}^*(n, F)$ are not asymptotically equal in general. For example, in [9] it was shown that asymptotically $\mathrm{ex}^*(n, C_6)$ is a constant multiplicative factor larger than $\mathrm{ex}(n, C_6)$. Thus, as in classical Turán theory, the difficult question is to determine rainbow Turán numbers for bipartite graphs. Various such problems have received attention; see, for example [2, 6, 7, 8]. Here, we focus on rainbow Turán numbers of paths.

We denote by P_ℓ the path on ℓ edges, i.e., $\ell + 1$ vertices. The study of $\mathrm{ex}^*(n, P_\ell)$ was first suggested by Keevash et al. [9], and has been considered by a number of authors, but previously has been determined asymptotically only for $\ell \leq 4$. For $\ell = 1$ and $\ell = 2$, the result $\mathrm{ex}^*(n, P_\ell) = \mathrm{ex}(n, P_\ell)$ is trivial, since any properly colored P_1 or P_2-copy is rainbow. For $\ell = 3$ and $\ell = 4$, Johnston, Palmer, and Sarkar [7] determined $\mathrm{ex}^*(n, P_\ell)$ asymptotically for all n, with exact values given certain divisibility criteria.

Theorem 1.1 (Johnston, Palmer, and Sarkar [7]). *If n is divisible by 4, then*

$$\mathrm{ex}^*(n, P_3) = \frac{3}{2}n.$$

If n is divisible by 8, then

$$\mathrm{ex}^*(n, P_4) = 2n.$$

We note that Keevash et al. [9] made the natural conjecture that $\mathrm{ex}^*(n, P_\ell)$ is achieved by disjoint copies of the clique on $c(\ell)$ vertices, where $c(\ell)$ is the largest constant such that $K_{c(\ell)}$ can be properly edge-colored without creating a rainbow P_ℓ-copy. However, Theorem 1.1 disproves this conjecture: since any proper edge-coloring of K_5 contains a rainbow P_4, the conjecture would be that $\mathrm{ex}^*(n, P_4)$ is asymptotically equal to $\frac{3}{2}n$, achieved by taking disjoint copies of a properly edge-colored K_4. Beyond Theorem 1.1, tight results on rainbow Turán numbers for paths have remained elusive, although various authors have made improvements to general bounds (see [3, 8]).

This history leaves P_5 as one of the smallest graphs whose rainbow Turán number has not been determined (the other notable example being C_4). Previously, the best known bounds on $\mathrm{ex}^*(n, P_5)$ were due to Johnston and Rombach [8] and Halfpap and Palmer [5], respectively.

Theorem 1.2 (Johnston, Rombach [8]; Halfpap, Palmer [5])**.**

$$\frac{5}{2}n + O(1) \leq \mathrm{ex}^*(n, P_5) \leq 4n.$$

The lower bound in Theorem 1.2 is a special case of a lower bound due to Johnston and Rombach [8], which is currently the best known in general. The upper bound in Theorem 1.2 is obtained by case analysis which does not easily generalize to longer paths; the best known general upper bound on $\mathrm{ex}^*(n, P_\ell)$ is due to Ergemlidze, Győri and Methuku [3].

Theorem 1.3 (Johnston, Rombach [8]; Ergemlidze, Győri, and Methuku [3])**.** *For $\ell \geq 3$,*

$$\frac{\ell}{2}n + O(1) \leq \mathrm{ex}^*(n, P_\ell) \leq \left(\frac{9\ell + 5}{7}\right) n.$$

The lower bound on $\mathrm{ex}^*(n, P_\ell)$ from [8] is achieved by taking disjoint copies of the following construction.

Construction 1. Let $Q_{\ell-1}$ be the $\ell - 1$ dimensional cube, i.e., the graph whose vertex set is the set of 01-strings of length $\ell - 1$ and two vertices are joined by an edge if and only if their Hamming distance is exactly 1.

We color the edges of $Q_{\ell-1}$ by the position in which their corresponding strings differ. For each vertex x of $Q_{\ell-1}$, let $\overline{x}$ be the *antipode* of x. That is, $\overline{x}$ is the unique vertex of Hamming distance $\ell - 1$ from x. Now add all edges $x\overline{x}$ to this graph and color these edges with a new color ℓ. Call these edges *diagonal* edges and denote the resulting edge-colored graph $D^*_{2^{\ell-1}}$. The underlying (uncolored) graph of $D^*_{2^{\ell-1}}$ is often referred to as a *folded cube graph.*

For a distinct construction due to Maamoun and Meyniel showing that $\mathrm{ex}^*(n, P_\ell) \geq \frac{\ell}{2}n + O(1)$ when ℓ is of the form $2^k - 1$, see [10].

Note that the lower bound from Theorem 1.3 is shown to be tight by Theorem 1.1 when $n = 3, 4$. While these small cases provide limited data, they suggest the following.

Conjecture 1.4. *For all $\ell \geq 3$, $\mathrm{ex}^*(n, P_\ell) = \frac{\ell}{2}n + O(1)$.*

The goal of this paper is to prove an upper bound on $\mathrm{ex}^*(n, P_5)$ which asymptotically matches the lower bound from Theorem 1.3, thereby adding further weight to the conjecture that this lower bound is asymptotically correct for $\ell > 2$.

Theorem 1.5. $\frac{5n}{2} + O(1) \leq \mathrm{ex}^*(n, P_5) \leq \frac{5n}{2}$.

Johnston and Rombach [8] also considered a rainbow version of the generalized Turán problems popularized by Alon and Shikhelman [1], which will be relevant to our approach. For fixed graphs H and F, let $\mathrm{ex}^*(n, H, F)$ denote the maximum possible number of rainbow H-copies in an n-vertex properly edge-colored graph which is rainbow-F-free. (For a different formulation combining rainbow Turán and generalized Turán problems, see [4].) We say that an n-vertex graph G *achieves* $\mathrm{ex}^*(n, H, F)$ if there exists a proper edge-coloring of G under which G is rainbow-F-free and contains $\mathrm{ex}^*(n, H, F)$ rainbow H-copies.

It may not be obvious that generalized rainbow Turán numbers have any connection to the solution of specific rainbow Turán problems. However, careful consideration of an appropriate generalized rainbow Turán number may yield insight into a corresponding "ordinary" rainbow Turán number, and vice versa. In one direction, examination of known extremal constructions for rainbow Turán problems may yield conjectures regarding certain generalized rainbow Turán numbers. For example, note that the known extremal constructions avoiding rainbow P_3 and P_4-copies contain many rainbow C_3 and C_4-copies. Intuitively, we expect a rainbow-P_ℓ-free graph with high average degree to contain many short rainbow cycles, since long rainbow walks are unavoidable. So, it is reasonable to conjecture that graphs achieving $\mathrm{ex}^*(n, P_\ell)$ also achieve $\mathrm{ex}^*(n, C_\ell, P_\ell)$. Indeed, Halfpap and Palmer [5] showed that the folded cube construction, which is known to achieve $\mathrm{ex}^*(n, P_\ell)$ for $\ell \in \{3, 4\}$, also achieves $\mathrm{ex}^*(n, C_\ell, P_\ell)$ for $\ell \in \{3, 4, 5\}$.

Theorem 1.6 (Halfpap, Palmer [5]). *Let $\ell \in \{3, 4, 5\}$. Then, when n is divisible by $2^{\ell-1}$, we have*

$$\mathrm{ex}^*(n, C_\ell, P_\ell) = \frac{(\ell-1)!}{2} n;$$

moreover, when n is divisible by $2^{\ell-1}$, the graph consisting of $\frac{n}{2^{\ell-1}}$ disjoint copies of $D^*_{2^{\ell-1}}$ achieves $\mathrm{ex}^*(n, C_\ell, P_\ell)$.

While Theorem 1.6 was suggested by consideration of known extremal constructions for $\mathrm{ex}^*(n, P_3)$ and $\mathrm{ex}^*(n, P_4)$, it was also motivated by the desire to understand $\mathrm{ex}^*(n, P_\ell)$ more deeply. The key step in the determination of $ex^*(n, P_5, C_5)$ was the proof of the following fact, which we present here as a separate lemma, and which will be crucial to our determination of $\mathrm{ex}^*(n, P_5)$.

Lemma 1.7 (Halfpap, Palmer [5]). *Let G be an n-vertex, properly edge-colored graph which is rainbow-P_5-free. Let $V' \subseteq V(G)$ be the set of vertices of G which are contained in at least one rainbow C_5-copy. Then*

$$\frac{\sum_{v \in V'} d(v)}{|V'|} \leq 5.$$

Now, Lemma 1.7 suggests an approach to finding $\mathrm{ex}^*(n, P_5)$. While we postpone a technical discussion of the approach to Section 2, note that Theorem 1.5 is equivalent to the statement that any n-vertex, properly edge-colored, rainbow-P_5-free graph has average degree at most 5. Since Lemma 1.7 already establishes an average degree bound for vertices which are contained in some rainbow C_5-copy, we shall be able to focus on vertices which are not contained in any rainbow C_5-copy.

The paper is organized as follows. In Section 2, we give technical details outlining the overall strategy of the paper, and prove a number of structural lemmas which will be required for our main result, Theorem 1.5. In Section 3, we employ these lemmas to prove Theorem 1.5. In Section 4, we offer some brief concluding remarks on the proof of Theorem 1.5 and some related open problems.

2 Preliminaries and Structural Lemmas

In this section, we give a detailed overview of the strategy which will be employed to prove Theorem 1.5, and establish a number of lemmas which will facilitate the proof of Theorem 1.5. We begin by setting up the graph with which we work in the remainder of the paper.

Denote by $d(G)$ and $\delta(G)$ the average degree and minimum degree, respectively, of a graph G. Note that, to prove Theorem 1.5, it would suffice to show that if G is an n-vertex, properly edge-colored, rainbow-P_5-free graph, then $d(G) \leq 5$. Indeed, this is the statement which we work to show.

Fix $n \geq 1$ and let G_0 be a properly edge-colored, n-vertex graph achieving $\mathrm{ex}^*(n, P_5)$. If $e(G_0) \leq \frac{5n}{2}$, then Theorem 1.5 holds, so for the sake of contradiction we assume that $e(G_0) > \frac{5n}{2}$. Under this assumption, $d(G_0) > 5$. We will modify G_0 by repeatedly pruning vertices of degree 1 or 2. By a standard argument, pruning in this way yields a subgraph with minimum degree at least 3, whose average degree is at least that of G_0. From this subgraph, we will moreover delete any components which have average degree at most 5. Denote by G the resulting subgraph of G_0. Observe that

$$d(G) \geq d(G_0) > 5.$$

Throughout, we work with this graph G. We will ultimately achieve a contradiction by arguing that in fact, $d(G) \leq 5$, completing the proof of Theorem 1.5.

Our strategy will be similar to that used to prove Lemma 1.7 (for details, see the proof of Theorem 3.5 in [5]). Lemma 1.7 is proved through a "vertex pairing" argument. Consider a properly edge-colored graph G which is rainbow-P_5-free. Partition the vertex set V of G as $V = V' \cup \overline{V'}$, where V' is the set of vertices of G which lie in some rainbow C_5-copy under the given edge-coloring. For a rainbow C_5-copy C in G, we let

$$S = \{v \in V(C) : d(v) \geq 6\}$$

and find $T \subseteq V(C) \setminus S$ such that there exists a matching between S and T with the property that if $v \in S$ and $u \in T$ are matched, then $\frac{d(v)+d(u)}{2} \leq 5$. We also require that if $u \in T$ has a neighbor w with $d(w) \geq 6$, then $w \in S$. We call the described set of vertices an S, T *pair*. It is clear that the average degree across an S, T pair is at most 5; the condition on the neighbors of vertices in T is necessary to conclude that the average degree across the union of all S, T pairs in V' is also at most 5. Since every vertex of degree greater than 5 is contained in an S, T pair, it follows that the average degree in V' is at most 5.

We shall use a similar approach to estimate the average degree of vertices in $\overline{V'}$. We shall say that $v \in \overline{V'}$ is a *high-degree* vertex if $d(v) \geq 6$, and is a *low-degree* vertex otherwise. For $v \in V$, let $N(v)$ denote the neighborhood of v. For each high-degree vertex v in $\overline{V'}$, the goal is to find a low-degree subset $L(v)$ of $N(v)$ to pair with v. Precisely, we define a *local pairing in* $\overline{V}'$ to be a set $\{v\} \cup L(v)$, where $d(v) \geq 6$, $L(v) \subseteq N(v)$, $\{v\} \cup L(v) \subseteq \overline{V}'$, and

$$\frac{d(v) + \sum_{u \in L(v)} d(u)}{|L(v)| + 1} \leq 5.$$

Thus, the average degree in a single local pairing is at most 5. The crux of the argument is to find a set of pairwise disjoint local pairings in $\overline{V'}$ which contain all high-degree vertices of $\overline{V'}$; this immediately implies that the average degree in $\overline{V'}$ is at most 5. The substance of Theorem 1.5 is to demonstrate the existence of such local pairings in $\overline{V'}$. However, to set up this argument, we shall first require a variety of lemmas on the structure of $\overline{V'}$.

The first step is to show that, if $v \in \overline{V'}$ has $d(v) \geq 6$, then the neighbors of v are also in $\overline{V'}$. Since we will eventually pair v to its neighbors, this will ensure that we are pairing v only to other vertices in $\overline{V'}$.

Lemma 2.1. *If $v \in \overline{V'}$ has $d(v) \geq 6$, then $N(v) \subset \overline{V'}$.*

Proof. Suppose for a contradiction that v has a neighbor, u, which is in V'. So u lies on a rainbow C_5-copy, which we shall call C. We may assume that the edges of C are colored from $\{1, 2, 3, 4, 5\}$, as pictured in Figure 1.

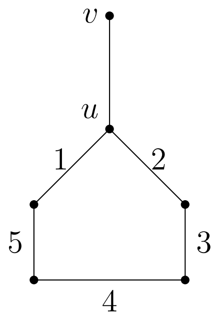

Figure 1

Observe that there is no edge with precisely one endpoint incident to C that is colored with a color not in $\{1, 2, 3, 4, 5\}$, as this immediately creates a rainbow P_5-copy. Observe also that since $d(v) \geq 6$, v is incident to an edge which is not colored from $\{1, 2, 3, 4, 5\}$. We call the color on this edge 6, and conclude that the other

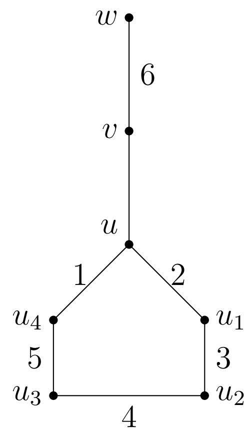

Figure 2

endpoint, say w, of this edge is not depicted in Figure 1. So the situation is as in Figure 2. We will also add, in Figure 2, labels to the remaining vertices on C.

Note now that $c(uv)$ must equal 4, or else either of $u_3u_2u_1uvw$ or $u_2u_3u_4uvw$ is a rainbow P_5-copy.

Now, we shall examine w. Recall that $\delta(G) \geq 3$. We will aim to arrive at a contradiction by showing that $d(w) \leq 2$.

First, we shall observe that w can be adjacent to no vertex of C. Indeed, if wu is an edge, then $c(wu) \in \{1, 2, 3, 4, 5\}$ to avoid an immediate rainbow P_5-copy. Since the coloring must be proper, this means that $c(uw) \in \{3, 5\}$. But either choice produces a rainbow P_5-copy (either $vwuu_4u_3u_2$ or $vwuu_1u_2u_3$). So wu is not an edge.

Next, observe that if wu_1 is an edge, then it must be colored 5, since otherwise either $vwu_1uu_4u_3$ or $vwu_1u_2u_3u_4$ is a rainbow P_5-copy. But if $c(wu_1) = 5$, then $u_4uvwu_1u_2$ is a rainbow P_5-copy. So wu_1 is not an edge. Analogously, wu_4 is not an edge.

Finally, if wu_2 is an edge, then it must be colored 1, since otherwise either $vwu_2u_3u_4u$ or $vwu_2u_1uu_4$ is a rainbow P_5-copy. But if $c(wu_2) = 1$, then wu_2u_1uvw is a rainbow C_5-copy containing v, a contradiction, since we assume $v \in \overline{V'}$. So wu_2 is not an edge. Analogously, wu_3 is not an edge.

Thus, w is not adjacent to any vertex on C. Moreover, if x is a vertex not yet considered such that wx is an edge, then observe that $c(wx)$ must equal 4, or else either $xwvuu_1u_2$ or $xwvuu_4u_3$ is a rainbow P_5-copy. Thus, to avoid a rainbow P_5-copy, we must have $d(w) \leq 2$, contradicting the minimum degree condition on G.

We conclude that v is not adjacent to any vertex in V'. □

We will also make use of the following lemma, which further restricts the subgraphs in which high-degree vertices of $\overline{V'}$ may appear. In a P_ℓ-copy P, the *endpoints* of P are the two vertices whose degree in P is 1.

Lemma 2.2. *Suppose $v \in \overline{V'}$ has $d(v) \geq 6$. Then v is not an endpoint of a rainbow P_4-copy.*

Proof. Suppose that $v \in \overline{V'}$ has $d(v) \geq 6$ and is an endpoint of a rainbow P_4-copy, say P. Our ultimate strategy is to obtain a contradiction by arguing that, to avoid a rainbow P_5-copy, G must contain a vertex of degree strictly less than 3. In order to identify this low-degree vertex, we will first need to perform some case analysis to understand the structure of G near v. We begin by drawing P in Figure 3, also labeling the other vertices and indicating edge colors.

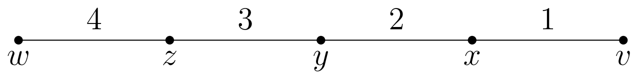

Figure 3

Now, $d(v) \geq 6$, so v is incident to at least two edges which are not colored from $\{1, 2, 3, 4\}$. Clearly, both endpoints of these edges must be on P to avoid a rainbow P_5-copy. Also, since we assume that v is not in a rainbow C_5-copy, vw cannot be such an edge. So, in fact, the two edges which are incident to v and not colored from $\{1, 2, 3, 4\}$ must be vy and vz. Without loss of generality, $c(vy) = 5$ and $c(vz) = 6$. Every other edge incident to v must be colored from $\{1, 2, 3, 4\}$; in particular, to achieve $d(v) \geq 6$, v must be incident to an edge colored 4. The other endpoint of this edge cannot be on P, since v is already adjacent to x, y, z via edges of different colors, and w is already incident to another edge of color 4. So v is incident to a vertex not yet drawn, say u, with $c(vu) = 4$. We update our drawing in Figure 4.

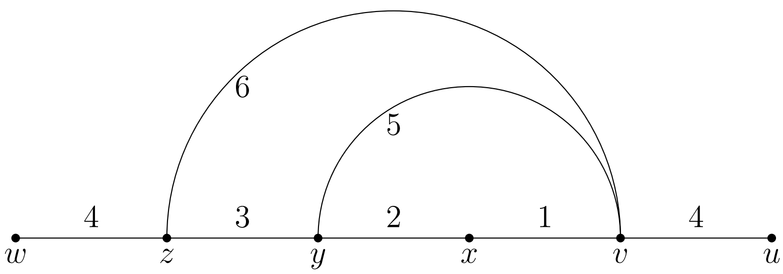

Figure 4

We will now investigate potential neighbors of w. Firstly, we claim that w can be adjacent only to vertices already depicted in Figure 3. Indeed, suppose w' is a vertex not yet represented, and that ww' is an edge. Obeying coloring rules, we must have $c(ww') \in \{1, 2, 3\}$, or else $vxyzww'$ is a rainbow P_5-copy. However, whichever of these colors we chose, either $xyvzww'$ or $xvyzww'$ is rainbow. Thus, $N(w) \subseteq \{x, y, z, v, u\}$. Since none of $vxyzww'$, $xyvzww'$, $xvyzww'$ use the vertex u, these observations also imply that wu is not an edge.

We also claim that wx is not an edge. Indeed, suppose edge wx is present. Then $vyxwzv$ is a C_5-copy containing v, so must not be rainbow. Thus, $c(wx)$ must be in $\{5, 6\}$ to obey coloring rules and avoid a rainbow C_5-copy. But then either $xvyzwx$ is a rainbow C_5-copy containing v, or $uvzyxw$ is a rainbow P_5-copy. We conclude that wx is not an edge.

Thus, the only possible neighbors of w are z, y, and v. Since $\delta(G) \geq 3$, w must be adjacent to all three of these vertices to avoid a contradiction. We can check that to obey coloring rules and avoid rainbow C_5-copies containing v, we must have $c(wv) \in \{2,3\}$ and $c(wy) \in \{1,6\}$. We will set $c(wy) = a$ and $c(wv) = b$.

Note also that we have now accounted for five neighbors of v; another must exist, and must be a vertex not depicted in Figure 4. We shall call this neighbor s. To ensure that $svxyzw$ is not a rainbow P_5-copy, we must have $c(sv) \in \{1,2,3,4\}$. By coloring rules, $c(sv) \in \{2,3\}$. We shall set $c(sv) = c$. Note also that $b \neq c$, so if one of b, c is determined, then the other is also.

We shall reflect our progress in Figure 5.

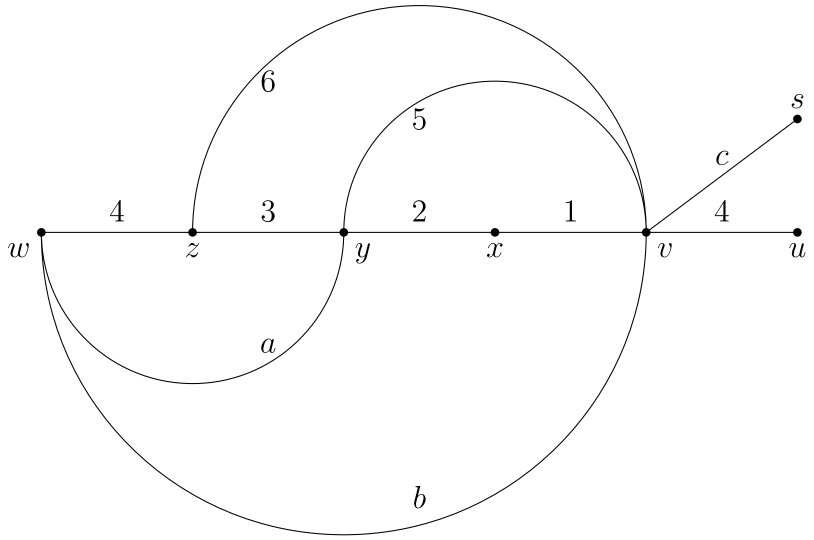

Figure 5

We shall next examine x. We begin by observing that x is not a neighbor of z or u. Recall that we have already shown that x is not a neighbor or w.

Suppose xz is an edge. By coloring rules, we either have $c(xz) = 5$ or $c(xz)$ is a new color, say 7. Therefore, $uvxzyw$ is a rainbow P_5-copy unless $a = 1$. Given $a = 1$, we have that $svzxyw$ is a rainbow P_5-copy unless $c = 2$. This implies $b = 3$. But now $uvwyxz$ is a rainbow P_5-copy. We conclude that xz is not an edge.

Next, suppose xu is an edge. By coloring rules, either $c(xu) \in \{3,5,6\}$ or $c(xu)$ is a new color, say 7. Observe that if $c(xu) \in \{3,7\}$, then $wzvyxu$ is a rainbow P_5-copy, and if $c(xu) \in \{6,7\}$, then $uxvyzw$ is a rainbow P_5-copy. Finally, if $c(xu) = 5$, then $vuxyzv$ is a rainbow C_5-copy containing v. We conclude that xu is not an edge.

Now, if $d(x) \geq 3$, then either xs is an edge, or x has a neighbor, say t, which is not depicted in Figure 5. We examine the cases separately; in each, we show that to avoid a rainbow P_5-copy, G must contain a vertex of degree at most 2.

Case 1: xs is an edge.

Observe that $c(xs) = 4$, else one of $sxvyzw$, $sxyvzw$, $sxyzvu$ is a rainbow P_5-copy. Consider potential neighbors of s. Observe that su is not an edge, since if so,

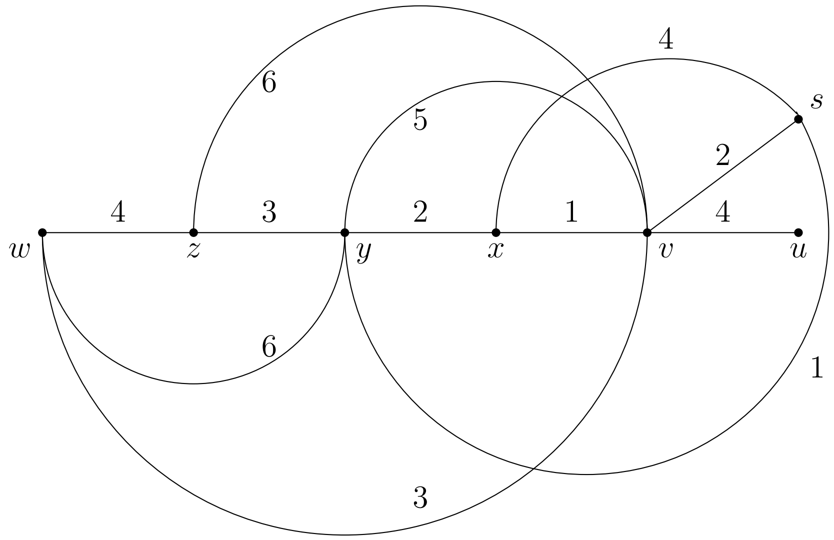

Figure 6

one of $usxyvz$, $usxvyz$, $usxyzv$ is a rainbow P_5-copy. Analogously, s can be adjacent to no vertex which is not depicted in Figure 5. We have also argued previously that w is only adjacent to z, y, and v, so sw is not an edge.

We next consider z. If sz is an edge, observe that $c(sz) \in \{1, 2\}$, since $uvxyzs$ is a rainbow P_5-copy under any other legal color assignment. Moreover, if $c(sz) = 2$, then $c = 3$, and $zsvywz$ is a rainbow C_5-copy containing v. So $c(sz) = 1$. Now, consider the paths $szwvyx$ and $xyvszw$. The colors on these paths are, respectively, $1, 4, b, 5, 2$ and $2, 5, c, 1, 4$. Recall that $b \in \{2, 3\}$ and $c \in \{2, 3\}$; moreover, $b \neq c$. The first path is rainbow unless $b = 2$, and the second path is rainbow unless $c = 2$; since both cannot simultaneously be true, we conclude that one path is a rainbow P_5-copy. Thus, sz is not an edge.

Now, to avoid a contradiction to the minimum degree condition on G, sy must be an edge. We first claim that $c(sy) = 1$. Observe that both $xvsyzw$ and $zysxvw$ are P_5-copies, respectively colored $1, c, c(sy), 3, 4$ and $3, c(sy), 4, 1, b$. Recall that either $b = 2$ or $c = 2$, so the colors on one of these two paths are (not in order) $1, 2, 3, 4, c(sy)$. By coloring rules, $c(sy)$ is not in $\{2, 3, 4\}$ (since $c(yx) = 2, c(yz) = 3$, and $c(sx) = 4$), so we must have $c(sy) = 1$. Since $c(sy) \neq a$ by coloring rules, and $a \in \{1, 6\}$, this means $a = 6$. Now, $zywvxs$ is a P_5-copy colored $3, 6, b, 1, 4$, so we must have $b = 3$ to avoid a rainbow P_5-copy, forcing $c = 2$.

Thus, if xs is an edge, then sy must also be an edge, and we can fix the colors of all edges indicated in Figure 5. We reflect this state of affairs in Figure 6.

Given the configuration in Figure 6, we shall observe that u cannot satisfy the minimum degree condition. We have seen already that uw and ux are not edges, and have seen that, given xs is an edge, su is not an edge. Observe that u is adjacent to no vertex not yet drawn (say t), since if so, one of $tuvxyz$, $tuvsyw$, $tuvwys$, $tuvwyx$ is a rainbow P_5-copy. Observe also that u cannot be adjacent to y; if so, by coloring rules, $c(uy)$ must be a color not yet used, say 7, and then $uysvzw$ is a rainbow

P_5-copy. Thus, $d(u) \leq 2$.

Case 2: xs is not an edge; thus, x has a neighbor t which is not depicted in Figure 5.

Observe that $c(xt) = 4$, else one of $txvyzw$, $txyvzw$, $txyzvu$ is a rainbow P_5-copy. Therefore, G contains the subgraph drawn in Figure 7.

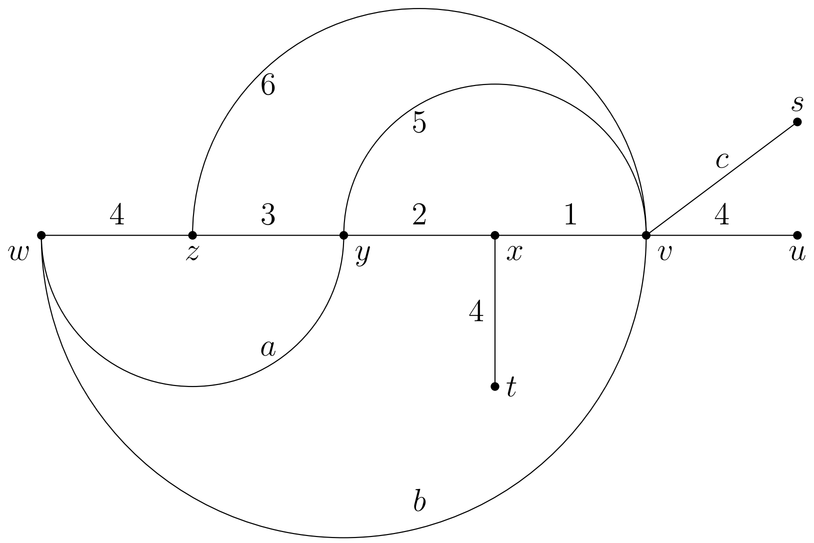

Figure 7

We argue that $d(t) \leq 2$.

Observe that t is not adjacent to any vertex which is not yet drawn (say r), else one of $rtxyvz$, $rtxvyz$, $rtxvzy$ is a rainbow P_5-copy. Analogously, t is not adjacent to s or u. We have already seen that w is not adjacent to t (or indeed, to any vertex not in $\{z, y, v\}$). v already has incident edges of every color from $\{1, 2, 3, 4, 5, 6\}$ (since $b, c \in \{2, 3\}$), so tv cannot be an edge, as it would receive a new color, say 7, making $tvxyzw$ a rainbow-P_5 copy. So, the only vertices to which t can be adjacent (aside from x) are y and z.

Suppose ty is an edge. By coloring rules, $c(ty)$ is not in $\{a, 2, 3, 4\}$. Observe that $svwytx$ is now a P_5-copy, with colors $c, b, a, c(ty), 4$. We know that b, c are in $\{2, 3\}$ and are not equal, and that $a \in \{1, 6\}$ is not equal to $c(ty)$. So, $c, b, a, c(ty)$, and 4 must all be distinct colors, and thus ty is not an edge, as its presence yields a rainbow P_5-copy. Thus, $d(t) \leq 2$, since its only neighbors are x and possibly z.

Thus, to avoid a rainbow P_5-copy in G, either $d(x) \leq 2$ or x has a neighbor of degree at most 2. In any case, we achieve a contradiction to the minimum degree hypothesis on G. We conclude that, if $v \in \overline{V'}$ has $d(v) \geq 6$, then v is not the endpoint of a rainbow P_4-copy. $\square$

From Lemma 2.2, we can quickly derive the following corollary, which will also be of use in our proof of the main result. The *distance* between vertices x, y in a graph G is the smallest ℓ such that G contains a P_ℓ-copy with endpoints x, y. (If no such ℓ exists, we say that x, y are at infinite distance.)

Corollary 2.3. *Suppose $v \in \overline{V'}$ has $d(v) \geq 6$ and u is a vertex at distance 2 from v with $d(u) \geq 6$. Then u is also in $\overline{V'}$.*

Proof. Suppose for a contradiction that u is contained in a rainbow C_5-copy, C. By assumption, there exists a P_2-copy connecting v and u, which is necessarily rainbow. We shall label the edges of P with colors $1, 2$, and the edges of C with colors a, b, c, d, e. Note that, since $d(u) \geq 6$, u must be incident to an edge of a color f which is not contained in $\{a, b, c, d, e\}$. The other endpoint of this edge must be on C, or a rainbow P_5-copy is immediately created. Recall also that, by Lemma 2.1, v is not adjacent to any vertex of C, so in particular, w is not contained in C. Thus, P and C intersect only in u. We depict this situation in Figure 8, adding labels to previously unnamed vertices for convenience.

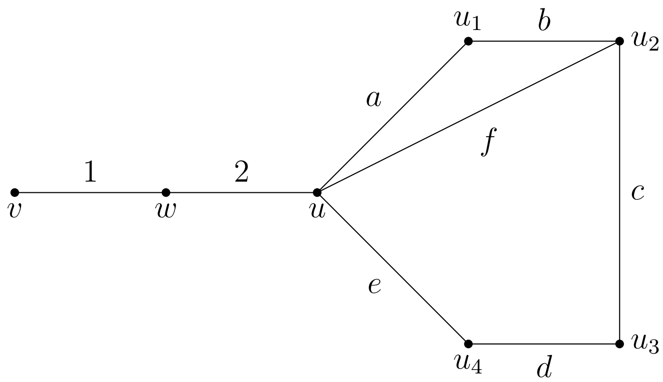

Figure 8

Now, since C is rainbow and f is distinct from a, b, c, d, e, at most two of a, b, c, d, e, f are in $\{1, 2\}$. Thus, one of $vwuu_1u_2$, $vwuu_2u_3$, $vwuu_4u_3$ is a rainbow P_4-copy ending at v, a contradiction by Lemma 2.2. $\square$

Finally, we will need the following result. Although the next lemma holds for any $v \in V$, we will apply it in particular to vertices of high degree in $\overline{V'}$, in order to more easily build rainbow P_4-copies.

Lemma 2.4. *Let $v \in V$. Then v is the endpoint of a rainbow P_3-copy.*

Proof. Suppose $v \in V$ is not the endpoint of a rainbow P_3-copy. We claim that the following holds. If u is a neighbor of v, then $d(u) = 3$ and $N(u) \subseteq N(v) \cup \{v\}$. If we can establish this claim, then the lemma is proved, as follows. Given that for any $u \in N(v)$, we have $N(u) \subseteq N(v) \cup \{v\}$, we observe that $\{v\} \cup N(v)$ must induce a component of G. Moreover, since every vertex in $N(v)$ has degree 3, the average degree in this component is

$$\frac{d(v) + 3d(v)}{d(v) + 1} < 5,$$

a contradiction, as every component of G has average degree greater than 5. Thus, we will have that every $v \in V$ is the endpoint of a rainbow P_3-copy.

To establish the claim, suppose $v \in V$ is not the endpoint of a rainbow P_3-copy, and let u be a neighbor of v. Without loss of generality, $c(uv) = 1$. Since $\delta(G) \geq 3$, u has another neighbor, say x, and $c(ux)$ cannot equal $c(uv)$. Say $c(ux) = 2$. Now, x has at least two more neighbors, so must have at least one neighbor not equal to v, say y. Since we assume v is not the endpoint of a rainbow P_3-copy, we must have $c(xy) = 1$. Observe that x can have no other neighbor except v without creating a rainbow P_3-copy ending in v, so xv must be an edge. We must have $c(xv) = 3$, since x is already incident to edges of colors 1 and 2.

We consider the neighbors of u; there must be at least one more. Either uy is an edge, or u is adjacent to some vertex w not already considered. We wish to show that there is no such edge uw. If there is, then $c(uw) = 3$ to avoid a rainbow P_3-copy ending at v. Thus, u is adjacent to only one new vertex w. Now, uy cannot be an edge, since then $c(uy)$ would be a new color, say 4, which would create a rainbow-P_3 copy ending at v. So y is adjacent to two more vertices, neither of which are u. If y is adjacent to a vertex z not already considered, then we must have $c(yz) = 3$ to avoid a rainbow P_3-copy ending at v. Note that y cannot be adjacent to w, as $c(yw)$ would be 2 or 4, creating a rainbow P_3-copy ending at v. So to achieve degree 3, y must be adjacent to v and a new vertex, z. We have $c(yz) = 3$, and must have $c(yv) = 2$ to avoid a rainbow P_3-copy ending at v.

Now, consider w. We have already noted that wy is not an edge; nor are wv or wx, since these would necessarily receive a new color and thus create a rainbow P_3-copy ending at v. Any legal coloring of wz also creates a rainbow P_3-copy ending at v, so neither is wz an edge. Finally, if w is adjacent to a new vertex, say s, then $c(ws) = 1$ to avoid a rainbow P_3-copy ending at v, so w has at most one neighbor not already considered. But this implies that $d(w) \leq 2$, a contradiction. We conclude that uw is not an edge.

Thus, to achieve degree 3, u is adjacent to y, and $c(uy) = 3$ is forced. Given this edge, y must be adjacent to v to achieve degree 3, with $c(yv) = 2$. Now, v, u, x, y form a properly colored K_4. It is clear that if any of u, x, y have another neighbor, the incident edge will create a rainbow P_3-copy ending at v, so in particular, $N(u) = \{v, x, y\} \subset N(v) \cup \{v\}$, and $d(u) = 3$. Since u was chosen from $N(v)$ arbitrarily, we are done. $\square$

3 Main Result

We are now ready to prove our main result, Theorem 1.5. We recall the statement for convenience. As in Section 2, we may assume that we work in a properly edge-colored, rainbow-P_5-free graph G with $\delta(G) \geq 3$ and such that every component of G has average degree greater than 5. The vertex set of G is again partitioned as $V' \cup \overline{V'}$, where V' is the set of vertices which lie in some rainbow C_5-copy in G.

Theorem 1.5. $\frac{5n}{2} + O(1) \leq \mathrm{ex}^*(n, P_5) \leq \frac{5n}{2}$.

Proof. Our goal is to show, for a contradiction, that $d(G) \leq 5$. By Lemma 1.7, we know that the average degree in V' is at most 5, so we will be done if we can also show that $\overline{V'}$ has average degree at most 5. Recall, we define a *local pairing* in $\overline{V'}$ to be a set $\{v\} \cup L(v)$, where $d(v) \geq 6$, $L(v) \subseteq N(v)$, $\{v\} \cup L(v) \subseteq \overline{V}'$, and

$$\frac{d(v) + \sum_{u \in L(v)} d(u)}{|L(v)| + 1} \leq 5.$$

Let us also define $H(\overline{V'}) := \{v \in \overline{V'} : d(v) > 5\}$. For each $v \in H(\overline{V'})$, we aim to find a set $L(v) \subseteq N(v)$ such that the following hold:

1. $\{v\} \cup L(v)$ is a local pairing in $\overline{V'}$
2. For any $u \in L(v)$, we have $d(u) \leq 5$ and $N(u) \cap H(\overline{V'}) = \{v\}$.

Suppose that for every $v \in H(\overline{V'})$, we can find such a set $L(v)$. Then it immediately follows that $\overline{V'}$ has average degree at most 5. Indeed, by the definition of a local pairing in $\overline{V'}$, condition (1) gives that each $\{v\} \cup L(v)$ is a subset of $\overline{V'}$ with average degree at most 5. By condition (2), each vertex in $\overline{V'}$ appears in at most one local pairing $\{v\} \cup L(v)$. Thus, setting S to be the set of vertices in $\overline{V'}$ which are not contained in any of the selected local pairings in $\overline{V'}$, we have

$$\begin{aligned} \sum_{v \in \overline{V'}} d(v) &= \sum_{v \in H(\overline{V'})} \left(d(v) + \sum_{u \in L(v)} d(u) \right) + \sum_{v \in S} d(v) \\ &\leq \sum_{v \in H(\overline{V'})} 5(|L(v)| + 1) + 5|S| \;=\; 5|\overline{V'}|. \end{aligned}$$

It thus only remains to show that such a set $L(v)$ can be found for every vertex $v \in H(\overline{V'})$. By Lemma 2.1, we have that $\{v\} \cup N(v) \subseteq \overline{V'}$ for any $v \in H(\overline{V'})$, so any potential local pairing will indeed be contained in $\overline{V'}$. Fix $v \in H(\overline{V'})$ and consider $N(v)$. We distinguish two cases. Throughout both, recall that by Lemma 2.2, we may assume that v is not the endpoint of a rainbow P_4-copy.

Case 1: v lies in a rainbow C_4-copy, C.

We shall label the vertices of C, so that $C = vxyzv$, and the edge colors on C are from $\{1, 2, 3, 4\}$. Since $d(v) \geq 6$, v is incident to at least two edges which are not colored from $\{1, 2, 3, 4\}$. One of these edges may be incident to y, the vertex on C to which we have not already specified an adjacency, but one must be incident to a new vertex, say w. We draw the situation in Figure 9.

Consider the possible neighbors of w. If w is adjacent to x, then $c(wx)$ must be 3, else $vwxyz$ is a rainbow P_4-copy ending at v. Similarly, if wz is an edge, then $c(wz) = 2$. If w is adjacent to a new vertex, say u, then we have $c(wu) \in \{2, 3\}$, or else one of $uwvxyz$, $uwvzyx$ is a rainbow P_5-copy. Thus, if w is adjacent to u, then it is not adjacent to one of x, z; if w is adjacent to two new vertices, then it is not adjacent to either x or z. Finally, w may be adjacent to y. We conclude from this analysis that $d(w) \leq 4$.

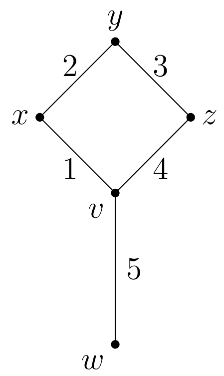

Figure 9

The vertex w is one of the neighbors of v which we will ultimately include in $L(v)$, so we must verify that w has no other neighbor of degree greater than 5. Observe first $d(x) \leq 5$, since any edge incident to x whose other endpoint is not on C must be colored from $\{1,2,3,4\}$ to avoid creating a rainbow P_4-copy ending at v. Analogously, $d(z) \leq 5$.

Suppose next that w is adjacent to a vertex u which is not on C. We have seen that $c(uw) \in \{2,3\}$; we set $c(uw) = a$. We now bound $d(u)$.

First, suppose u is adjacent to a vertex not yet described, say s. If $a = 2$, then $c(us) \in \{3,4,5\}$, since otherwise $suwvzy$ is a rainbow P_5-copy. Similarly, if $a = 3$, then $c(us) \in \{1,2,5\}$, since otherwise $suwvxy$ is a rainbow P_5-copy. Note that if $a = 2$, then u is not adjacent to z, since if so, either $vxyzu$ or $vwuzy$ is a rainbow P_4-copy ending at v. Analogously, if $a = 3$, then u is not adjacent to x. If $a = 2$ and ux is an edge, then $c(ux) \in \{3,4\}$, since otherwise $vzyxu$ is a rainbow P_4-copy ending at v. Analogously, if $c(ux) = 3$ and uz is an edge, then $c(uz) \in \{1,2\}$. Thus, $d(u) \leq 5$, since u is adjacent to w, may be adjacent to y, and any other incident edge to u must be colored from $\{3,4,5\}$ if $a = 2$ and $\{1,2,5\}$ if $a = 3$.

Finally, w may be adjacent to y. We wish to show that, if wy is an edge, then $d(y) \leq 5$. Suppose for a contradiction that wy is an edge and $d(y) \geq 6$. By Corollary 2.3, $y \in \overline{V'}$, and so by Lemma 2.2, y is not the endpoint of a rainbow P_4-copy.

We re-draw the situation in Figure 10, setting $c(yw) = c$.

Now, consider w. If w is adjacent to a vertex not depicted in Figure 10, say s, then one of $yxvws$, $yzvws$ is a rainbow P_4-copy ending at y. So w is adjacent to no vertex which is not drawn in Figure 10. If wx is an edge, then either $vwxyz$ is a rainbow P_4-copy ending at v, or $yxwvz$ is a rainbow P_4-copy ending at y. Finally, if wz is an edge, then either $yzwvx$ is a rainbow P_4-copy ending at y or $vwzyx$ is a rainbow P_4-copy ending at v. We conclude that $d(w) = 2$, a contradiction. So, if wy is an edge, then $d(y) \leq 5$.

We have thus shown that $d(w) \leq 4$ and v is the only vertex in $N(w)$ of degree greater than 5. We are now ready to build $L(v)$.

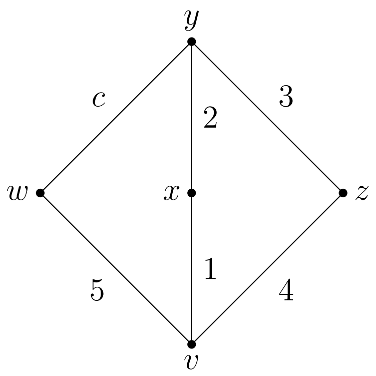

Figure 10

Observe that v has at least $d(v) - 5$ neighbors of the same type as w, namely, neighbors which do not lie on C and are incident to v by an edge whose color is not from $\{1, 2, 3, 4\}$. Let $L(v)$ be the set of such vertices. The above argument then shows that the maximum degree in $L(v)$ is at most 4, and that if w is any vertex in $L(v)$, then v is the only neighbor of w with degree greater than 5.

We now observe that the average degree in $L(v) \cup \{v\}$ is at most 5. Say $|L(v)| = d(v) - k$; we have noted that $k \leq 5$. The average degree in $L(v) \cup \{v\}$ is then

$$\frac{d(v) + \sum_{u \in L(v)} d(u)}{|L(v)| + 1} \leq \frac{d(v) + 4(d(v) - k)}{d(v) - k + 1} = \frac{5(d(v) - 4k/5)}{d(v) - (k-1)}.$$

Observe that

$$\frac{5(d(v) - 4k/5)}{d(v) - (k-1)} \leq 5$$

as long as $d(v) - (k-1) \geq d(v) - 4k/5$, i.e., $k - 1 \leq 4k/5$, which holds precisely when $k \leq 5$.

So, $\{v\} \cup L(v)$ is a local pairing as desired.

Case 2: v does not lie in a rainbow C_4-copy.

Using Lemma 2.4, we know that v is the endpoint of a rainbow P_3-copy, say $P = vxyz$, with edge colors $1, 2, 3$. Moreover, since $d(v) \geq 6$, v is incident to at least three edges which are not colored from $\{1, 2, 3\}$. Clearly, one of these does not have its other endpoint on P. So there exists a new vertex w such that vw is an edge which receives a new color, say 4.

Consider the possible neighbors of w. If w is adjacent to a vertex not on P, say u, then $c(wu) \in \{1, 2, 3\}$ in order to obey coloring rules and ensure that $uwvxyz$ is not a rainbow P_5-copy. If wx is an edge, then $vwxyz$ is a P_4-copy ending at v, so must not be rainbow, which means $c(wx) = 3$. If wy is an edge, then $wyxvw$ is a C_4-copy containing v, so is not rainbow, meaning $c(wy) = 1$. If wz is an edge, then $vwzyx$ is a P_4-copy ending at v, so cannot be rainbow, meaning $c(wz) = 2$. Thus, w can only be incident to edges colored from $\{1, 2, 3, 4\}$, so we conclude that $d(w) \leq 4$.

Our goal is to use w as one of the low-degree vertices in $L(v)$, so we must verify that w has no neighbor other than v of degree greater than 6.

Firstly, we examine z. If z is adjacent to a vertex not in $\{w, v, x, y\}$, say u, then $c(zu) \in \{1, 2\}$, else $vxyzu$ is a rainbow P_4-copy ending at v. We have seen that if zw is an edge, then $c(zw) = 2$. If vz is an edge, then $c(vz) = 2$, or else $vxyzv$ is a rainbow C_4-copy containing v. So, there is at most one vertex, namely x, which is incident to z via an edge which is not colored from $\{1, 2, 3\}$. Thus, $d(z) \leq 4$.

Next, consider x. Suppose wx is an edge and (for a contradiction) that $d(x) \geq 6$. Since x is adjacent to v, we know by Lemma 2.1 that $x \in \overline{V'}$. Thus, by Lemma 2.2, x is not the endpoint of a rainbow P_4-copy.

We have seen that $c(wx)$ must be 3. Observe that, since $d(v) \geq 6$, v must be incident to an edge of a new color, say 5, whose other endpoint is a vertex not yet considered, say u. We draw this in Figure 11.

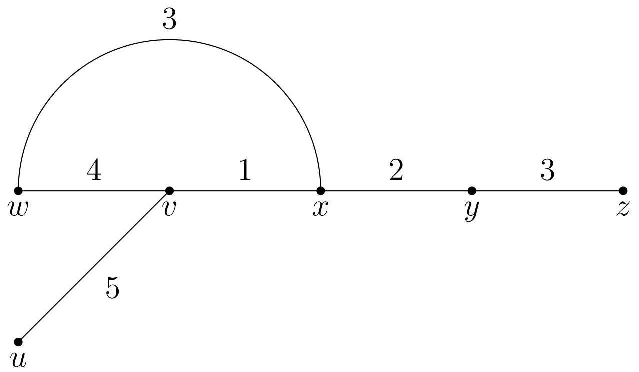

Figure 11

We shall achieve a contradiction by showing that $d(u) \leq 2$. First, we argue that u is not adjacent to any vertex from $\{w, x, y, z\}$. Observe that uw is not an edge, since if so, either $vuwxy$ is a rainbow P_4-copy ending at v, or $vuwxv$ is a rainbow C_4-copy containing v. Next, ux is not an edge, since if so, any legal choice for $c(ux)$ makes $vuxyz$ a rainbow P_4-copy ending at v. Observe that uy is not an edge, since if so, either $vuyxwv$ is a rainbow C_5-copy containing v, or $vuyxv$ is a rainbow C_4-copy containing v. Finally, uz is not an edge, since if so, either $vuzyx$ is a rainbow P_4-copy ending at v or $xvuzy$ is a rainbow P_4-copy ending at x.

Thus, u is not adjacent to w, x, y, or z. Suppose now that u has a neighbor s which is not yet considered. Then $suvwx$ is a P_4-copy ending in x, so cannot be rainbow, which means $c(us) \in \{3, 4\}$. We also have that $suvxyz$ is a P_5-copy, so cannot be rainbow, which means $c(us) \neq 4$. Thus, if u is adjacent to a new vertex, the edge used must be colored 3, meaning that u has at most one neighbor not yet drawn. So $d(u) \leq 2$, a contradiction. We conclude that if wx is an edge, then $d(x) \leq 5$.

Next, suppose wy is an edge. We wish to show that $d(y) \leq 5$. Suppose for a contradiction that $d(y) \geq 6$. We have seen that $c(wy) = 1$. Note that by Corollary 2.3, y is in $\overline{V'}$, so is not the endpoint of a rainbow P_4-copy.

We now examine z. Note first that if z has a neighbor not in $\{w, v, x\}$, say u, then $c(zu) = 1$, or else one of $vxyzu$, $vwyzu$ is a rainbow P_4-copy ending at v. Thus, to achieve degree 3, z must have at least one neighbor among $\{w, v, x\}$. We shall show that, to the contrary, z cannot be adjacent to any of these vertices.

Suppose wz is an edge. We have seen that $c(wz)$ must be 2. But now $yzwvx$ is a rainbow P_4-copy ending in y, a contradiction. So wz is not an edge. Observe that zv is not an edge, since for any legal choice of $c(zv)$, $vzywv$ is a rainbow C_4-copy containing v. Finally, suppose xz is an edge. We must have $c(xz) = 4$, else $vwyzx$ is a rainbow P_4-copy ending at v. Now, since $d(v) = 6$, v must be incident to an edge of a new color, say 5, whose other endpoint is not in $\{w, x, y, z\}$. Say this edge is vu. Observe that $yzxvu$ is now a rainbow P_4-copy ending at y. We conclude that z is adjacent to none of v, y, w, so $d(z) \leq 2$, a contradiction. Thus, if wy is an edge, then $d(y) \leq 5$.

Finally, we must show that if w is adjacent to a vertex not in $\{v, x, y, z\}$, say u, then $d(u) \leq 5$. Suppose to the contrary that $d(u) \geq 6$. We can again apply Corollary 2.3 to give that u is in $\overline{V'}$, so u is not the endpoint of a rainbow P_4-copy. We must have $c(uw) \in \{1, 2\}$, else $uwvxy$ is a rainbow P_4-copy ending at u; however, unlike the other cases, we cannot at the moment fix $c(uw)$. We shall set $c(uw) = a$.

Now, since $d(u) \geq 6$, u is incident to an edge of a new color, 5. It is simple to check that the other endpoint of this edge must be a vertex not in $\{v, x, y, z\}$, say s. Similarly, v must be incident to an edge of another new color, 6, and the endpoint of this edge must be a vertex not in $\{s, u, w, x, y, z\}$, say t. We illustrate the situation in Figure 12.

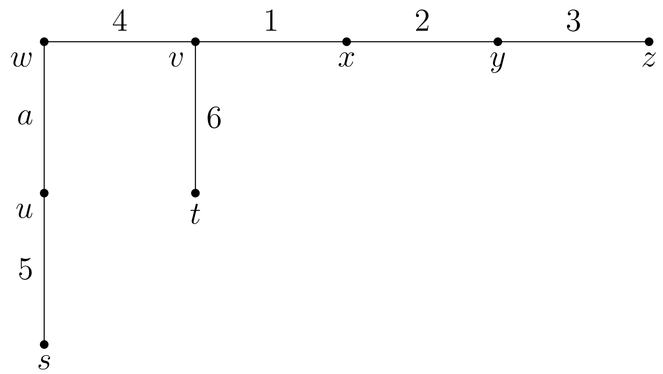

Figure 12

We will argue that $d(t) \leq 2$. We claim first that t is not adjacent to s, u, w, or x. Indeed, if ts is an edge, then one of the paths $ustvw, ustvx$ is a rainbow P_4-copy ending at u. If tu is an edge, then either v is in a rainbow C_4-copy or $utvxy$ is a rainbow P_4-copy ending at u. If tw is an edge, then either $vtwus$ is a rainbow P_4-copy ending at v, or $wtvxyz$ is a rainbow P_5-copy. And if tx is an edge, then either $vtxyz$ is a rainbow P_4-copy ending at v, or $uwvtx$ is a rainbow P_4-copy ending at u. Thus, the only possible neighbors of t are y, z, and vertices not depicted in Figure 12.

Observe, if ty is an edge, then $c(ty) \in \{a, 4\}$, else $uwvty$ is a rainbow P_4-copy ending at u. But $c(ty) \neq 4$, since then $vtyxv$ is a rainbow C_4-copy containing v. Thus, if ty is an edge, then $c(ty) = a$. Similarly, if tz is an edge, then $c(tz) = a$. And if t is adjacent to a vertex not yet drawn, then the incident edge also must be colored a to avoid forming either a rainbow P_5-copy with $tvxyz$ or a rainbow P_4-copy ending at u.

Thus, if t is adjacent to any vertex other than v, the incident edge must be colored a, so $d(t) \leq 2$, a contradiction.

We have thus argued that $d(w) \leq 4$ and that v is the only vertex in $N(w)$ of degree greater than 5. We are now ready to build $L(v)$.

Note that v has at least $d(v)-4$ neighbors of the same type as w, that is, neighbors which are not on the path $vxyz$ and which are incident to v by an edge whose color is not from $\{1, 2, 3\}$. (Note that if vz is an edge, then $c(vz) = 2$, else v is in a rainbow C_4-copy, and thus we can find $d(v) - 4$ vertices of the same type as w, instead of the expected $d(v) - 5$.) Let $L(v)$ be the set of such vertices. As in the previous case, it follows that the average degree in $L(v) \cup \{v\}$ is at most 5, (and actually will be strictly less than 5, as $|L(v)| > d(v) - 5$). □

4 Concluding remarks

While the details of the proofs are largely routine case analysis, it is worth reiterating the central idea which makes this approach to our problem tractable. The choice to partition V as $V' \cup \overline{V'}$ is not arbitrary. Since rainbow C_ℓ-copies arise naturally when we attempt to avoid rainbow P_ℓ-copies, it is much easier to show that rainbow P_ℓ-copies are unavoidable if we start with a vertex in $\overline{V'}$ than with a vertex whose situation we do not know. On the other hand, every vertex in V' lies on a rainbow C_5-copy, meaning that we can begin any case analysis concerning vertices in V' with a rainbow C_5-copy, rather than with a single vertex. This additional guaranteed structure likewise makes case analysis arguments tractable where they seem not to be if we pick a vertex from V without assuming that it lies (or does not lie) in a rainbow C_5-copy. Similar partitioning ideas may be applicable for other rainbow Turán problems, although of course there is an upper limit to the amount of case analysis which can reasonably done by hand, even with such expedients.

Given the increase of required analysis from previous results (the proof that $\mathrm{ex}^*(n, P_4) = 2n + O(1)$ is about two pages long), it seems clear that these naive methods will not be sufficient to obtain exact results for longer paths. However, the result for P_5 alone adds weight to the conjecture that $\mathrm{ex}^*(n, P_\ell) = \frac{\ell}{2}n + O(1)$ in general.

We also remark that the proof of Theorem 1.5 does not in any way establish Construction 1 as uniquely optimal; in fact, it is known that Construction 1 does *not* uniquely achieve $\mathrm{ex}^*(n, P_5)$. Johnston and Rombach [8] note the existence of a geometric proper edge-coloring of K_6 which avoids rainbow P_5-copies, and ask when K_k admits a proper edge-coloring avoiding a rainbow K_{k-1}-copy. The question is worth reiterating, given that we have now exhibited a case in which equally dense complete and non-complete constructions are extremal. If $\mathrm{ex}^*(n, P_\ell) = \frac{\ell}{2}n + O(1)$ in general, the construction of Maamoun and Meyniel [10] demonstrates that cliques give rise to extremal constructions in an infinite number of cases (whenever $\ell = 2^k - 1$ for some $k \geq 2$). On the other hand, Johnston, Palmer, and Sarkar [7] note that any proper edge-coloring of K_5 contains a rainbow P_4-copy, so it is not true that $\mathrm{ex}^*(n, P_\ell)$ is always achievable (modulo a constant error term) by a union of cliques.

Acknowledgements

The author would like to thank Cory Palmer for his advice and support in the preparation of this paper. The author would also like to thank the anonymous referees for their thorough and helpful comments, which improved the clarity and presentation of this paper.

References

[1] N. Alon and C. Shikhelman, Many T copies in H-free graphs, *J. Combin. Theory Ser. B* **121** (2016), 146–172.

[2] S. Das, C. Lee and B. Sudakov, Rainbow Turán problem for even cycles, *European J. Combin.* **34** (2013), 905–915.

[3] B. Ergemlidze, E. Győri and A. Methuku, On the Rainbow Turán number of paths, *Electron. J. Combin.* **26** (2019), #P1.17.

[4] D. Gerbner, T. Mészáros, A. Methuku and C. Palmer, Generalized rainbow Turán problems, *Electron. J. Combin.* **29** (2022), #P2.44.

[5] A. Halfpap and C. Palmer, Rainbow cycles versus rainbow paths, *Australas. J. Combin.* **81** (2021), 152–169.

[6] O. Janzer, Rainbow Turán number of even cycles, repeated patterns and blow-ups of cycles, `arXiv:2006.01062` (2020).

[7] D. Johnston, C. Palmer and A. Sarkar, Rainbow Turán Problems for Paths and Forests of Stars, *Electron. J. Combin.* **24** (2017), #P1.34.

[8] D. Johnston and P. Rombach, Lower bounds for rainbow Turán numbers of paths and other trees, *Australas. J. Combin.* **78** (2020), 61–72.

[9] P. Keevash, D. Mubayi, B. Sudakov and J. Verstraëte, Rainbow Turán problems, *Combin. Probab. Comput.* **16** (2007), 109–126.

[10] M. Maamoun and H. Meyniel, On a problem of G. Hahn about coloured Hamiltonian paths in K_{2t}, *Discrete Math* **51** (1984), 213–214.

(Received 11 Jan 2023; revised 26 Aug 2023)