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2-Tone coloring of chordal and outerplanar graphs

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Abstract

A 2-tone coloring of a graph assigns two distinct colors to each vertex with the restriction that adjacent vertices have no common colors, and vertices at distance two have at most one common color. The 2-tone chromatic number of a graph is the minimum number of colors in any 2-tone coloring. We determine bounds and some exact formulas for the 2-tone chromatic number of powers of graphs, chordal graphs, and outerplanar graphs. We also determine an exact formula for the 2-tone chromatic number of maximal outerplanar graphs.

1 Introduction

There are many generalizations of classical vertex coloring of graphs. (See [3] for basic terminology and notation.) Some assign each vertex a set of colors, while others impose restrictions on colors of vertices at distance at least 2. These definitions have been generalized to define a coloring that assigns k colors to each vertex with restrictions on which sets may appear within distance k of each other.

Definition 1.1. [13] Let G be a graph, $k, t \in \mathbb{N}$, $[k] = \{1, 2, \ldots, k\}$, and let $\binom{[k]}{t}$ denote the set of t-element subsets of $[k]$. A function $f : V(G) \to \binom{[k]}{t}$ is called a *proper t-tone k-coloring* (or sometimes just a *t-tone coloring*) of G if $|f(u) \cap f(v)| < d(u, v)$ for all distinct vertices u and v of G. A graph is *t-tone k-colorable* if it has a proper t-tone k-coloring. The *t-tone chromatic number of G*, denoted by $\tau_t(G)$, is the smallest positive integer k for which G has a proper t-tone k-coloring.

This definition was first studied in a class project [13] in 2009. The 2-tone chromatic number has been determined for complete multipartite graphs, trees [13], cycles, theta graphs [7], Mobius ladders, wheels, fans, products of complete graphs, some products of cycles [2], Sierpinski triangle graphs, Hanoi graphs [4], and most cactus graphs [5]. General upper bounds were found in [2, 9, 10, 11, 12], and lower bounds were studied in [16].

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One group [1] considered the 2-tone chromatic number of the random graph. Several authors [2, 14, 8] have studied 2-tone coloring for graph products. For general t, t-tone coloring has been studied for cycles [18, 10], grids [10], and some hypercubes [19].

Note that for $t = 1$, $\tau_1(G) = \chi(G)$, the usual chromatic number of a graph G. This paper is solely concerned with the 2-tone chromatic number.

We shall often call $f(v)$ the *label* associated with the vertex v of the coloring f, and the elements of $f(v)$ will be called *colors*. Thus, in a 2-tone coloring, each vertex has a label of 2 distinct colors. Adjacent vertices have no common colors, and vertices distance two apart have at most one common color. When the context is clear, the label $\{a, b\}$ will be denoted ab. Vertices distance two apart are called *second-neighbors*.

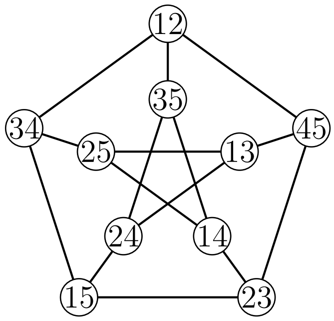
A *color class* is the set of all vertices with the same color in some coloring of the graph. A *k-chord* of a cycle is a pair of vertices of the cycle distance $k \geq 2$ apart. If a k-chord has its pair of vertices appear in a color class of a 2-tone coloring, we say the color class (and the coloring) *uses the k-chord*. The *center of a 2-chord* is a vertex adjacent to its two vertices.

Some basic results are immediate. The 2-tone chromatic number exists for all graphs. If H is a subgraph of G then $\tau_2(H) \leq \tau_2(G)$. We have $\tau_2(K_n) = 2n$. If G has components G_i, then $\tau_2(G) = \max \tau_2(G_i)$. Also, we see $2n(G) \leq \alpha(G) \cdot \tau_2(G)$ since each color class is an independent set, so $\tau_t(G) \geq \frac{2 \cdot n(G)}{\alpha(G)}$.

For a nontrivial tree T with maximum degree Δ, $\tau_2(T) = \left\lceil \frac{5+\sqrt{1+8\Delta}}{2} \right\rceil$ [13], so for stars, $\tau_2(K_{1,s}) = \left\lceil \frac{5+\sqrt{1+8s}}{2} \right\rceil$. For the cycle C_n, $\tau_2(C_n) = \begin{cases} 6 & n = 3, 4, 7 \\ 5 & n \neq 3, 4, 7 \end{cases}$ [7].

If $\tau_2(G) = k$, we call a 2-tone k-coloring of G a *minimum coloring*. Two colorings of a graph are *distinct* if they cannot be made the same by a permutation of the colors and an automorphism of the graph. A 2-tone k-coloring is *unique* if there are no two distinct k-colorings. The minimum colorings of C_n are unique for $n \in \{3, 4, 5, 6, 8, 9\}$ [7].

The *Kneser graph* $KG(r, k)$ has as its vertices all k-element subsets of $[r]$, with edges joining disjoint sets. A graph G is 2-tone r-colorable if and only if it has a homomorphism f from G to a graph $KG(r, 2)$ with the property that adjacent edges of G do not map onto the same edge of $KG(r, 2)$. Note that $KG(5, 2)$ is the Petersen graph (see below).



Definition 1.2. A *pair k-coloring* of a graph G is a 2-tone k-coloring in which every label is distinct. A graph is *pair k-colorable* if it has a pair k-coloring. The *pair chromatic number* of G, $pc(G)$, is the smallest k for which it has a pair k-coloring.

Some results on the pair chromatic number are immediate. We have $pc(G) \geq \tau_2(G)$, and if $diam(G) \leq 2$, then this is an equality. This implies that for a join $G+H$, $\tau_2(G+H) = pc(G+H) = pc(G) + pc(H)$. If H is a subgraph of G, then $pc(H) \leq pc(G)$. A graph G is pair r-colorable if and only if it is a subgraph of $KG(r,2)$. Thus if $n(G) > \binom{k}{2}$, $pc(G) > k$. Equivalently, $pc(G) \geq \frac{1+\sqrt{1+8n(G)}}{2}$.

Theorem 1.3. *[2] We have*

$$pc(P_n) = \begin{cases} 5 & 3 \leq n \leq 10 \\ \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil & n \geq 11 \end{cases}.$$

Definition 1.4. A graph G is *k-degenerate* if the vertices of G can be successively deleted, so that when each vertex v is deleted, it has degree at most k in the remaining graph. The *degeneracy* $D(G)$ is the smallest k such that G is k-degenerate. A *deletion sequence* of a graph G is an ordering $v_1, \ldots, v_n$ of $V(G)$ such that each v_i has minimum degree in the induced subgraph $G[\{v_i, v_{i+1}, \ldots, v_n\}]$.

Cranston et al. [9] found an upper bound for 2-tone coloring based on degeneracy and maximum degree. The following bound is a slight improvement proved using the same technique.

Theorem 1.5. *[2] Let G be a graph with degeneracy k and maximum degree $\Delta = \Delta(G)$. Then*

$$\tau_2(G) \leq 2k + \left\lceil \frac{1+\sqrt{9+8\left(2\Delta k - \Delta - k^2\right)}}{2} \right\rceil.$$

In Section 2, we prove an upper bound for the 2-tone chromatic number of chordal graphs, and for the class of simple k-trees. In Section 3, we prove an exact formula for maximal outerplanar graphs, and use this to find an improved upper bound for all outerplanar graphs. In Section 4, we consider powers of paths, cycles, and trees, some of which are chordal.

2 Chordal Graphs

Cranston et al. [9] found a bound on the 2-tone chromatic number of chordal graphs. In this section, we prove a bound that is usually stronger than theirs. We also prove a better bound for a special class of chordal graphs, the simple k-trees.

Definition 2.1. A graph G is *chordal* if every cycle of length more than three has a chord, that is, G contains no induced cycle other than C_3. A *simplicial vertex* is a vertex whose neighbors induce a clique. A *simplicial elimination ordering* is formed by successively deleting a simplicial vertex of G until none remain.

Any chordal graph has a simplicial elimination ordering.

Proposition 2.2. *(Cranston et al. [9]) a. If G is a chordal graph, then $\tau_2(G) \leq \left\lceil \left(1+\frac{\sqrt{6}}{2}\right)\Delta \right\rceil + 1$.*

b. For every $\epsilon > 0$, there exists an r_0 such that whenever $r > r_0$, if G is a chordal graph with maximum degree r, then $\tau_2(G) \leq (2+\epsilon) r$.

Since $D(G) \leq \Delta(G)$, the following result is almost always better.

Theorem 2.3. *Let G be a chordal graph with $\Delta = \Delta(G)$ and degeneracy $k \leq \frac{9}{10}\Delta$. Then*

$$\tau_2(G) \leq 2k + \left\lceil \frac{1+\sqrt{5+8k(\Delta-k)}}{2} \right\rceil.$$

Proof. We color G using the reverse of a simplicial elimination ordering. Let v be a vertex and S be the set of its previously colored neighbors. Let $r = |S|$, $r \leq k$, so S requires $2r$ colors. Each $u \in S$ has at most $\Delta - r$ neighbors outside $S \cup \{v\}$. Thus v has at most $r(\Delta - r)$ second-neighbors. Thus we need s extra colors, where $\binom{s}{2} > r(\Delta - r)$. Thus $s^2 - s - 2r(\Delta - r) - 1 \geq 0$, so $s \geq \frac{1+\sqrt{5+8r(\Delta-r)}}{2}$. We need at least $f(r) = 2r + \frac{1+\sqrt{5+8r(\Delta-r)}}{2}$ colors to guarantee a label for v. Using calculus, it is easily verified that $f(r)$ is increasing when $r \leq \frac{9}{10}\Delta$, so it is maximized by $r = k$. Thus G can be colored using at most $2k + \left\lceil \frac{1+\sqrt{5+8k(\Delta-k)}}{2} \right\rceil$ colors. □

The bound is also true when $k = \Delta$ (and G is complete). The assumption $k \leq \frac{9}{10}\Delta$ is almost always true for chordal graphs of interest. For example, any k-tree with order $n \geq 2k+1$ has $k \leq \frac{1}{2}\Delta$.

Definition 2.4. A *k-tree* is a graph that can be formed by starting with K_{k+1} and iterating the operation of adding a new vertex adjacent to all the vertices of a k-clique of the existing graph. A *k-leaf* is a degree-k vertex of a k-tree. A *simple k-tree* is defined recursively by starting with K_{k+1} and iteratively adding a vertex adjacent to all vertices of a k-clique not previously used as the neighborhood of a k-leaf.

Thus K_{k+1} is the only k-tree of order $k+1$, and $K_{k+2} - e$ is the only k-tree of order $k+2$. A survey of k-trees, simple k-trees, and related graphs is in [6].

Theorem 2.5. *Let G be a simple k-tree with maximum degree $\Delta = \Delta(G)$. Then*

$$\tau_2(G) \leq 2k + \left\lceil \frac{1+\sqrt{1+8((k-1)(\Delta-k)+2)}}{2} \right\rceil.$$

Proof. Let G be a simple k-tree, and hence k-degenerate. The result is obvious for order $n = k+1$.

Note that deleting all k-leaves of G produces another k-tree, or K_k. Since any k-leaf is contained in k k-cliques, any simple k-tree with order $n \geq k+2$ has a k-leaf adjacent to a vertex u with degree at most $2k$.

Form a deletion sequence by successively deleting a k-leaf adjacent to a vertex with degree at most $2k$, and then the final k vertices. Let $S : v_1, \ldots, v_n$ be the reverse of this sequence, and color G using S. The first $k+1$ vertices all receive mutually disjoint labels.

When colored, v_i, $i \geq k+2$, has a set R of k mutually adjacent neighbors already colored, one of which has at most $2k-1$ colored neighbors. The set R excludes $2k$ colors. The vertices in R have a common neighbor other than v_i, since any k-clique of a k-tree is contained in a $k+1$-clique. Thus v_i has at most $(k-1)(\Delta - k) + (2k-k) - (k-1) = (k-1)(\Delta - k) + 1$ second-neighbors already colored. Thus we need r extra colors, where $\binom{r}{2} \geq (k-1)(\Delta - k) + 2$. Solving, we find $r \geq \frac{1+\sqrt{1+8((k-1)(\Delta-k)+2)}}{2}$. □

This theorem is not sharp. Simple 2-trees are equivalent to maximal outerplanar graphs, for which we prove an exact formula for τ_2 in the next section. Simple 3-trees are also known as Apollonian networks, a class of maximal planar graphs. For them, the bound becomes $\tau_2(G) \leq 6 + \left\lceil \frac{1+\sqrt{16\Delta-31}}{2} \right\rceil$.

3 Outerplanar Graphs

In this section, we prove an exact formula for the 2-tone chromatic number of maximal outerplanar graphs. We then use this, along with some structural results on outerplanar graphs, to prove an improved bound on the 2-tone chromatic number of outerplanar graphs.

Since every outerplanar graph is 2-degenerate, Theorem 1.5 implies that for any outerplanar graph G with maximum degree $\Delta = \Delta(G) \geq 1$, $\tau_2(G) \leq 4 + \left\lceil \frac{1+\sqrt{24\Delta-23}}{2} \right\rceil$. Cranston and LaFayette [10] improved on this by using information about the structure of outerplanar graphs.

Theorem 3.1. *[10] Let G be an outerplanar graph with maximum degree $\Delta = \Delta(G)$. Then $\tau_2(G) \leq \left\lfloor \sqrt{2\Delta + 4.25} + 5.5 \right\rfloor$.*

A *maximal outerplanar graph* (MOP) cannot have any edge added while remaining outerplanar. The previous bound is not sharp, even for MOPs, as we will see in Theorem3.4. One special class of MOPs are the fans, $P_n + K_1$. By Theorem 1.3,

$$\tau_2(P_n + K_1) = \begin{cases} 7 & 3 \leq n \leq 10 \\ \left\lceil \frac{5+\sqrt{1+8n}}{2} \right\rceil & n \geq 11 \end{cases}.$$

Note that greedy coloring need not be optimal for MOPs, or even for fans. For example, $\tau_2(K_1 + P_{10}) = 7$. However, coloring the center (the K_1) with 67 and greedily coloring the path starting at one end produces 12-34-15-23-14-25-13-24-35-, and then the only remaining label from $\{1, 2, 3, 4, 5\}$, 45, cannot be used on the last vertex.

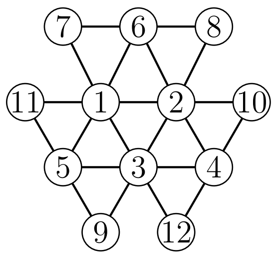
To optimally color MOPs, we successively color fans, coloring the vertices of each fan simultaneously. We need a way to construct a MOP by adding fans so that each does not overlap with the existing graph too much.

Lemma 3.2. *Any MOP can be constructed by starting with a fan and iteratively adding another fan overlapping on $K_1 + P_3$ or $K_1 + P_4$.*

Call this process a *fan construction* of the MOP.

Proof. Let G be a MOP, and consider sequence S, the reverse of a deletion sequence for G. We start with a fan centered at the first vertex in S, and iteratively add all remaining vertices of a fan F_v when its center v appears in S. Now v is adjacent to at most two previous vertices in S, and so is contained in at most two fans whose centers already appeared in S. Now v can have at most one extra neighbor in each fan, so it has at most four neighbors before F_v is completed. Thus when F_v is completed, it overlaps the existing graph on $K_1 + P_3$ or $K_1 + P_4$. □

For example, consider the graph below, for which $12, 11, \ldots, 1$ is a deletion sequence. A fan construction starts with the fan F_1, which includes vertices $\{7, 6, 2, 3, 5, 11\}$. Next we add F_2, which includes vertices $\{8, 6, 1, 3, 4, 10\}$, and overlaps F_1 on $K_1 + P_3$. Then add F_3, which includes vertices $\{9, 5, 1, 2, 4, 12\}$, and overlaps $F_1 \cup F_2$ on $K_1 + P_4$. Now F_4 overlaps the existing graph on $K_1 + P_4$, and no new vertices are added. The same is true for all remaining vertices.



To show that it is possible to color each fan in order, we prove a slightly stronger lemma.

Lemma 3.3. *When $r \geq 6$, there is a pair r-coloring of a cycle with $\binom{r}{2}$ vertices that extends any possible coloring of a copy of P_4 (denoted H) in the cycle so that the ends of H are the centers of 2-chords used in the pair k-coloring.*

Proof. Suppose the cycle contains vertices $abcdef$ in order, and $bcde$ are the vertices of H. There are multiple distinct colorings for H. They may use 2, 1, or 0 2-chords centered on c or d. There may also be a common color on b and e, or not. With symmetry, there are six possible cases, which we denote A-F (see below).

A	$-34-15-27-68-$
B	$-34-15-23-67-$
C	$-34-15-23-56-$
D	$-34-15-27-46-$
E	$-34-15-23-46-$
F	$-34-15-23-45-$

With 6 colors on the path, there are three possible colorings (cases C, E, and F). We align vertices 2 through 5 of the 15-cycles below with the existing path of length 4.

$-12-34-15-23-56-24-16-25-46-13-45-36-14-26-35-$

$-12-34-15-23-46-25-16-24-56-13-45-36-14-26-35-$

$-12-34-15-23-45-36-24-35-16-25-46-13-26-14-56-$

With 7 colors on the path, there are five possible colorings (cases B-F). We align vertices 2 through 5 of the 21-cycles below with the existing path of length 4.

$-12-34-15-23-67-25-37-24-17-46-27-16-47-56-13-45-36-57-14-26-35-$

$-12-34-15-23-56-24-37-16-47-25-17-46-27-13-67-45-36-57-14-26-35-$

$-12-34-15-27-46-25-37-16-47-23-17-24-56-13-67-45-36-57-14-26-35-$

$-12-34-15-23-46-25-37-16-57-24-17-56-27-13-67-45-36-14-26-47-35-$

$-12-34-15-23-45-36-17-24-67-35-27-16-47-25-37-46-57-13-26-14-56-$

With 8 colors on the path, all six cases are possible. Aligning vertices 2 through 5 of the 28-cycle below covers case A. It is easy to check that this 28-cycle also contains paths that work for cases B-F, permuting the colors as necessary.

$-12-34-15-27-68-25-18-67-38-24-17-28-46-23-14-58-47-56-48-13-45-36-57-16-37-26-78-35-$

For larger values of r, we seek a Hamiltonian cycle in the Kneser graph $KG(r,2)$ containing the labels of H in order. Note that $KG(r,2)$ has $\binom{r}{2}$ vertices, each with degree $\binom{r-2}{2}$. We delete the vertices with the same labels as $\{b,c,d,e\}$ from $KG(r,2)$. The new graph G has minimum degree at most four less than before. Now

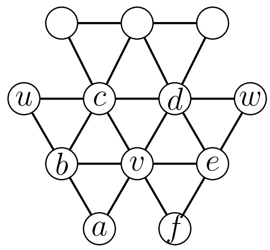
$$\frac{n(G)}{\delta(G)} \leq \frac{\binom{r}{2}-4}{\binom{r-2}{2}-4} = \frac{\frac{r(r-1)}{2}-4}{\frac{(r-2)(r-3)}{2}-4} = \frac{r(r-1)-8}{(r-2)(r-3)-8} < 2$$

when $r \geq 9$. Then $\delta(G) > \frac{n(G)}{2}$, so G is Hamiltonian-connected [15] (for every pair of vertices u, v there is a Hamiltonian $u-v$ path) when $r \geq 9$. Thus we can extend $bcde$ through a spanning path of G to form a Hamiltonian cycle of $KG(r,2)$ containing the labels of H in order. $\square$

Theorem 3.4. *Let G be a MOP with order $n \geq 4$ and $\Delta = \Delta(G)$. Then* $\tau_2(G) = \max\left\{7, \left\lceil \frac{5+\sqrt{1+8\Delta}}{2} \right\rceil\right\}$.

Proof. Any MOP G with $n \geq 4$ contains $K_4 - e$, and so has $\tau_2(G) \geq 7$. Also, G contains $K_{1,\Delta}$, so $\tau_2(G) \geq \left\lceil \frac{5+\sqrt{1+8\Delta}}{2} \right\rceil$. We show that G has a 2-tone coloring with $\max\left\{7, \left\lceil \frac{5+\sqrt{1+8\Delta}}{2} \right\rceil\right\}$ colors.

We color a MOP G using a fan construction. Say we color all the vertices of a fan F_v centered at v. By Lemma 3.2, we may assume that v neighbors c and d, the centers of previously colored fans. Thus there may be as many as 4 neighbors of v already colored, and they induce a path $bcde$ (see the example below).

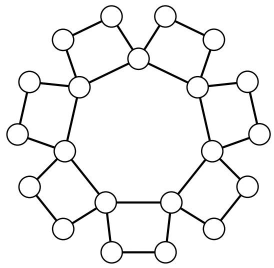


When $\Delta \leq 10$, we use 7 colors, and there is only one 2-tone 7-coloring of $K_1 + P_4$ up to permutation of colors. Now any 5-coloring of P_r, $3 \leq r \leq 10$, uses every 2-chord of the path. Thus F_v can be colored consistently with the existing coloring. There may be a previously colored vertex u not on F_v that is distance 2 from the center v of F_v and from a newly colored vertex a of F_v. However, u and v must share a color, so u and a cannot share a label. Similarly, w and f do not share a label.

Suppose $\Delta > 10$, and let r be so that $\binom{r-1}{2} < \Delta \leq \binom{r}{2}$. Then $r \geq 6$, so we need at least 6 colors on the path of some fan. By Lemma 3.3, there is a cycle with $\binom{r}{2}$ vertices that extends any possible coloring of a copy of P_4 in the cycle. Further, there are common colors on a and c, and hence no common label for u and a. Similarly, w and f do not share a label. Now F_v can be colored using the pair coloring of this cycle (deleting an edge, and possibly one or more vertices). Iterating, G can be colored using $\max\left\{7, \left\lceil \frac{5+\sqrt{1+8\Delta}}{2} \right\rceil\right\}$ colors. □

Theorem 3.4 implies a bound on τ_2 for any outerplanar graph with the same maximum degree. Note that not all outerplanar graphs are contained in a MOP with the same maximum degree (e.g. most paths and cycles). Identifying each edge of C_r, $r \geq 7$, with an edge of a copy of C_4 (see below for $r = 7$) produces an outerplanar graph that cannot be made into a MOP without increasing its maximum degree by at least 3. (See figure overleaf.)

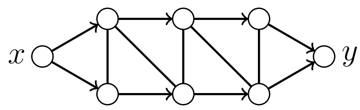
Thus it is of interest to determine how much the maximum degree must increase when an outerplanar graph is made maximal. Note that the problem of determining $\tau_2(G)$ can be reduced to finding τ_2 of proper subgraphs of G when G is disconnected or contains a bridge [7]. Thus we can limit our interest to 2-edge-connected outerplanar graphs.



Lemma 3.5. *If G is a 2-edge-connected outerplanar graph containing vertex x, then G is contained in a MOP H so that $d_H(x) = d_G(x)$ and $d_H(v) \leq d_G(v) + 3$ for every vertex v.*

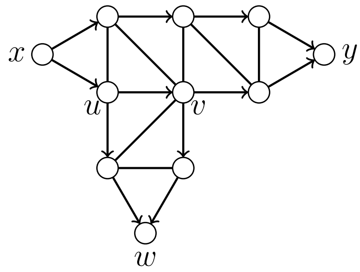
Proof. Any 2-edge-connected outerplanar graph can be formed by successively identifying chordless cycles on copies of K_2 or K_1 (this can be verified by induction on order). Note that any cycle can be made into a MOP (triangulated) by adding edges in its interior that form a path. For an r-cycle, we increase the degrees of $r - 4$ vertices by 2, 2 vertices 1, and leave two degrees unchanged. When we triangulate a cycle sharing an edge with a previously triangulated cycle, we triangulate it so that the two overlapping vertices have their degrees increased by 0 and 1.

We need to avoid adding 1 to the degree of the same vertex more than once unless its degree was not previously increased by 2. We can do this by orienting each edge of G. Make the initial cycle C_0 one that contains x. When C_0 has length at least 4, add a path inside C_0 so that the degree of x does not increase. There must be another vertex y on C_0 whose degree does not change. When $C_0 = K_3$, make y either vertex that isn't x. Orient the edges of the $x - y$ paths of C_0 toward y (see example below).

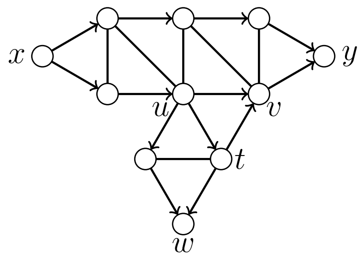


Say we add a new cycle C overlapping on directed edge uv. When C has length at least 4, add a path inside C so that v has degree increased by 1 and the degree of u does not change. Then there must be another vertex w on C whose degree does not change. Orient the edges of the $u - w$ and $v - w$ paths of C toward w. This guarantees that no vertex of C other than perhaps w will have its degree increased by 1 more than once when additional cycles are added, and the degree of w will be increased by 1 at most twice (see example overleaf).

Suppose we add a new cycle C overlapping only on vertex u. We may assume that there is a directed edge uv on the exterior region where no cycle will be overlapped (else we add that cycle first). Also, we may assume that there is a vertex t on C that immediately precedes u and v on the exterior region, and no cycle with be overlapped on tu (else we redefine C to be that cycle).



When C has length at least 4, add a path inside C so that t has degree increased by 1 and the degree of u does not change. Then there must be another vertex w on C whose degree does not change. Orient the edges of the $u-w$ paths of C toward w. Also, add directed edge tv. This guarantees that no vertex of C other than perhaps w will have its degree increased by 1 more than once when additional cycles are added, and the degree of w will be increased by 1 at most twice. The degrees of t and v cannot be increased further (see example below).



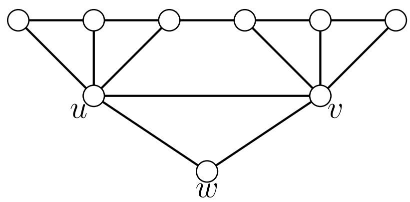
Thus no vertex has its degree increased by more than 3 no matter how many cycles are triangulated, and $d(x)$ is unchanged. □

When $\Delta(G)$ is large (relative to n), H can have the same maximum degree.

Lemma 3.6. *Let G be an outerplanar graph with degrees $d_1 \geq d_2 \geq \ldots \geq d_n$. If $d_1 = \frac{n+2+r}{2}$, then $d_2 \leq \frac{n+2-r}{2}$.*

Proof. Suppose G is outerplanar, $d_1 = \frac{n+2+r}{2}$, and to the contrary, $d_2 > \frac{n+2-r}{2}$. Then $d_1 + d_2 > \frac{n+2+r}{2} + \frac{n+2-r}{2} = n + 2$. Then two vertices have at least three common neighbors. Then G contains $K_{2,3}$, so it is not outerplanar. □

Thus when $\Delta(G) \geq \frac{n+5}{2}$ (and $r = 3$), G is contained in a MOP H with $\Delta(G) = \Delta(H)$. This result is nearly best possible. Let G be formed from $P_{2r} = v_1 \ldots v_{2r}$ by adding three vertices u, v, and w, with u adjacent to v_i, $i \in \{1, \ldots, r\}$, v adjacent to v_i, $i \in \{r+1, \ldots, 2r\}$, and add edges uv, uw, and vw (see below for $r = 3$). Then $\Delta(G) = \frac{n+1}{2}$, and any MOP H containing G has $\Delta(H) = \frac{n+3}{2}$.



Corollary 3.7. *Let G be outerplanar with order $n \geq 4$ and $\Delta = \Delta(G)$. Then* $\tau_2(G) \leq \max\left\{7, \left\lceil \frac{5+\sqrt{1+8(\Delta+3)}}{2} \right\rceil\right\}$.

Proof. The problem of determining $\tau_2(G)$ can be reduced to finding τ_2 of proper subgraphs of G when G is disconnected or contains a bridge [7]. When G is 2-edge-connected, Lemma 3.5 shows that G is contained in a MOP H with $\Delta(H) \leq \Delta(G)+3$. Then by Theorem 3.4, $\tau_2(G) \leq \tau_2(H) \leq \max\left\{7, \left\lceil \frac{5+\sqrt{1+8(\Delta(G)+3)}}{2} \right\rceil\right\}$. □

This bound is superior to Theorem 3.1 for all outerplanar graphs.

Note that although we know τ_2 for all MOPs, it may still be difficult to determine τ_2 for an outerplanar graph. A *cactus graph* has every block a cycle or edge. In another publication [5], I determined τ_2 for all cactus graphs with $\Delta \neq 6$. When $\Delta = 6$, there are some cactus graphs for which determining τ_2 is difficult.

4 Powers of Graphs

Since 2-tone coloring has been studied for graph products and other graph operations, it is natural to consider it for graph powers. In this section, we find exact formulas for the 2-tone chromatic number of powers of paths and squares of trees and cycles. The first two classes are chordal, while the last usually is not.

Definition 4.1. The k^{th} *power* G^k of a graph G adds all edges between pairs of vertices with distance at most k. The graph G^2 is the *square* of G.

We can determine the 2-tone chromatic number of some powers of graphs.

Proposition 4.2. *For $n \geq k+2$, we have $\tau_2\left(P_n^k\right) = 2k+3$.*

Proof. For $n \geq k+2$, $\tau_2\left(P_n^k\right) \geq \tau_2(K_{k+2} - e) = 2k+3$. Repeating the coloring $12 - 34 - 56 - \cdots - \{2k+3, 1\} - 23 - \cdots - \{2k+2, 2k+3\} -$ as long as necessary on the path provides a 2-tone $2k+3$-coloring. □

For sufficiently long cycles, the same formula holds.

Theorem 4.3. *For $n \geq (2k+2)(k^2+3k+1)$, we have $\tau_2\left(C_n^k\right) = 2k+3$.*

Before giving a formal proof, we provide an example. When $k=2$, we find 2-tone 7-colorings of C_7 and C_{12} that use no 2-chord or 3-chord and have no repeated label. Expressed as broken cycles, these are:

$$-12 - 34 - 56 - 71 - 23 - 45 - 67-$$

$$-12 - 34 - 56 - 71 - 32 - 54 - 16 - 37 - 52 - 14 - 36 - 57-$$

We "splice" these cycles together to produce the 2-tone 7-coloring C_{19} below.

$$-12-34-56-71-23-45-67-12-34-56-71-32-54-16-37-52-14-36-57-$$

For cycles $v_1v_2\dots v_rv_1$ and $u_1u_2\dots u_su_1$, we can *splice* them together to create the cycle $v_1v_2\dots v_ru_1u_2\dots u_sv_1$. Note that 2-tone k-colorings of two cycles can produce a 2-tone k-coloring of the spliced cycle if the first two vertices after the break in both cycles have the same labels (see also [10]). For the squares of cycles, we need the first four labels to agree.

Proof. There is a 2-tone $(2k+3)$-coloring of C_{2k+3} which is formed by letting each pair of vertices at distance $k+1$ on the cycle be a distinct color class. There is a 2-tone $(2k+3)$-coloring of $C_{(k+1)(k+2)}$ formed by letting $k+1$ color classes be $k+2$ equally spaced vertices around the cycle and letting $k+2$ color classes be $k+1$ equally spaced vertices around the cycle. These both provide 2-tone $(2k+3)$-colorings of the k^{th} powers of C_{2k+3} and $C_{(k+1)(k+2)}$. Now these colorings can be made to agree on the first $2k+2$ vertices, so they can be spliced together to form longer cycles.

Furthermore, the numbers $2k+3$ and $(k+1)(k+2)$ are relatively prime since $2k+3 \equiv 1 \mod k+1$ and $2k+3 \equiv -1 \mod k+2$. A theorem on Diophantine equations [17] guarantees that when a and b are relatively prime, there is a linear combination $ax+by=N$ of a and b with nonnegative coefficients x and y whenever $N \geq (a-1)(b-1)$. Thus when $N \geq (2k+2)((k+1)(k+2)-1)$, cycles of length $2k+3$ and $(k+1)(k+2)$ can be spliced together to obtain a 2-tone $2k+3$-coloring of an N-cycle. □

The bound on n is definitely not the best possible, since the proof uses only two cycles.

Hence only for the shorter cycles must the 2-tone chromatic number be determined. Certainly if $n \leq 2k+1$, $\tau_2\left(C_n^k\right) = 2n$. If $n = 2k+2$, $\tau_2\left(C_n^k\right) = \tau_2\left(\overline{\frac{n}{2}K_2}\right) = \frac{3}{2}n$, and if $n = 2k+3$, $\tau_2\left(C_n^k\right) = \tau_2\left(\overline{C}_n\right) = n$ [7]. For C_n^2, filling in the rest is not too difficult.

Theorem 4.4. *We have*

$$\tau_2\left(C_n^2\right) = \begin{cases} 6 & n = 3 \\ 7 & n = 7, 12, 14, 15, 16, n \geq 18 \\ 8 & n = 4, 8, 9, 10, 11, 13, 17 \\ 9 & n = 6 \\ 10 & n = 5 \end{cases}$$

Proof. This is obvious for $3 \leq n \leq 6$. For $n \geq 5$, we have $\tau_2\left(C_n^2\right) \geq \tau_2\left(P_n^2\right) \geq 7$. Each color class of C_n^2 has size at most $\left\lfloor \frac{n}{3} \right\rfloor$. Thus $\tau_2\left(C_n^2\right) \geq \frac{2n}{\left\lfloor \frac{n}{3} \right\rfloor} > 7$ for $n \in \{8, 11\}$.

Any 2-tone coloring of C_n^2 can use each of the n 3-chords of C_n at most once. For $n = 9$, a 7-coloring would require at least four color classes of size 3, which would use $4 \cdot 3 = 12$ 3-chords. For $n = 10$, a 7-coloring would require six color classes of size 3, which would use $6 \cdot 2 = 12$ 3-chords. For $n = 13$, a 7-coloring would require at least five color classes of size 4, which would use $5 \cdot 3 = 15$ 3-chords. For $n = 17$, a 7-coloring would require at least six color classes of size 5, which would use at least $6 \cdot 3 = 18$ 3-chords.

The colorings below provide upper bounds. Note that 7-colorings exist for $n \in \{7, 12, 15, 16, 18, 20\}$ and 8-colorings exist for $n \in \{8, 9, 10, 11, 13\}$. Splicing broken cycles together works for the larger cycles.

$$-12-34-56-71-23-45-67-$$

$$-12-34-56-78-13-24-57-68-$$

$$-12-34-56-17-23-45-16-37-58-$$

$$-12-34-56-71-23-68-15-24-38-57-$$

$$-12-34-56-17-24-36-58-14-26-38-57-$$

$$-12-34-56-71-32-54-16-37-52-14-36-57-$$

$$-12-34-56-71-32-54-16-37-58-14-32-57-68-$$

$$-12-34-56-17-32-46-15-37-24-16-35-27-14-36-57-$$

$$-12-34-56-17-23-54-16-27-35-14-26-37-15-24-36-57-$$

$$-12-34-56-17-23-45-16-73-25-14-76-23-15-74-26-13-45-76-$$

$$-12-34-56-17-23-45-16-37-25-14-36-27-45-13-26-47-15-23-46-57-$$

□

Analysis of 3-chords shows that the minimum colorings are unique for C_n, $n \in \{3, 4, 5, 6, 7, 12, 15, 16\}$.

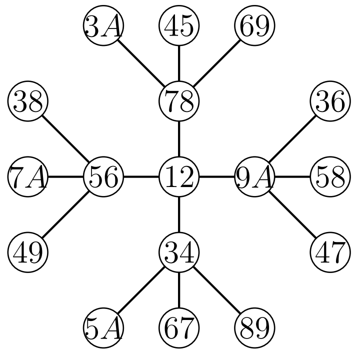
We have seen that $\tau_2\left(P_n^2\right) = 7$ for $n \geq 4$. Since a tree with $\Delta = 2$ is a path, we consider trees with $\Delta \geq 3$.

Theorem 4.5. *Let T be a tree with maximum degree $\Delta = \Delta(T) \geq 3$. Then $\tau_2\left(T^2\right) = 2(\Delta + 1)$.*

Before giving a formal proof, we provide an example. Let T be a tree with all internal vertices having degree 4. To color T^2, we start with a vertex v and color all its neighbors and second-neighbors in T. We then iteratively color all second-neighbors of a given vertex simultaneously. Say v gets label 12, and its neighbors get labels 34, 56, 78, and 9A (see the graph below). If u gets label 34, its other neighbors must use disjoint labels from $\{5, 6, 7, 8, 9, A\}$ without repeating any labels. Similar statements are true for the other neighbors of v.

A *branch* of T (with respect to v) is a component of $T - v$. We make a table (below left) with (say) the smaller of the two colors of each neighbor of v on the left. In each column, we list the other color used on a label of each branch of T. (The last column gives labels 34, 5A, 67, and 89 for the branch containing u.) There are four branches, so four columns. Each color appears exactly once in each row and column, so the table is a Latin square. Further, there must be a selection of four cells, one in each row and column, so that the four colors in these cells are distinct. (This is called a *transversal* of the Latin square.) The Latin square is shown below left with the squares of the transversal in boxes. The corresponding coloring is shown at right.

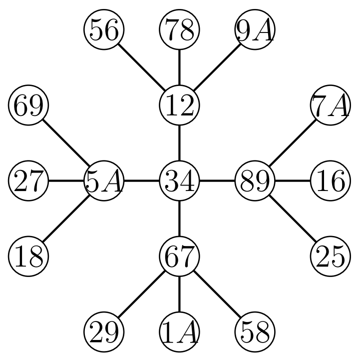
3	A	6	8	4
5	4	8	6	A
7	8	4	A	6
9	6	A	4	8



Next we extend this coloring to a neighbor of v, say u, which has 9 second-neighbors that must be colored consistent with the existing coloring. We use the same Latin square and swap in new colors. In the Latin square below left, we put 9, 7, 1, and 5 on the left, and 8, 6, 2, A in the transversal squares. To complete the Latin square, we fill in the same pattern as before using the new colors (put 8 in the same positions as 4 held in the first Latin square, etc.). This extends the coloring to the second-neighbors of u with no conflicts. A portion of T, now centered at u, is shown below right.

9				8
7			6	
1	2			
5		A		

9	A	6	2	8
7	8	2	6	A
1	2	8	A	6
5	6	A	8	2



Proof. Since $K_{\Delta+1} \subseteq T^2$, $\tau_2(T^2) \geq 2(\Delta+1)$. We may assume that all of T's internal vertices have degree Δ.

Begin coloring T^2 with an internal vertex v. It is well-known that there is an $n \times n$ Latin square with a transversal for all $n \neq 2$. (For all $n \notin \{2, 6\}$, there are two mutually orthogonal Latin squares, a stronger property.) Put $n = \Delta$ colors to the left margin of the Latin square, and use n colors to fill the Latin square. Each label is one color to the left and one from the same row in the Latin square. The labels in the transversal are used on the neighbors of v, and the labels from a given column are used on a single branch. Each branch has no common color, and no label is repeated, so we have a valid coloring so far.

Next we show how to extend this coloring when T^2 has diameter more than 2. We choose a neighbor of v, say u, that is an internal vertex of T^2. Then u has at most $(\Delta - 1)^2$ second-neighbors in T that must be colored consistent with the existing coloring. We use the same Latin square and swap in new colors. Choose one color from each of the Δ neighbors of u to put on the left margin. For each label of a neighbor of u, put the other color in the square of the transversal that is in the same row. To complete the Latin square, replace each old color with the same new color that it was replaced with in a square of the transversal. This extends the coloring to the second-neighbors of u with no conflicts. Iterating this process, the coloring can be extended to all vertices, so $\tau_2(T^2) \leq 2(\Delta + 1)$. □

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