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# Enumerating spanning trees of a graph with edge constraints

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**Abstract**

Suppose that $F \cup H$ is a spanning subgraph of a complete graph $K_n$ of order $n$, where $F$ is a forest with $s$ components of orders $n_1, n_2, \ldots, n_s$ and $H$ is a subgraph of $K_n$. In this paper we prove that the number of spanning trees of $K_n$ containing all edges in $F$ and avoiding (containing no) edges in $H$ equals $n^{s-2}\left(\prod_{i=1}^{s} n_i\right)\prod_{\alpha}(n-\alpha)$, where the second product ranges over all Laplacian eigenvalues $\alpha$ of $H$; this generalizes a well-known result by Moon.

## 1 Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, and let $G^c$ be the complement of $G$. Denote the number of spanning trees of $G$ by $t(G)$. If $G$ is an edge-weighted graph, we also use $t(G)$ to denote the sum of the weights of spanning trees of $G$, where the weight of a spanning tree $T$ in $G$ is the product of weights of the edges in $T$. Enumeration of spanning trees in a graph is an important and popular topic in mathematics, physics, computer science, and so on, and has been studied extensively for a long time (see, for example, [1, 5, 6, 7, 8, 9, 11, 12, 16]).

The well-known Cayley's formula states that a complete graph $K_n$ with $n$ vertices has $n^{n-2}$ spanning trees [2]. This result is extended to a complete multipartite graph $K_{n_1,n_2,\ldots,n_s}$ [1], and it is proved that

$$t(K_{n_1,n_2,\ldots,n_s}) = n^{s-2}\prod_{i=1}^{s}(n-n_i)^{n_i-1},$$

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where $n = n_1 + n_2 + \cdots + n_s$.

For the enumeration of spanning trees of a complete graph $K_n$ with some constraints, Moon [13] first proved that the number of spanning trees of $K_n$ containing all edges in a spanning forest $F = T_1 \cup T_2 \cup \cdots \cup T_c$ with $c$ components, denoted by $t_F(K_n)$, can be expressed by

$$t_F(K_n) = n^{c-2} \prod_{i=1}^{c} n_i, \tag{1}$$

where $n_1, n_2, \ldots, n_c$ are the numbers of vertices of $T_1, T_2, \ldots, T_c$, respectively.

Recently, Dong and Ge [4] extended this result to a complete bipartite graph, and obtained an interesting formula as follows. If $F = T_1 \cup T_2 \cup \cdots \cup T_k$ is a spanning forest of a complete bipartite graph $K_{m,n}$ with a bipartition $(X, Y)$ satisfying $m_i = |X \cap V(T_i)|$ and $n_i = |Y \cap V(T_i)|$ for $1 \le i \le k$, then the number of spanning trees of $K_{m,n}$ containing edges in $F$ can be expressed by

$$t_F(K_{m,n}) = \frac{1}{mn} \left[ \prod_{i=1}^{k} (m_i n + n_i m) \right] \left( 1 - \sum_{j=1}^{k} \frac{m_j n_j}{m_j n + n_j m} \right), \tag{2}$$

which generalizes the results in [15] by Zhang and Yan, and in [10] by Ge and Dong. Furthermore, Cheng, Chen and Yan [3] extended Eq. (2) to complete multipartite graphs.

On the other hand, Weinberg [14] first considered the enumerative problem of spanning trees of a complete graph $K_n$ containing no edges in a subgraph $H$ of $K_n$. This is equivalent to enumerating spanning trees of the graph $G = K_n - E(H)$ obtained from $K_n$ by deleting all edges in $H$. Weinberg [14] obtained the following formulae:

$$t(K_n - pK_2) = n^{n-2} \left( 1 - \frac{2}{n} \right)^p \tag{3}$$

and

$$t(K_n - E(K_{1,k})) = (n - 1 - k)(n - 1)^{k-1} n^{n-k-2}, \tag{4}$$

where $pK_2$ is a matching of $K_n$ ($2p \le n$), and $K_{1,k}$ is a star with $k+1$ vertices which is a subgraph of $K_n$ ($k \le n-1$).

More generally, if $H$ is a subgraph of $K_n$ with $s$ vertices and has Laplacian eigenvalues $0 = \alpha_1 \le \alpha_2 \le \cdots \le \alpha_s$, then the Laplacian eigenvalues of $G = K_n - E(H)$ are $0, \overbrace{n, \ldots, n}^{n-s}, n - \alpha_2, n - \alpha_3, \ldots, n - \alpha_s$, which implies that the number of spanning trees of $K_n$ containing no edges in $H$ can be expressed by

$$t(K_n - E(H)) = n^{n-s-2} \prod_{i=1}^{s} (n - \alpha_i), \tag{5}$$

where the product is over all Laplacian eigenvalues of $H$ [1]. Obviously, Eq. (5) generalizes Eqs. (3) and (4).

A natural problem is to enumerate spanning trees of $K_n$ containing all edges in $F$ and avoiding (containing no) edges in $H$, where $F$ is a forest of $K_n$ and $H$ is a subgraph of $K_n$ such that $V(F) \cap V(H) = \emptyset$ and $V(K_n) = V(F) \cup V(H)$; that is, to calculate $t_F(G)$, where $G = K_n - E(H)$.

Ge and Dong [10] first considered a similar problem, and obtained the following result. Let $M = A \cup B$ be a matching of a complete bipartite graph $K_{m,n}$ with $k$ edges, and $A \cap B = \emptyset$, $|A| = k - i$ and $|B| = i$. Then the number of spanning trees of $K_{m,n}$ containing all edges in $A$ and avoiding edges in $B$ can be expressed by

$$t_{k-i,i}(K_{m,n}) = \\ (m+n)^{k-i-1}(mn-m-n)^{i-1}[(m+n-k)(mn-m-n)+imn]m^{n-k-1}n^{m-k-1}.$$

In this paper, we solve the problem above, and prove mainly the following result, whose proof will be given in the next section.

**Theorem 1.1.** *Let $G$ be a simple graph with $n$ vertices, which is the graph obtained from a complete graph $K_n$ by deleting all edges of a subgraph $H$ of $K_n$. Suppose that $F = T_1 \cup T_2 \cup \cdots \cup T_s$ is a subforest of $G$ with $s$ components satisfying $V(F) \cap V(H) = \emptyset$ and $V(G) = V(F) \cup V(H)$. Then the number of spanning trees of $G$ containing all edges in $F$ is*

$$t_F(G) = n^{s-2} \left( \prod_{i=1}^{s} n_i \right) \prod_{\alpha} (n - \alpha), \tag{6}$$

*where $n_i = |V(T_i)|$ for $1 \le i \le s$, and the second product ranges over all Laplacian eigenvalues $\alpha$ of $H$.*

**Remark 1.1.** Obviously, Eqs. (1), (3), (4) and (5) are special cases of Eq. (6) for $V(H) = \emptyset$, $H = pK_2$ (a matching with $p$ edges) and $F = (n-2p)K_1$ ($n-2p$ isolated vertices), $H = K_{1,k}$ and $F = (n-k-1)K_1$, and $F = (n-s)K_1$, respectively.

**Remark 1.2.** In the theorem above, if $H$ is the vertex disjoint union of some complete bipartite graphs $K_{a_1,b_1}, K_{a_2,b_2}, \ldots, K_{a_t,b_t}$, i.e., the components of $H$ are $K_{a_1,b_1}, K_{a_2,b_2}, \ldots, K_{a_t,b_t}$, then

$$t_F(K_n - E(H)) = n^{s+t-2} \left( \prod_{i=1}^{s} n_i \right) \prod_{j=1}^{t} (n - a_j - b_j) a_j^{b_j - 1} b_j^{a_j - 1}.$$

In particular, if $H = pK_2$ is a matching of $K_n$, then

$$t_F(K_n - pK_2) = n^{s+2p-2} \left( 1 - \frac{2}{n} \right)^p \prod_{i=1}^{s} n_i.$$

## 2 Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. Firstly, we need to introduce some lemmas as follows.

**Lemma 2.1** (Matrix-Tree Theorem, [1]). *Let $L_G$ be the Laplacian matrix of a graph $G$. Then*

$$t(G) = (-1)^{i+j} \det(L_G)_{ij},$$

*where $(L_G)_{ij}$ is the submatrix of $L_G$ obtained from $L(G)$ by deleting the $i$-th row and $j$-th column.*

Let $G \vee H$ be the join of two vertex-disjoint graphs $G$ and $H$. That is, $G \vee H$ has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{(u, v)|u \in V(G), v \in V(H)\}$.

Given a vertex-weighted graph $G = (V(G), E(G))$ with a vertex weight function $\omega : V(G) \to \mathcal{R}$, this results in an edge-weighted graph, also denoted by $G$, in which each edge $e = uv \in E(G)$ has weight $\omega(u)\omega(v)$. Let $G^*$ be the weighted graph obtained from $G$ by replacing the induced subgraph $H$ of $G$ with $\{o\} \vee H^c$, where the weight of an edge in $G^*$ is defined as

$$\begin{cases} \omega(v_iv_j) = -\omega(v_i)\omega(v_j), & \text{if } v_iv_j \in E(H^c), \\ \omega(v_iv_j) = \omega(v_i)\omega(v_j), & \text{if } v_iv_j \in E(G)\backslash E(H), \\ \omega(ov_i) = \omega(v_i) \sum\limits_{v \in V(H)} \omega(v), & \text{if } v_i \in V(H). \end{cases}$$

Zhou and Bu [16] used the Schur complement formula to give the mesh-star transformation in vertex-weighted version as follows, which will play an important role in the proof of the main result.

**Lemma 2.2** ([16]). *Let $G$ be a weighted graph with vertex set $V(G)$ and edge set $E(G)$ and let $\omega : V(G) \to \mathcal{R}$ be the vertex-weighted function, where each edge $v_iv_j \in E(G)$ has weight $\omega(v_i)\omega(v_j)$. Keeping the notation above,*

$$t(G) = \frac{1}{\left(\sum\limits_{v \in V(H)} \omega(v)\right)^2} t(G^*). \tag{7}$$

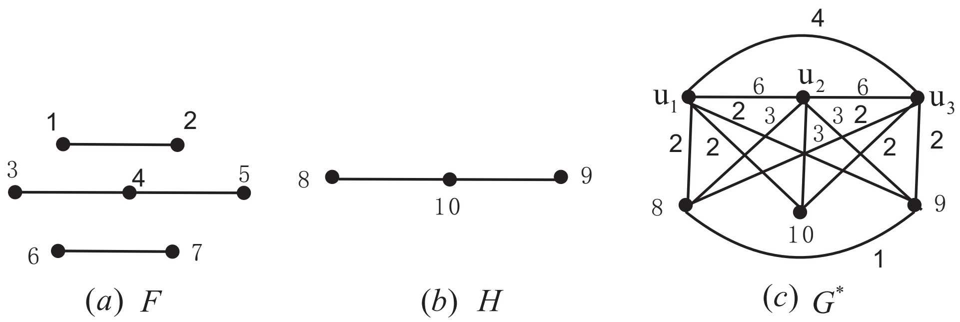


Figure 1: (a) The subforest $F = P_2 \cup P_3 \cup P_2$ in $K_{10}$.
(b) The subgraph $H$ of $K_{10}$. (c) The graph $G^*$.

Now, we can give the proof of Theorem 1.1 as follows.

**Proof of Theorem 1.1.** Note that $F \cup H$ is a spanning subgraph of $K_n$ and $F = T_1 \cup T_2 \cup \cdots \cup T_s$ is a subforest of $G = K_n - E(H)$ with $s$ components of $n_1, n_2, \ldots, n_s$ vertices and $H$ is a subgraph of $K_n$ such that $V(F) \cap V(H) = \emptyset$. Hence $H$ has $n - n_1 - n_2 - \cdots - n_s$ vertices. Contracting each component $T_i$ of $T$ in $G = K_n - E(H)$ into a new vertex $u_i$ for $i = 1, 2, \ldots, s$, we get a new edge-weighted graph $G^*$ with vertex set $V(G^*) = \{u_1, u_2, \ldots, u_s\} \cup V(H)$ and edge set $E(G^*) = \{u_iu_j \mid 1 \le i < j \le s\} \cup \{u_iv \mid 1 \le i \le s, v \in V(H)\} \cup E(H^c)$, and the edge weight function $\omega$ satisfies:

$$\begin{aligned} \omega(u_iu_j) &= n_in_j \ \text{ for } \ 1 \le i < j \le s, \\ \omega(u_iv) &= n_i \ \text{ for } \ 1 \le i \le s, v \in V(H), \\ \text{and} \quad \omega(e) &= 1 \ \text{ for all edges } \ e \in E(H^c), \end{aligned}$$

where $H^c$ is the complement of $H$. For example, if $n = 10$, $F = P_2 \cup P_3 \cup P_2$ and $H = P_3$, are illustrated in Figures 1(a) and (b), and the corresponding edge-weighted graph $G^*$ is illustrated in Figure 1(c).

Obviously, the number of spanning trees of $K_n$ containing all edges in $F$ and no edge in $H$ equals the sum of weights of spanning trees in $G^*$, that is,

$$t_F(G) = t_F(K_n - E(H)) = t(G^*). \tag{8}$$

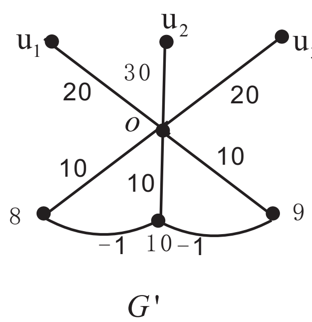


Figure 2: The edge-weighted graph $G'$.

Define a vertex weight function $\omega^* : V(G^*) \to \mathcal{R}$ such that $\omega^*(u_i) = n_i$ for $1 \le i \le s$ and $\omega^*(v) = 1$ for $v \in V(H)$. Obviously, the edge weight function $\omega$ of $G^*$ satisfies: $\omega(u_iu_j) = \omega^*(u_i)\omega^*(u_j)$ for any $1 \le i, j \le s$, $\omega(u_pv) = \omega^*(u_p)\omega^*(v)$ for $1 \le p \le s$, and $\omega(xy) = \omega^*(x)\omega^*(y)$ for any $x, y \in V(H)$.

Let $G'$ be the edge-weighted graph obtained from $G^*$ by replacing $G^*$ by $\{o\} \vee (G^*)^c$, where the weight of each edge in $G'$ is defined as

$$\begin{cases} \omega(xy) = -\omega^*(x)\omega^*(y) = -1, & \text{for } xy \in E((G^*)^c), \\ \omega(u_iu_j) = \omega^*(u_i)\omega^*(u_j) = n_in_j, & \text{for } 1 \le i \ne j \le s, \\ \omega(ou_i) = \omega^*(u_i) \sum\limits_{x \in V(G^*)} \omega^*(x) = n_i(\sum_{p=1}^{s} n_p + |V(H)|) = n_in, & \text{for } 1 \le i \le s, \\ \omega(ov) = \omega^*(v) \sum\limits_{x \in V(G^*)} \omega^*(x) = \sum_{p=1}^{s} n_p + |V(H)| = n, & \text{for } v \in V(H). \end{cases}$$

For the edge-weighted graph $G^*$ illustrated in Figure 1(c), the corresponding edge-weighted graph $G'$ is illustrated in Figure 2.

By Lemma 2.2,

$$t(G^*) = \frac{1}{\left(\sum_{x\in V(G^*)} \omega^*(x)\right)^2} t(G') = \frac{1}{n^2} t(G'). \tag{9}$$

Note that the induced subgraph $G_1$ of $G'$ with vertex set $\{o, u_1, u_2, \ldots, u_s\}$ is an edge-weighted star in which each edge $ou_i$ has weight $nn_i$ for $i = 1, 2, \ldots, s$, and the induced subgraph $G_2$ of $G'$ with vertex set $\{o\} \cup V(H)$ is an edge-weighted graph in which each edge $ov$ has weight $n$ for each $v \in V(H)$ and each edge $e \in E(H)$ has weight $-1$. Particularly, $V(G_1) \cap V(G_2) = o$ (i.e., $o$ is a cut vertex of $G'$). Thus

$$t(G') = t(G_1)t(G_2) = n^s \left(\prod_{i=1}^{s} n_i\right) t(G_2). \tag{10}$$

Note that if we delete the row and column corresponding to vertex $o$ of the Laplacian matrix $L_{G_2}$ of $G_2$, then we obtain the matrix $nI - L_H$, where $I$ is the $n \times n$ identity matrix and $L_H$ is the Laplacian matrix of $H$. By Lemma 2.1,

$$t(G_2) = \det(nI - L_H) = \prod_{\alpha} (n - \alpha), \tag{11}$$

where the product ranges over all Laplacian eigenvalues $\alpha$ of $H$.

The theorem is immediate from Eqs. (8)–(11). □

## 3 Discussion

In this paper, by the so-called mesh-star transformation in the vertex-weighted version by Zhou and Bu [16], we obtain an enumerative formula for the number of spanning trees in a complete graph $K_n$ containing all edges in a subforest $F$ and no edge in a subgraph $H$ of $K_n$, where $F \cup H$ is a spanning subgraph of $K_n$ which satisfies $V(F) \cap V(H) = \emptyset$. This result generalizes Moon's formula (i.e., Eq. (1)) and Weinberg's formulae (i.e., Eqs. (3) and (4)). Note that Dong and Ge [4] generalized Moon's formula to the case of the complete bipartite graph and Ge and Dong [10] obtained a formula for the number of spanning trees of a complete bipartite graph $K_{m,n}$ containing all edges in a matching $M_1$ of $K_{m,n}$ and avoiding all edges in a matching $M_2$ in $K_{m,n}$, where $V(M_1) \cap V(M_2) = \emptyset$. A natural problem is: if $F \cup H$ is a spanning subgraph of a complete multipartite graph $G = K_{n_1,n_2,\ldots,n_s}$ for $s \geq 2$, find a formula for the number of spanning trees of $G$ containing edges in $F$ and avoiding edges in $H$, where $F$ is a forest of $G$ and $H$ is a subgraph of $G$ such that $V(F) \cap V(H) = \emptyset$.

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