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Three classes of inversion sequences counted by large Schröder numbers

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Abstract

In this paper, analytically and combinatorially, we reprove that the number of inversion sequences that avoid $\{100, 101, 201, 210\}$ (respectively, $\{100, 110, 201, 210\}$) is given by the large Schröder number, as shown by Martinez and Savage. Moreover, we show that the number of inversion sequences that avoid $\{101, 110, 201, 210\}$ is also given by the large Schröder number.

1 Introduction

An *inversion sequence* of length n is an integer sequence $e = e_0e_1 \cdots e_n$ such that $0 \le e_i \le i$ for each $0 \le i \le n$. Let $\mathbf{I}_n$ denote the set of all the inversion sequences of length n. We say that an inversion sequence $e \in I_n$ contains the word $\tau = \tau_1 \cdots \tau_k$ if there is a subsequence of length k in e that is order isomorphic to τ; otherwise, we say that e avoids τ. Define $\mathbf{I}_n(\tau)$ to be the set of inversion sequences of length n that avoid τ. In this context, τ is called a *pattern*. More generally, for any set B of patterns, we define $\mathbf{I}_n(B) = \cap_{\tau\in B}\mathbf{I}_n(\tau)$. Two sets of patterns B_1 and B_2 are said to be *Wilf equivalent* if $|I_n(B_1)| = |I_n(B_2)|$, for all $n \ge 0$.

The systematic study of pattern-avoidance for inversion sequences was initiated by Mansour and Shattuck [6] for the patterns of length three with non-repeating

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letters, and by Corteel et al. [3] for repeating and non-repeating letters. In particular, [6] gives an analytic and [3] a bijective proof that

$$\sum_{n\geq 0}|\mathbf{I}_n(021)|x^n=\mathrm{Sch}(x)=\frac{1-x-\sqrt{1-6x+x^2}}{2x}.$$

See also [1] for a bijective proof involving the little Schröder numbers. Yan and Lin [8] completed the classification of the Wilf-equivalences for inversion sequences avoiding pairs of length-three patterns. Martinez and Savage [7] generalized and extended the notion of pattern-avoidance for the inversion sequences to triples of binary relations. In particular, they showed that

$$\sum_{n\geq 0}|\mathbf{I}_n(\{100,101,201,210\})|x^n=\sum_{n\geq 0}|\mathbf{I}_n(\{100,110,201,210\})|x^n=\mathrm{Sch}(x). \quad (1.1)$$

Hong and Li [4] completed the Wilf classification for patterns of length four. More recently, the authors with Jelínek [2] considered the classification of the Wilf-equivalences for inversion sequences avoiding triples of length-three patterns.

The aim of this paper is to prove analytically and combinatorially the following result.

Theorem 1.1. *Let $B_1=\{100,101,201,210\}$, $B_2=\{100,110,201,210\}$, and $B_3=\{101,110,201,210\}$. Then, for all $j=1,2,3$,*

$$\sum_{n\geq 0}|\mathbf{I}_n(B_j)|x^n=\mathrm{Sch}(x). \quad (1.2)$$

In particular, we reprove (1.1) by two different proofs.

2 Analytical proofs

In this section, we present an analytical proof for (1.2). To do that, we first use the algorithm presented in [5] to guess and then to prove the rules of each generating tree (see [9]) for all inversion sequences in $\bigcup_{n\geq 0}\mathbf{I}_n$ that avoid B_j, where $j=1,2,3$. In each case, we translate the rules of the generating tree to a system of functional equations and then solve to obtain an explicit formula for the generating function $\sum_{n\geq 0}|\mathbf{I}_n(B_j)|x^n$. In the next three subsections, we consider the enumeration problem of counting elements of $\mathbf{I}_n(B_j)$, where $j=1,2,3$.

2.1 Avoiding B_3

As we said, by using the algorithm of [5], we obtain the following result.

Lemma 2.1. *The generating tree for $\bigcup_{n\geq 0}\mathbf{I}_n(B_3)$ can be characterized as a generating tree $\mathcal{T}_3$ defined by a root 0 and the following rules*

$$\begin{aligned}&0^m\rightsquigarrow 0^{m+1},0^m1,\ldots,0^mm,\\&0^mj\rightsquigarrow (0^{m+2-j})^{j+1},0^{m+1}(j+1),\ldots,0^{m+1}(m+1).\end{aligned}$$

Proof. Fix $0 \in \mathbf{I}_0(B_3)$ to be the root of $\mathcal{T}_3$. Thus, it remains to show that the rules hold. By the definitions, the children of $0^m \in \mathbf{I}_m(B_3)$ are $0^{m+1}, 0^m1, \ldots, 0^m m$. Thus, $0^m \rightsquigarrow 0^{m+1}, 0^m1, \ldots, 0^m m$. Also, the children of $0^m j \in \mathbf{I}_{m+1}(B_3)$, $1 \le j \le m$, are $0^m j0, 0^m j1, \ldots, 0^m j(m+1)$. Note that

- the subtree $\mathcal{T}(B_3; 0^m ji)$ of $\mathcal{T}_3$ with $0 \le i \le j$ is isomorphic (in the sense of plane trees) to the subtree $\mathcal{T}(B_3; 0^{m+2-j})$. To see that let $0^m ji\pi' \in \mathbf{I}_n(B_3)$, then π' does not contain any letter belong to the set $\{0, 1, \ldots, i-1, i+1, \ldots, j\}$. Thus, $0^m ji\pi' \in \mathbf{I}_n(B_3)$ if and only if $0^{m+2-j}\pi'' \in I_{n+2-j}$, where π'' is the word that is obtained from π' after replacing the letter k of π' with $k-j$, where $k \ge j+1$.
- the subtree $\mathcal{T}(B_3; 0^m ji)$ of $\mathcal{T}_3$ with $j+1 \le i \le m+1$ is isomorphic to $\mathcal{T}(B_3; 0^{m+1}i)$. This holds because $0^m ji\pi' \in \mathbf{I}_n(B_3)$ if and only if $0^{m+1}i\pi' \in \mathbf{I}_n(B_3)$.

Thus, $0^m j \rightsquigarrow (0^{m+2-j})^{j+1}, 0^{m+1}(j+1), \ldots, 0^{m+1}(m+1)$, which completes the proof. $\square$

Define $A_m(x)$ (respectively, $B_{m,j}(x)$) to be the generating functions for the number of nodes at level n in the subtree of $\mathcal{T}_3$ with root 0^m (respectively, $0^m j$) of $\mathcal{T}_3$, where the root of this subtree is the vertex 0^m (respectively, $0^m j$) that stays at level 0. Thus, Lemma 2.1 gives

$$A_m(x) = x + xA_{m+1}(x) + x\sum_{j=1}^{m} B_{m,j}(x),$$
$$B_{m,j}(x) = x + (j+1)xA_{m+2-j}(x) + x\sum_{i=j+1}^{m+1} B_{m+1,i}(x).$$

Define $B_m(x;u) = \sum_{j=1}^m B_{m,j}(x)u^{j-1}$. Then, by multiplying the recurrence for $B_{m,j}(x)$ by u^{j-1} and summing over $j = 1, 2, \ldots, m$, we obtain

$$\begin{aligned} A_m(x) &= x + xA_{m+1}(x) + xB_m(x;1),\\ B_m(x;u) &= \frac{x(1-u^m)}{1-u} + x\sum_{j=1}^{m}(j+1)A_{m+2-j}(x)u^{j-1}\\ &\quad + \frac{x}{1-u}(B_{m+1}(x;1) - B_{m+1}(x;u)). \end{aligned}$$

Let $A(x;v) = \sum_{m\ge1} A_m(x)v^{m-1}$ and $B(x;v,u) = \sum_{m\ge1} B_m(x;u)v^{m-1}$. Then, by multiplying these two recurrences by v^{m-1} and summing over $m \ge 1$, we obtain

$$A(x;v) = \frac{x}{1-v} + \frac{x}{v}(A(x;v) - A(x;0)) + xB(x;v,1), \tag{2.1}$$
$$\begin{aligned} B(x;v,u) &= \frac{x}{(1-v)(1-vu)} + \frac{x(2-vu)}{(1-vu)^2}(A(x;v) - A(x;0))\\ &\quad + \frac{x}{v(1-u)}(B(x;v,1) - B(x;v,u)). \end{aligned} \tag{2.2}$$

In order to solve (2.1)–(2.2), we *assume* that the generating functions $A(x;v)$ and $B(x;v,u)$ satisfy two extra equations:

$$\frac{A(x;v)}{B(x;v,1)} + \frac{1}{2}A(x;0) = 1 - v - \frac{1}{2}x, \tag{2.3}$$

$$A(x;0) = \frac{1 - x - \sqrt{1-6x+x^2}}{2}. \tag{2.4}$$

Note that the relation (2.4) is guessed from the fact that we wish to prove $\sum_{n\geq 0}|\mathbf{I}_n(B_2)|x^{n+1} = x\mathrm{Sch}(x)$. The relation (2.3) is hard to guess directly from the problem, but here we used the first terms of these generating functions and we tested several relations such as $A(x;v) - B(x;v,1)$, $A(x;v)/B(x;v,v)$, and $A(x;v)/B(x;v,1)$.

Since solution of the system (2.1)–(2.4) is a solution of the system (2.1)–(2.2), we only need to solve the system (2.1)–(2.4). By (2.1), (2.3), and (2.4), we obtain that

$$A(x;v) = \frac{x((v^2+vx-2v+1)\sqrt{x^2-6x+1} + v^2x+vx^2-3v^2-5vx+4v+x-1)}{2(v-1)(v^2+vx-v+x)} \tag{2.5}$$

and

$$B(x;v,1) = \frac{x((v+x-2)\sqrt{x^2-6x+1} + vx + x^2 - 3v - 5x + 2)}{2(1-v)(v^2+vx-v+x)}.$$

Now, by (2.2), we obtain

$$\begin{aligned} B(x;v,u) = {} & \frac{x(uv^3+xuv^2-2uv^2+uv-2v^2-2xv+4v+x-2)\sqrt{x^2-6x+1}}{2(v^2+vx-v+x)(1-uv)^2(1-v)} \\ & + \frac{xv(uv^2(x-3)+uv(x-4)(x-1)+u(x-1)+2v(2-x))}{2(v^2+vx-v+x)(1-uv)^2(1-v)} \\ & + \frac{x(-2v(x^2-5x+3)+x^2-5x+2)}{2(v^2+vx-v+x)(1-uv)^2(1-v)}. \end{aligned} \tag{2.6}$$

By using the expressions of $A(x;v)$ and $B(x;v,u)$, we see that (2.1)–(2.4) hold. Hence, we can state the following result.

Theorem 2.2. *We have*

$$\sum_{n\geq 0}|\mathbf{I}_n(B_3)|x^n = \mathrm{Sch}(x).$$

Moreover, the generating functions $A(x;v)$ and $B(x;v,u)$ are given by (2.5) *and* (2.6)*, respectively.*

2.2 Avoiding B_2

Again, by using the algorithm of [5], we obtain the following result.

Lemma 2.3. *The generating tree for $\bigcup_{n\geq 0}\mathbf{I}_n(B_2)$ can be characterized as a generating tree $\mathcal{T}_2$ defined by a root* 0 *and the following rules:*

$$\begin{aligned}
&0^m \rightsquigarrow 0^{m+1}, 0^m1, \ldots, 0^m m,\\
&0^m j \rightsquigarrow (0^{m+2-j})^{j+1}, 0^{m+1}(j+1), \ldots, 0^{m+1}(m+1).
\end{aligned}$$

By Lemma 2.1, Theorem 2.2, and Lemma 2.3, we have the following result.

Theorem 2.4. *We have*

$$\sum_{n\geq 0} |\mathbf{I}_n(B_2)|x^n = \mathrm{Sch}(x).$$

2.3 Avoiding B_1

Lemma 2.5. *The generating tree for $\bigcup_{n\geq 0}\mathbf{I}_n(B_1)$ can be characterized as a generating tree $\mathcal{T}_1$ defined by a root* 0 *and the following rules:*

$$\begin{aligned}
&0^m \rightsquigarrow 0^{m+1}, 0^m1, \ldots, 0^m m,\\
&0^m j \rightsquigarrow (0^{m+1-j})^{j}, 0^{m+1}j, \ldots, 0^{m+1}(m+1).
\end{aligned}$$

Define $A_m(x)$ (respectively, $B_{m,j}(x)$) to be the generating functions for the number of nodes at level n in the subtree of $\mathcal{T}_1$ with root 0^m (respectively, 0^mj) of $\mathcal{T}_1$, where the root of this subtree is the vertex 0^m (respectively, 0^mj) that stays at level 0. Thus, Lemma 2.5 gives

$$\begin{aligned}
A_m(x) &= x + xA_{m+1}(x) + x\sum_{j=1}^{m} B_{m,j}(x),\\
B_{m,j}(x) &= x + jxA_{m+1-j}(x) + x\sum_{i=j}^{m+1} B_{m+1,i}(x).
\end{aligned}$$

Define $B_m(u) = \sum_{j=1}^{m} B_{m,j}u^{m-j}$. Then, by multiplying the recurrence for $B_{m,j}(x)$ by u^{m-j} and summing over $j = 1, 2, \ldots, m$, we obtain:

$$\begin{aligned}
A_m(x) &= x + xA_{m+1}(x) + xB_m(x;1),\\
B_m(x;u) &= \frac{x(1-u^m)}{1-u} + x\sum_{j=1}^{m} jA_{m+1-j}(x)u^{m-j} + x\sum_{j=1}^{m}\sum_{i=j}^{m+1} B_{m+1,i}(x)u^{m-j}.
\end{aligned}$$

Let $A(x;v) = \sum_{m\geq 1} A_m(x)v^{m-1}$ and $B(x;v,u) = \sum_{m\geq 1} B_m(x;u)v^{m-1}$. Then, by multiplying these two recurrences by v^{m-1} and summing over $m \geq 1$, we obtain:

$$A(x;v) = \frac{x}{1-v} + \frac{x}{v}(A(x;v) - A(x;0)) + xB(x;v,1), \tag{2.7}$$

$$\begin{aligned}
B(x;v,u) &= \frac{x}{(1-v)(1-vu)} + \frac{x}{(1-v)^2}A(x;vu)\\
&\quad + \frac{x}{vu(1-u)}(B(x;v,u) - uB(x;vu,1)) - \frac{x}{vu}B(x;v,0).
\end{aligned} \tag{2.8}$$

Again, in order to solve (2.7)–(2.8), we *assume* that the generating functions $A(x;v)$ and $B(x;v,u)$ satisfy:

$$\frac{A(x;v)}{B(x;v,1)}+\frac{1}{2}A(x;0)=1-v-\frac{1}{2}x, \tag{2.9}$$

$$A(x;0)=x\,\mathrm{Sch}(x)=\frac{1-x-\sqrt{1-6x+x^2}}{2}, \tag{2.10}$$

$$B(x;v,0)=-\frac{A(x;0)}{(1-v)^2}(v+1+A(x;0)). \tag{2.11}$$

By (2.7), (2.9), and (2.10), we obtain that

$$A(x;v)=\frac{x((v^2+vx-2v+1)\sqrt{x^2-6x+1}+v^2x+vx^2-3v^2-5vx+4v+x-1)}{2(v-1)(v^2+vx-v+x)} \tag{2.12}$$

and

$$B(x;v,1)=\frac{x((v+x-2)\sqrt{x^2-6x+1}+vx+x^2-3v-5x+2)}{2(1-v)(v^2+vx-v+x)}.$$

Now, by (2.8), (2.11), and (2.12), we obtain

$$\begin{aligned}B(x;v,u)=&\frac{x(u^2v^3-2u^2v^2+xuv^2-2uv^2-2xuv+4uv+v+x-2)\sqrt{1-6x+x^2}}{2(v-1)^2(1-uv)(u^2v^2+xuv-uv+x)}\\&+\frac{x((x-3)u^2v^3+2(2-x)u^2v^2+(x-4)(x-1)uv^2)}{2(v-1)^2(1-uv)(u^2v^2+uvx-uv+x)}\\&+\frac{x(-2x^2uv+10xuv-6uv+vx+x^2-v-5x+2)}{2(v-1)^2(1-uv)(u^2v^2+uvx-uv+x)}.\end{aligned} \tag{2.13}$$

By using the expressions for $A(x;v)$ and $B(x;v,u)$, we see that (2.7)–(2.11) all hold. Hence, we can state the following result.

Theorem 2.6. *We have*

$$\sum_{n\geq 0}|\mathbf{I}_n(B_1)|x^n=\mathrm{Sch}(x).$$

Moreover, the generating functions $A(x;v)$ and $B(x;v,u)$ are given by (2.12) *and* (2.13)*, respectively.*

3 Combinatorial proofs

In this section, we present combinatorial proofs of Theorems 2.2, 2.4, and 2.6. First, let us recall UDF and Schröder paths. A UDF path is is a lattice path of upsteps $U=(1,1)$, downsteps $D=(1,-1)$ and flatsteps $F=(2,0)$. The horizontal line through the initial point is *ground level* and heights are measured relative to ground level. A *Schröder path*[1] is a UDF path that ends at ground level and never dips

[1] See `http://oeis.org/A006318`

below ground level. Its size is measured as the number of Us plus the number of Fs. We use the term *weak-fall* step to mean an F or a D. Since $\# U\text{s} = \# D\text{s}$ in a Schröder path, the size of a Schröder path is the number of weak-fall steps it contains. Let $\mathcal{R}_n$ denote the set of Schröder paths of size n. An *elevated* Schröder path is one that starts at U, ends at D, and has no vertex at ground level other than the endpoints. The *elevation* of a Schröder path P is $U.P.D$, where dots denote concatenation. An *ascent* in a path is a maximal run of contiguous upsteps and similarly for a *descent*. The terminal descent of a Schröder path is the run of Ds that terminates the path; it is of length 0 if the path ends with a flatstep.

3.1 Bijection ϕ from $\mathbf{I}_n(B_1)$ to $\mathcal{R}_n$.

Recall that $B_1 = \{100, 101, 201, 210\}$. The set $\mathbf{I}_n(B_1)$ can be characterized as the set of inversion sequences $e_0e_1 \cdots e_n$ such that for each $i \geq 2$, if $e_i < e_{i-1}$ then $e_j > e_{i-1}$ for all $j \geq i+1$. In particular, for an avoider $e_0e_1 \cdots e_n$ with $n \geq 1$, we have $\max(e_i)_{i=0}^n = e_n$ if $e_n \geq e_{n-1}$, and $= e_{n-1}$ otherwise.

It will be convenient to say a vertex in a Schröder path is a *key* vertex if it is a valley vertex—the vertex separating a contiguous DU—or the terminal vertex of a flatstep F (regardless of the next step).

Define $\phi(0) = \epsilon$, the empty path. Now, suppose given $e = e_0e_1 \cdots e_n \in \mathbf{I}_n(B_1)$ with $n \geq 1$. By induction, set $P = \phi(e_0 \cdots e_{n-1}) \in \mathcal{R}_{n-1}$ and set $i = n - e_n$. Also, set $Q = P.D$. Note that Q contains n weak-fall steps and ends at height -1. Define $S = \phi(e)$ as follows. If $i = n$, set $S = U.Q$, thus forming the elevation of P. If $0 \leq i \leq n-1$, locate the i-th weak-fall step in Q measuring from the end of the path, and insert a U immediately before this weak-fall step to obtain an intermediate Schröder path $R \in \mathcal{R}_n$. In case $i = 0$, this is interpreted to mean that U is inserted at the end of Q, thereby introducing a key vertex at height -1. In the path R, if the step immediately preceding the inserted U is a D such that the valley vertex formed by this DU is at odd height, delete this D and change the inserted U to an F to obtain S. Otherwise, set $S = R$. For example, with $n = 4$ and $e = 00102$, we have $P = \phi(0010) = UUDUDD$ (induction), $i = 2$, and P, Q, R, S are shown in Figure 1 below, where a DU in R is changed to an F in S.

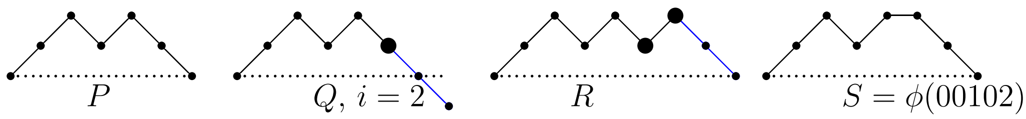

Figure 1: The path $\phi(00102)$

As another example, $e = 01131$ yields $P = FUDUD$, $i = 3$, and Q, S are shown in Figure 2 below, where a DU at ground level in Q is raised to odd height in $R = S$.

We have $\phi(00) = UD$, $\phi(01) = F$ and the action of ϕ on avoiders for $n = 2$ is shown in Figure 3 below.

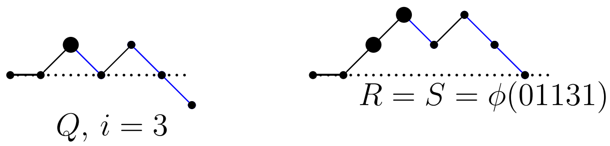

Figure 2: The path $\phi(01131)$

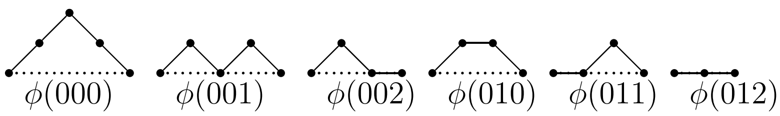

Figure 3: The paths for $\mathbf{I}_2(B_1)$

For purposes of inverting ϕ, note that whenever ϕ introduces a new key vertex in S that was not present in P, it is at even height.

To reverse the map, it is only necessary to determine where the inserted U/F appears in S for then we can retrieve i by counting the weak-fall steps after this U/F, we can retrieve P by deleting the last step of S and deleting the inserted U or, in case the inserted U became an F, changing this F to a D. Then, by induction, we get the first n terms of the inverse avoider as $e_0 \cdots e_{n-1} = \phi^{-1}(P)$, and $e_n = n - i$.

Here is the procedure to identify the location of the inserted U/F.

- If S has no key vertices at even height (so S must be an elevated path or else there is a valley vertex or F step at ground level, thus at even height), then a U was inserted at the end of the run of (0 or more) upsteps at the start of Q to get $R = S = \phi(e)$. Delete the first and last steps of S to get P and set $i = n$.

- If S has at least one key vertex at even height, locate the last such vertex V. If V is immediately followed by a U, the ascent started by this U is where a U was inserted, and P and i can be recovered. Otherwise, V terminates an F step that either ends the path or is immediately followed by a D (it cannot be followed by another F for then V would not be the last key vertex at even height and it is not followed by U by assumption) and this F step replaced a DU in R to get S. So change the F to D and delete the last step to get P, and count weak-fall steps after V to get i. (If the path S ends with a flatstep F, the net effect is to delete F, and $i = 0$.)

To explain why this reversal procedure works, we begin with a useful lemma.

Lemma 3.1. *In an avoider* $e = e_0 e_1 \cdots e_n \in \mathbf{I}_n(B_1)$,
(i) *the number of consecutive* U*s that start* $\phi(e)$ *is the number of noninitial 0s in* e, *and*
(ii) *the number of consecutive* D*s that terminate* $\phi(e)$ *is* $n - \max\{e_i\}$.

Proof. We prove both parts by simultaneous induction on n. The base cases are clear.

(i) Suppose first that $e_0 \cdots e_{n-1}$ is weakly increasing so that all its 0 s occur at the start. By induction, the initial ascent length of $P = \phi(e_0 \cdots e_{n-1})$ is the number of initial 0s in $e_1 \cdots e_{n-1}$. If $e_n = 0$, then P is elevated to get $\phi(e)$ and the conclusion holds. If $e_n \neq 0$, then $i < n$ and in the construction of $\phi(e)$ from P, the initial ascent of P is undisturbed. Next suppose that $e_0 \cdots e_{n-1}$ has a descent and that $\max\{e_i\}_{i=0}^{n-1} = a$. By induction, P ends with $(n-1) - a$ Ds, so Q ends with $n - a$ Ds and, since e is an avoider, $e_n > a$. Hence, $i = n - e_n < n - a$, and the insertion of the step U in P occurs within the terminal descent of P and so does not affect the initial ascent.

(ii) Again, let $a = \max\{e_i\}_{i=0}^{n-1}$. By induction, $P = \phi(e_0 \cdots e_{n-1})$ ends with $(n-1) - a$ Ds, so Q ends with $n - a$ Ds. If $e_n \leq a$, then $\max\{e_i\}_{i=0}^{n} = a$ and $i = n - e_n \geq n - a$ and the insertion of U leaves the $n - a$ Ds that terminate P undisturbed. If $e_n > a$, then $\max\{e_i\}_{i=0}^{n} = e_n$ and $i = n - e_n < n - a$. Since U is inserted just before the ith-from-end step on the terminal descent of $n - a$ Ds in Q, we see that $\phi(e)$ ends with precisely $i = n - e_n = n - \max\{e_i\}_{i=0}^{n}$ Ds, as required. $\square$

The pyramid paths $U^n D^n$, $n \geq 0$, are the only Schröder paths with no key vertices at all and they correspond under ϕ to the all-zeros avoiders 0^{n+1}. Otherwise, the last (rightmost) key vertex in a Schröder path is the very last vertex if the path ends with F and the first vertex of the last ascent if the path ends with D.

For $n \geq 2$, consider an avoider e and the construction for ϕ described above. Set $a = \max(e_j)_{j=0}^{n-1}$ $(= \max(e_{n-2}, e_{n-1}))$. By Lemma 3.1, P ends with $n - 1 - a$ downsteps and so Q ends with $n - a$ downsteps. If $e_n \geq e_{n-1}$, then certainly $e_n \geq a$ and so $i = n - e_n \leq n - a$. Hence, the construction will insert U at one of the vertices *on the terminal descent* of Q (since Q ends with D, there are at least two such vertices). If $e_n = e_{n-1}$, then $i = n - a$, so U is inserted at the top vertex of the terminal descent (Figure 4a), no new key vertex is introduced and no key vertex of P is disturbed. If $e_n > e_{n-1}$, then $i < n - a$ and a new key vertex (at even height) is introduced, and existing key vertices of P are undisturbed (Figure 4b).

(4a) $n - a = i = 3$ (4b) $n - a = 4$, $i = 3$

Figure 4

If $e_n < e_{n-1}$, then $a = e_{n-1}$ and $i > n - a$ and the U must be inserted at a vertex strictly before the last key vertex of P and all key vertices after the inserted U are raised 1 unit (example in Figure 2 above).

Next, for an avoider $e = e_0 e_1 \cdots e_n \in \mathbf{I}_n(B_1)$, consider the paths P_k, Q_k, R_k, S_k, with $P_k = \phi(e_0 \cdots e_k)$ used in the recursive construction of $\phi(e)$ described above as k increases from 1 to $n - 1$. As long as $e_k \geq e_{k-1}$, all key vertices are at even level and everything is fine. At the first descent (if there is one) $e_k < e_{k-1}$, so by the previous paragraph, one or more of the key vertices at the end of S_k will be at odd height and the inserted U can be recovered from the last key vertex at even height. The next term e_{k+1} must satisfy $e_{k+1} > e_{k-1}$, and so a new last key vertex is introduced, not disturbing its predecessor key vertex at odd height, and this last key vertex is at even height. As long as the terms of e continue to weakly increase, existing key vertices are undisturbed and any new key vertex is at even height, the last key vertex serving to recover the inserted U.

At the next descent, say $e_{k+m} < e_{k+m-1}$ with $m \geq 2$, the inserted U will be placed *before* the last key vertex, which is at even height. The crucial point is that it will be placed *after* the last of the run of one or more key vertices at odd height that were introduced by the previous descent. To see this, note that the number i of weak-valley steps to be counted off from the end of Q_{k+m} to obtain the insert location for U is given by $i = k + m - e_{k+m}$. But there are more than this number of weak-valley steps in Q_{k+m} after the last key vertex at odd height: the terminal descent of Q_k contains $k - \max(e_j)_{j=0}^{k} = k - e_{k-1}$ D steps, which all appear in Q_{k+m}, and each of $Q_{k+1}, \ldots, Q_{k+m}$ introduces one additional D step (while the inserted U may change an existing D to an F) for a total of $k-e_{k-1}+m$ steps, and $k-e_{k-1}+m > k+m-e_{k+m}$. Thus, if a key vertex reaches odd height due to the insertion of a U, it remains henceforth at that height, and the procedure for inverting ϕ works.

A consequence of the preceding analysis is that ϕ takes the number of descents in an avoider e to the number of runs of key vertices at odd height in the corresponding Schröder path. (The key vertices occur in runs of one height parity followed by a run of the other height parity.)

3.2 Bijection from $\mathbf{I}_n(B_1)$ to $\mathbf{I}_n(B_2)$.

Recall that $B_1 = \{100, 101, 201, 210\}$ and $B_2 = \{100, 110, 201, 210\}$.

The B_2-avoiders can be characterized as the inversion sequences $e_0 \cdots e_n$ that satisfy for $i \geq 1$ (i) $e_i \geq \max(\text{descent tops in } e_0 \cdots e_{i-1})$, and (ii) $e_i \geq \max(\text{repeated entries in } e_0 \cdots e_{i-1})$. To define the bijection, given $e_0 \cdots e_n \in \mathbf{I}_n(B_1)$, split the sequence into segments after each descent bottom and place each segment in a box. Each segment is weakly increasing except for a descent bottom in the last position (the last segment may or may not end with a descent), and all entries in each non-initial box exceed all entries in the preceding box. Within each box, place a circle around the descent bottom (if there is one) and around the second occurrence of each repeated entry that is greater than the descent bottom. Then within each box, rotate the circled entries to the right, and erase circles and boxes. An example is shown below.

$$\boxed{0\,1\,1\,2\,2\,2\,2\,3\,3\,4\,1}\ \boxed{6\,8\,8\,8\,5}\ \boxed{10\ \ 9}\ \boxed{11\ 11\ 13} \ \rightarrow$$

$$\boxed{0\,1\,1\,2\,1\,2\,2\,3\,2\,4\,3}\ \boxed{6\,8\,5\,8\,8}\ \boxed{10\ \ 9}\ \boxed{11\ 11\ 13} \ =$$

$$0\,1\,1\,2\,1\,2\,2\,3\ \,2\ \,4\ \,3\quad 6\,8\,5\,8\,8\quad 10\ \ 9\quad 11\ 11\ 13$$

To describe the inverse map, we use the following lemma.

Lemma 3.2. *For $e = e_0 \cdots e_n \in \mathbf{I}_n(B_2)$, if bab is an instance of a 101 pattern in e, then (i) a is immediately preceded by b in e, (ii) a occurs only once as part of a 101 pattern and such a's increase left to right.*

Proof. (i) Suppose, for a contradiction, that $bcab$ appears in e with $b > a$ (so bab is a 101) and $c \neq b$. If $c > b$, then cab is a forbidden 201; if $a < c < b$, then bca is a 210; if $c < a$, then bca is a 201. (ii) After the first occurrence of the descent ba all later entries are greater than or equal to b. □

Here is the inverse map. Given $e_0 \cdots e_n \in \mathbf{I}_n(B_2)$, place a circle around the "0" of each occurrence of a 101 pattern bab. From each circled a, draw an arrow to the last occurrence of the corresponding b and circle it. Then rotate left each set of circled entries connected by arrows.

3.3 Bijection from $\mathbf{I}_n(B_3)$ to $\mathbf{I}_n(B_2)$.

Recall that $B_2 = \{100, 110, 201, 210\}$ and $B_3 = \{101, 110, 201, 210\}$. We show that the bijection $\psi : \mathbf{I}_n(101, 110, 201) \to \mathbf{I}_n(100, 110, 201)$ of Theorem 5.1 in [2] restricts to send 210-avoiding inversion sequences in $\mathbf{I}_n(101, 110, 201)$ to 210-avoiding inversion sequences in $\mathbf{I}_n(100, 110, 201)$ and inversely. This follows from

Proposition 3.3. *If $e_i e_j$ starts a 210 pattern in $e \in \mathbf{I}_n(101, 110, 201)$, then $\psi(e)_i \psi(e)_j$ starts a 210 pattern in $\psi(e) \in \mathbf{I}_n(100, 110, 201)$, and similarly for ψ^{-1}.*

We prove the assertion for ψ; the ψ^{-1} case is analogous. Recall that ψ replaces each entry e_k that serves as the second "0" of a 100 pattern, say $e_k = a$, with the *largest* entry preceding the *last* occurrence of a *before* e_k, and changes no other entry.

Lemma 3.4. *If $e_i > e_j$ for some $i < j$ in $e \in \mathbf{I}_n(101, 110, 201)$, that is, $b := e_i$ is an inversion top, then b occurs only once in e.*

Proof. If b occurs twice before e_j, then bbe_j is a forbidden 110, and if there is a b after e_j, then be_jb is a 101. □

Now suppose $e_i e_j$ starts a 210 in $e \in \mathbf{I}_n(101, 110, 201)$. Then e_i and e_j each occurs only once in e by Lemma 3.4 and so are left unchanged by ψ. Take k minimal such that $e_i e_j e_k$ is a 210. If e_k is not the second "0" of a 100, then e_k is unchanged by ψ and $e_i e_j e_k$ is a 210 in $\psi(e)$. If $a := e_k$ is the second "0" of a 100 pattern, say baa, then the middle a occurs before e_j because by minimality of k there is no a between e_j and e_k, and there is no a between e_i and e_j because then $e_i a e_j$ would be a 201. Now $b < e_j$ because $b = e_j$ would imply bae_j is a 101, and $b > e_j$ would imply bae_j is a 201. In $\psi(e)$, the entry e_k becomes b and then $e_i e_j b$ is a 210 in $\psi(e)$. □

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