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Some notes about the odd area of unit discs centered at points on a circle

Amir Carmel   Rom Pinchasi*

Mathematics Department
Technion – Israel Institute of Technology
Haifa 32000, Israel
amircarmel@campus.technion.ac.il   room@tx.technion.ac.il

Abstract

Let $\mathcal{F}$ be a family of n unit discs in the plane, where n is odd. A well-known open problem seeks to determine the minimum (over all such families $\mathcal{F}$) possible area $\mathrm{OA}(\mathcal{F})$ of all the points in the plane that belong to an odd number of discs in $\mathcal{F}$. In this paper we show that if $\mathcal{F}$ is a family of n unit discs in the plane whose centers lie on a circle of radius r where $0 \leq r \leq 1$, centered at the origin, then $\mathrm{OA}(\mathcal{F}) \geq \pi$. Furthermore, we show that as we push the discs in $\mathcal{F}$ towards the origin, keeping their centers on a circle centered at the origin, the function $\mathrm{OA}(\mathcal{F})$ decreases. Additionally, we provide a separate proof for the interesting case of $r = 1$ using completely different ideas.

One of the key tools we use is a new trigonometric inequality that is of independent interest. For a fixed odd integer $n \geq 3$ and $0 \leq \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_{n-1} \leq 2\pi$, we show

$$\sum_{0 \leq i < j < n} (-1)^{j-i+1} \sin(\alpha_j - \alpha_i) \geq 0.$$

1 Introduction

For a family $\mathcal{F}$ of measurable sets in the plane we define $\mathrm{OA}(\mathcal{F})$ to be the area of the set of all points in the plane that belong to an odd number of members in $\mathcal{F}$ (see Figure 1).

With a slight abuse of notation, for a measurable set B we denote by $\mathrm{OA}(B)$ the *odd area* of B. This is defined to be the infimum of $\mathrm{OA}(\mathcal{F})$ over all families $\mathcal{F}$ consisting of an odd number of translates of the set B. The notion of odd area was defined in [3]. It emerged from an Olympiad problem suggested by Uri Rabinovich

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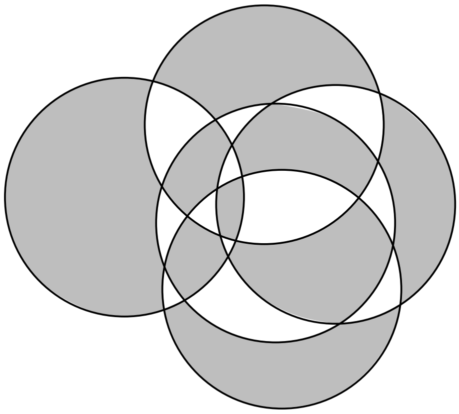

Figure 1: A family $\mathcal{F}$ of 5 unit discs. The area in gray is covered an odd number of times.

in [6]. See also [1, 5] for more results related to the notion of odd area of sets in the plane.

The question of finding the odd area of the unit disc in the plane was suggested by Igor Pak. It appears as an (unsolved) Exercise 15.14 in [2]. This is perhaps the most exciting problem about odd area. The reason for this is because for discs the problem is very symmetric and also because practically nothing until now is known about the odd area of the unit disc, despite considerable effort.

Problem A. *Given a family $\mathcal{F}$ of an odd number of unit discs in the plane, is it true that $OA(\mathcal{F}) \geq \pi$?*

The work of the second author in [4] gives necessary geometric and analytic conditions for a family of discs $\mathcal{F}$ such that $\mathrm{OA}(\mathcal{F})$ has a local minimum in the sense that any small shift of any of the discs in $\mathcal{F}$ will increase $\mathrm{OA}(\mathcal{F})$. It is further shown in [4] that if $\mathcal{F}$ is a family of an odd number of unit discs such that the intersection of all discs in $\mathcal{F}$ is a convex region whose boundary is composed of boundary parts of **all** discs in $\mathcal{F}$, then $\mathrm{OA}(\mathcal{F}) \geq \pi$. A special case of this result is the case where the centers of the discs in $\mathcal{F}$ lie on a circle of radius $0 < r < 1$. In the current paper we concentrate only on this special case and present several contributions beyond the results in [4]. We provide a different proof showing that $\mathrm{OA}(\mathcal{F}) \geq \pi$ if the centers of the discs in $\mathcal{F}$ lie on a circle of radius $0 < r < 1$. Our proof, which relies on the first part of [4], in fact preceded the work in [4] showing this very fact.

Our main theorem, which we prove in Section 3, is in fact stronger and shows much more than what is shown in [4] about the case where the centers of the discs in $\mathcal{F}$ lie on a circle of radius $0 < r < 1$. We show that the odd area of the family $\mathcal{F}$ is monotone increasing in r in the interval $0 < r < 1$. Interestingly, this monotonicity does not necessarily exist for $r > 1$. More precisely, our main theorem is the following:

Theorem 1. *Let n be an odd positive integer and $0 \leq \alpha_0, \ldots, \alpha_{n-1} \leq 2\pi$ be fixed. For every $0 \leq r$ define $f(r)$ to be equal to $OA(\mathcal{F})$, where $\mathcal{F}$ is the family of unit discs centered at $(r\cos\alpha_j, r\sin\alpha_j)$ for $j = 0, \ldots, n-1$. Then $f(r)$ is a monotone increasing function of r for $0 < r < 1$.*

We remark that an immediate consequence of Theorem 1 is that $\mathrm{OA}(\mathcal{F}) \geq \pi$ for any family $\mathcal{F}$ of odd number of unit discs centered at points on a circle of radius $0 \leq r \leq 1$. This is because the function $f(r)$ in Theorem 1 is continuous and clearly $\lim_{r \to 0^+} f(r) = \pi$ because n is odd.

In particular we conclude that $f(1) \geq \pi$. In Section 4 we give an independent proof of this fact using completely different ideas, not related in any way to those in [4], that we hope may help in obtaining future progress on Problem A. The proof is more elementary and very elegant and suggests further ideas for study.

Theorem 2. *Let $\mathcal{F}$ be a family of an odd number of unit discs centered at points on a circle of radius* 1 *(see Figure 2). Then $OA(\mathcal{F}) \geq \pi$. Equality is possible only if $\mathcal{F}$ consists of pairs of identical unit discs plus another unit disc.*

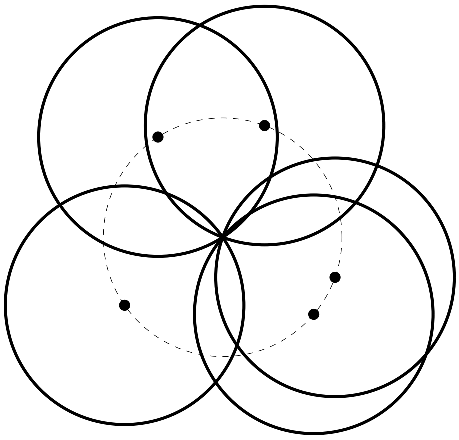

Figure 2: A family $\mathcal{F}$ of 5 unit discs whose centers lies on a common unit circle.

Our proof of Theorem 1 is based on the following independently interesting trigonometric inequality whose proof we bring in Section 2.

Lemma 1. *Let $n \geq 3$ be a fixed odd integer. Assume $0 \leq \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_{n-1} \leq 2\pi$. Then*

$$\sum_{0 \leq i < j < n} (-1)^{j-i+1} \sin(\alpha_j - \alpha_i) \geq 0.$$

We conclude this article with two examples. The first is an example of a family $\mathcal{F}$ of discs whose centers lie on a unit circle and which satisfies the analytic conditions required for a local minima as proved in [4]. This example motivated our study of collections of discs whose centers are located on a circle. The second example shows that $f(r)$ is not necessarily monotone when $r > 1$ and encourages further work on this case.

2 Proof of Lemma 1.

We prove the lemma by induction on n. For $n = 3$ the statement can be interpreted geometrically by considering the three unit vectors v_0, v_1, and v_2 in directions

α_0, α_1, and α_2, respectively. Then the sum $\sum_{0 \leq i<j<n} \sin(\alpha_j - \alpha_i)(-1)^{j-i+1}$ is equal to $\sin(\alpha_1 - \alpha_0) + \sin(\alpha_2 - \alpha_1) - \sin(\alpha_2 - \alpha_0)$. It is enjoyable to verify that as long as $0 \leq \alpha_0 \leq \alpha_1 \leq \alpha_2$ this is twice the area of the triangle with vertices v_0, v_1, and v_2.

Assume $n > 3$ (hence, $n \geq 5$). Consider $0 \leq \alpha_0 \leq \ldots \leq \alpha_{n-1} \leq 2\pi$ such that $\sum_{0 \leq i<j<n} \sin(\alpha_j - \alpha_i)(-1)^{j-i+1}$ is minimum. Notice that this minimum must be attained when $\alpha_0 < \ldots < \alpha_{n-1}$. This is because if say $\alpha_k = \alpha_{k+1}$, then we can omit both indices k and $k+1$ and conclude by the induction hypothesis for the case $n-2$. This is because if $\alpha_k = \alpha_{k+1}$, then the terms involving α_k and α_{k+1} in the sum $\sum_{0 \leq i<j<n} \sin(\alpha_j - \alpha_i)(-1)^{j-i+1}$ cancel each other.

Consider now any fixed index $0 \leq k < n$. For convenience define $\alpha_{-1} = \alpha_{n-1} - 2\pi$ and $\alpha_n = \alpha_0 + 2\pi$. We think of α_k as a variable x where $\alpha_{k-1} \leq x \leq \alpha_{k+1}$. Let $f_k : [\alpha_{k-1}, \alpha_{k+1}] \to \mathbb{R}$ be the function defined by

$$
\begin{aligned}
f_k(x) \quad &= \sum_{0 \leq i<j<n, i,j \neq k} \sin(\alpha_j - \alpha_i)(-1)^{j-i+1} \\
&\quad + \sum_{k<j} \sin(\alpha_j - x)(-1)^{j-k+1} + \sum_{i<k} \sin(x - \alpha_i)(-1)^{k-i+1} \\
&= \sum_{0 \leq i<j<n, i,j \neq k} \sin(\alpha_j - \alpha_i)(-1)^{j-i+1} \\
&\quad + \sum_{k<j} \sin(\alpha_j - x)(-1)^{j-k+1} - \sum_{i<k} \sin(\alpha_i - x)(-1)^{k-i+1}.
\end{aligned}
$$

Notice that we would like to show that $f_k(\alpha_k) \geq 0$. We will show something even stronger. We will show that f_k cannot have a minimum in the open interval $(\alpha_{k-1}, \alpha_{k+1})$. It will follow then that f_k must have a minimum either at $x = \alpha_{k-1}$ or at $x = \alpha_{k+1}$.

By our assumption, f_k has a minimum at $x = \alpha_k$. Because $\alpha_{k-1} < \alpha_k < \alpha_{k+1}$, we must have $f_k'(\alpha_k) = 0$. Moreover, we must also have $f_k''(\alpha_k) \geq 0$.

We observe that

$$f_k'(\alpha_k) = \sum_{k<j} \cos(\alpha_j - \alpha_k)(-1)^{j-k} - \sum_{i<k} \cos(\alpha_i - \alpha_k)(-1)^{k-i}. \tag{1}$$

$$f_k''(\alpha_k) = \sum_{k<j} \sin(\alpha_j - \alpha_k)(-1)^{j-k} - \sum_{i<k} \sin(\alpha_i - \alpha_k)(-1)^{k-i}. \tag{2}$$

The equations above allow us to conclude that $f_k''(\alpha_k) > 0$ (strictly greater). Indeed, this is because the function f_k has periodic derivatives of order 4 and $f^{(1)}(\alpha_k) = -f^{(3)}(\alpha_k) = 0$ and $f^{(2)}(\alpha_k) = -f^{(4)}(\alpha_k)$. If we assume to the contrary that $f_k''(\alpha_k) = 0$, then it will follow that all the higher derivatives of f_k vanish and therefore f_k is constant. In such a case we are done.

In order to reach a contradiction we consider the unit vectors $v_0, \ldots, v_{n-1} \in \mathbb{R}^2$ defined by $v_j = (\cos \alpha_j, \sin \alpha_j)$.

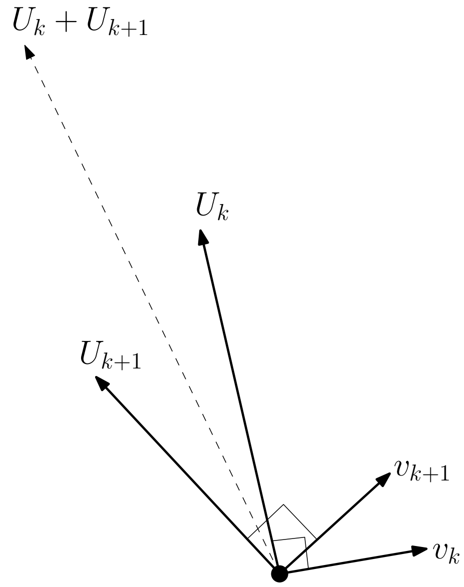

Figure 3: The vectors U_k, U_{k+1}, v_k, and v_{k+1}.

Equations (1) and (2) and the fact that $f'_k(\alpha_k) = 0$ and $f''_k(\alpha_k) > 0$ imply that if we rotate the vector $U_k = \sum_{k<j} v_j(-1)^{j-k} - \sum_{j<k} v_j(-1)^{j-k}$ by angle α_k about the origin in the clockwise direction, the resulting vector will have x-coordinate equal to 0 and positive y-coordinate. This implies that if we rotate U_k by angle $\frac{\pi}{2}$ about the origin in the clockwise direction, the resulting vector is a positive multiple of v_k.

Consider now also the vector

$$U_{k+1} = \sum_{k+1<j} v_j(-1)^{j-(k+1)} - \sum_{j<(k+1)} v_j(-1)^{j-(k+1)}.$$

Here we may conclude in the same way that if we rotate U_{k+1} by angle $\frac{\pi}{2}$ about the origin in the clockwise direction, the resulting vector is a positive multiple of v_{k+1} (see Figure 3).

Observe that $U_k + U_{k+1} = v_k - v_{k+1}$. This is geometrically impossible. It is easily seen when $\alpha_{k+1} - \alpha_k < \frac{\pi}{2}$, which we may indeed assume as we can choose such k because $n \geq 5$. In this case $U_k + U_{k+1}$ is to the left of v_{k+1} (in the sense that $\det(v_{k+1}, U_k + U_{k+1}) > 0$) while $v_k - v_{k+1}$ is to the right of v_{k+1} (that is, $\det(v_{k+1}, v_k - v_{k+1}) < 0$). $\square$

3 Proof of Theorem 1.

The proof of Theorem 1 goes by computing $f'(r)$ and using Lemma 1 to show that $f'(r) \geq 0$. Let $g(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$ be the function of $2n$ variables that is equal to the odd area of the family of n unit discs centered at $(x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$, respectively.

We use the result in [4] to calculate the partial derivatives of g at $(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) = (r\cos\alpha_0, r\sin\alpha_0, \ldots, r\cos\alpha_{n-1}, r\sin\alpha_{n-1})$.

For a circle C and two points A and B on C we denote by $C(A, B)$ the arc on C that starts at A and goes counterclockwise to B. We define a function g by letting $g(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$ denote the odd area of the n unit discs centered at $(x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$. The main theorem in [4] states as follows:

Theorem 3 ([4]). *Let $\mathcal{C} = \{C_0, \ldots, C_{n-1}\}$ be a family of unit circles centered at the pairwise distinct points $v_0 = (x_0, y_0), \ldots, v_{n-1} = (x_{n-1}, y_{n-1})$, respectively. Consider a fixed index $0 \leq i \leq n-1$ and let m be the number of circles from $\mathcal{C} \setminus \{C_i\}$ that intersect C_i. Let $u_0, \ldots, u_{2m-1}$ denote the intersection points, with multiplicities, of C_i with the m circles in $\mathcal{C} \setminus \{C_i\}$ that intersect C_i (notice that the number of these intersection points must indeed be $2m$, as we count touching points on C_i twice, if there are any). We index the points $u_0, \ldots, u_{2m-1}$ according to their counterclockwise cyclic order on C_i.*

Then $(-\frac{\partial g}{\partial y_i}, \frac{\partial g}{\partial x_i})$ is equal to $2M\sum_{j=0}^{2m-1}(-1)^j u_j$, where M is equal either to $+1$, or to -1. $M = +1$ if the region bounded by C_i and adjacent to the arc of $C_i(u_0, u_1)$ belongs to an even number of discs bounded by $C_1, \ldots, C_n$. Otherwise, $M = -1$.

In the statement of Theorem 3 we assume without loss of generality that $u_0 \neq u_1$, otherwise $m = 0$, or $m = 1$ and $u_0 = u_1$ and the theorem is almost trivially true.

To be consistent with the notation in Theorem 3, for $0 \leq j < n$ denote by C_j the unit circle centered at $(r\cos\theta_j, r\sin\theta_j)$. Fix $0 \leq i < n$. Consider the intersection points of C_i with the other unit circles C_j. The crucial observation is that because we assume $0 < r < 1$, then C_i intersects at two points with every C_j where $j \neq i$.

Let Z denote the point $((r+1)\cos\theta_i, (r+1)\sin\theta_i)$. Notice that Z lies on C_i and it does not belong to unit discs bounded by C_j for $j \neq i$.

For every $j \neq i$ denote by A_j and B_j the two intersection points of C_i and C_j such that Z, B_j, A_j is the counterclockwise cyclic order of these three points on C_i. We observe (and leave the reader to verify this easy fact, which is also shown in more generality in [4]) that the following counterclockwise cyclic order $Z, B_{i+1}, B_{i+2}, \ldots, B_{n-1}, B_0, B_1 \ldots, B_{i-1}, A_{i+1}, A_{i+2}, \ldots, A_{n-1}, A_0, A_1 \ldots, A_{i-1}$ is the counterclockwise cyclic order of these points on C_i (see Figure 4).

We take $u_0 = B_{i+1}$ in Theorem 3. Then $M = 1$ in Theorem 3 because $u_1 = B_{i+2}$ and points bounded by C_i very close to the arc $C_i(u_0, u_1) = C_i(B_{i+1}, B_{i+1})$ are covered precisely twice: by the unit discs bounded by C_i and C_{i+1}, respectively.

Theorem 3 gives

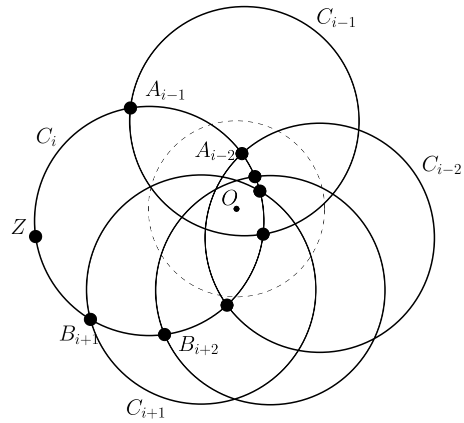

Figure 4: The intersection points of C_i with the other unit circles.

$$
\begin{aligned}
(-\frac{\partial g}{\partial y_i}, \frac{\partial g}{\partial x_i}) &= 2M \sum_{j=0}^{2m-1} (-1)^j u_j \\
&= 2 \sum_{j=i+1}^{n-1} (-1)^{j-i-1} B_j + 2 \sum_{j=0}^{i-1} (-1)^{j-i} B_j \\
&+ 2 \sum_{j=i+1}^{n-1} (-1)^{j-i-1} A_j + 2 \sum_{j=0}^{i-1} (-1)^{j-i} A_j. \qquad (3)
\end{aligned}
$$

Notice that for every $j \neq i$ we have $(A_j - v_i) + (B_j - v_i) = v_j - v_i$. Hence, $A_j + B_j = v_j + v_i$. Inserting this in (3) we get:

$$
\begin{aligned}
(-\frac{\partial g}{\partial y_i}, \frac{\partial g}{\partial x_i}) &= 2 \sum_{j=i+1}^{n-1} (-1)^{j-i-1} (v_j + v_i) + 2 \sum_{j=0}^{i-1} (-1)^{j-i} (v_j + v_i) \\
&= 2 \sum_{j=i+1}^{n-1} (-1)^{j-i-1} v_j + 2 \sum_{j=0}^{i-1} (-1)^{j-i} v_j. \qquad (4)
\end{aligned}
$$

We note that we used the fact that n is odd in the second equality in (4). Because $v_j = (x_j, y_j)$ we conclude

$$\frac{\partial g}{\partial x_i} = 2 \sum_{j=i+1}^{n-1} (-1)^{j-i-1} y_j + 2 \sum_{j=0}^{i-1} (-1)^{j-i} y_j, \qquad (5)$$

and

$$\frac{\partial g}{\partial y_i} = -2 \sum_{j=i+1}^{n-1} (-1)^{j-i-1} x_j - 2 \sum_{j=0}^{i-1} (-1)^{j-i} x_j. \qquad (6)$$

Observe that $f(r) = g(r\cos\alpha_0, r\sin\alpha_0, \ldots, r\cos\alpha_{n-1}, r\sin\alpha_{n-1})$. We would like to show that $f'(r) \geq 0$ for $0 < r < 1$. The chain rule gives:

$$f'(r) = \sum_{i=0}^{n-1}\left(\frac{\partial g}{\partial x_i}\cos\alpha_i + \frac{\partial g}{\partial y_i}\sin\alpha_i\right). \tag{7}$$

Combining (5), (6) and (7) and keeping in mind that $(x_j, y_j) = (r\cos\alpha_j, r\sin\alpha_j)$, we get

$$\begin{aligned}
f'(r) &= 2r\sum_{i=0}^{n-1}\cos\alpha_i\sum_{j=i+1}^{n-1}(-1)^{j-i-1}\sin\alpha_j + 2r\sum_{i=0}^{n-1}\cos\alpha_i\sum_{j=0}^{i-1}(-1)^{j-i}\sin\alpha_j \\
&\quad -2r\sum_{i=0}^{n-1}\sin\alpha_i\sum_{j=i+1}^{n-1}(-1)^{j-i-1}\cos\alpha_j - 2r\sum_{i=0}^{n-1}\sin\alpha_i\sum_{j=0}^{i-1}(-1)^{j-i}\cos\alpha_j \\
&= 2r\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}(-1)^{j-i-1}(\cos\alpha_i\sin\alpha_j - \sin\alpha_i\cos\alpha_j) \\
&\quad +2r\sum_{i=0}^{n-1}\sum_{j=0}^{i-1}(-1)^{j-i}(\cos\alpha_i\sin\alpha_j - \sin\alpha_i\cos\alpha_j) \\
&= 2r\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}(-1)^{j-i-1}\sin(\alpha_j - \alpha_i) + 2r\sum_{i=0}^{n-1}\sum_{j=0}^{i-1}(-1)^{j-i}\sin(\alpha_j - \alpha_i) \\
&= 2r\sum_{i=0}^{n-1}\sum_{j>i}^{n-1}(-1)^{j-i-1}\sin(\alpha_j - \alpha_i) + 2r\sum_{i=0}^{n-1}\sum_{j<i}(-1)^{j-i-1}\sin(\alpha_i - \alpha_j) \\
&= 4r\sum_{j>i}(-1)^{j-i-1}\sin(\alpha_j - \alpha_i).
\end{aligned}$$

We can now use Lemma 1 to conclude that $f'(r) \geq 0$ for $0 < r < 1$, as desired. □

4 The case $r = 1$ revisited: Proof of Theorem 2.

In this section, we study the case where $\mathcal{F}$ is a family of an odd number of unit discs whose centers lie on a fixed **unit** circle. We will prove Theorem 2 and show that $\mathrm{OA}(\mathcal{F}) \geq \pi$. As mentioned before, the result already follows from Theorem 1 because of the continuity of the function $f(r)$ in Theorem 1. The main reason for providing a separate proof for Theorem 2 is because we can prove this case using completely different ideas. We hope that these new ideas will find a use in other problems related to odd area and in particular the odd area of the unit disc.

Let $\mathcal{F}$ be a family of an odd number of unit discs whose set of centers $\mathcal{P} = \{P_1, \ldots, P_n\}$ is contained in a unit circle C centered at the origin. (This is precisely the case $r = 1$ in Theorem 1.) Let W denote the set of all points in the plane that belong to an odd number of discs from $\mathcal{F}$. Observe that a point X belongs to W if and only if the unit disc centered at X contains an odd number of points from $P_1, \ldots, P_n$. For every $0 < \alpha < \pi$ let S_α denote the set of all points X in the plane such that the unit disc centered at X intersects C at an arc of length α (see Figure 5). Notice that S_α is a circle of radius $r(\alpha) = 2\cos\frac{\alpha}{2}$ centered at the origin.

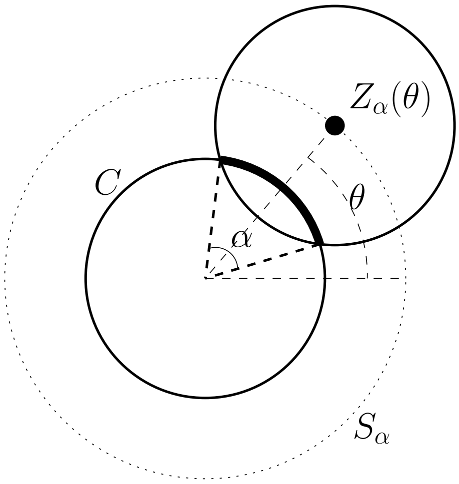

Figure 5: The definition of S_α and $Z_\alpha(\theta)$.

For every $0 < \alpha < \pi$ and $0 < \theta < 2\pi$ we denote by $Z_\alpha(\theta)$ the point $(r(\alpha)\cos\theta, r(\alpha)\sin\theta)$. This is the point on S_α with argument θ.

We observe that for every $0 < \alpha < \frac{\pi}{2}$ and $0 < \theta < \pi$ at least one of the four points $Z_\alpha(\theta), Z_{\pi-\alpha}(\theta + \frac{\pi}{2}), Z_\alpha(\theta + \pi)$, and $Z_{\pi-\alpha}(\theta + \frac{3\pi}{2})$ must belong to W. This is because the four unit discs centered at these four points partition C into four arcs that are pairwise disjoint except at their endpoints. At least one of these arcs must contain an odd number of points from $\mathcal{P}$, because n is odd and the points $P_1, \ldots, P_n$ lie on C, as illustrated in Figure 6 (we neglect here cases where a point P_j is a common endpoint of two of the arcs, as these cases have measure 0).

Let I_α be the set of all $0 < \theta < \pi$ such that at least one of the points $Z_\alpha(\theta)$ and $Z_\alpha(\theta + \pi)$ is in W. Denote by $|I_\alpha|$ the one-dimensional measure (length) of I_α. We get a contribution of

$$|I_\alpha| r(\alpha) dr(\alpha) = |I_\alpha| 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} d\alpha$$

to the calculated area of W. For every $0 < \theta < \pi$ not in I_α (the one-dimensional measure of this set is $\pi - |I_\alpha|$) it is true that at least one of $Z_{\pi-\alpha}(\theta + \frac{\pi}{2})$ and $Z_{\pi-\alpha}(\theta + \frac{3\pi}{2})$ is in W. This contributes

$$\begin{array}{rc} & (\pi - |I_\alpha|) r(\pi - \alpha) dr(\pi - \alpha) \\ = & (\pi - |I_\alpha|) 2\cos(\frac{\pi}{2} - \frac{\alpha}{2})\sin(\frac{\pi}{2} - \frac{\alpha}{2}) d\alpha \\ = & (\pi - |I_\alpha|) 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} d\alpha \end{array}$$

to the calculated area of W.

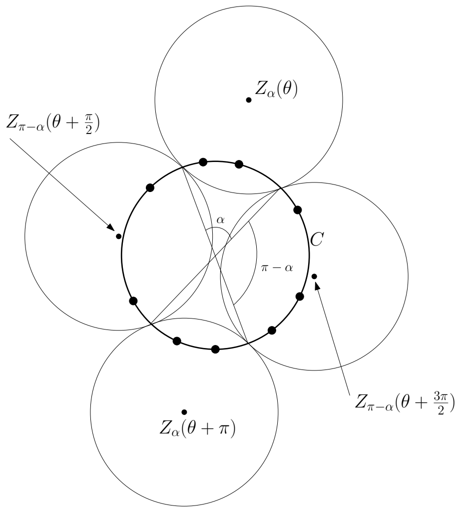

Figure 6: The unit discs centered at $Z_\alpha(\theta)$, $Z_{\pi-\alpha}(\theta+\frac{\pi}{2})$, $Z_\alpha(\theta+\pi)$, and $Z_{\pi-\alpha}(\theta+\frac{3\pi}{2})$ partition C into four pairwise disjoint arcs.

Altogether, we get a contribution of $\pi 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}d\alpha$ to the calculated area of W. Therefore, the area of W is at least $\pi\int_0^{\frac{\pi}{2}}\sin\alpha d\alpha=\pi$. This proves that $\mathrm{OA}(\mathcal{F})\geq\pi$.

This proof allows us to analyze easily the case of equality. It follows from the proof that the area of W is equal to π only if for almost (that is, up to measure zero) every $0<\alpha<\frac{\pi}{2}$ and $0<\theta<\pi$ *exactly* one of the four points $Z_\alpha(\theta), Z_{\pi-\alpha}(\theta+\frac{\pi}{2}), Z_\alpha(\theta+\pi)$, and $Z_{\pi-\alpha}(\theta+\frac{3\pi}{2})$ belongs to W. This means that precisely one of the four arcs that are the intersections of C with the four unit discs centered at $Z_\alpha(\theta), Z_{\pi-\alpha}(\theta+\frac{\pi}{2}), Z_\alpha(\theta+\pi)$, and $Z_{\pi-\alpha}(\theta+\frac{3\pi}{2})$, contains an odd number of points from $\mathcal{P}$. We may assume that the points $P_1,\ldots,P_n$ are pairwise distinct. If not, then we remove pairs that are equal, as this does not change the set W. The number of points n remains odd. If we remain with $n>1$ distinct points $P_1,\ldots,P_n$, then we can divide C into two half-circles, each of which contains a nonempty subset of $\mathcal{P}$. Let H_1 (respectively, H_2) denote the half-circle of odd (respectively, even) cardinality. We notice that the half-circles H_1 that we can choose have positive measure among all possible half-circles of C.

Because H_2 contains a nonempty subset of $\mathcal{P}$, there is α such that the arc s of length between $\alpha-\epsilon$ and $\alpha+\epsilon$ (for $\epsilon>0$ small enough) starting at one endpoint of H_2 contains an odd number of points of $\mathcal{P}$. The partition of C into the four arcs $s, H_2\setminus s, -s, -(H_2\setminus s)$ has precisely three arcs that contain an odd number of points from $\mathcal{P}$. Indeed, each of the arcs s and $H_2\setminus s$ contains an odd number of points from $\mathcal{P}$. In addition, the union of the two arcs $-s$ and $-(H_2\setminus s)$ is equal to H_1. The half-circle H_1 contains an odd number of points of $\mathcal{P}$. Therefore, necessarily one of $-s$ and $-(H_2\setminus s)$ must contain an odd number of points of $\mathcal{P}$.

Therefore, if the discs in $\mathcal{F}$ are distinct, then unless $n=1$, we must have a set of positive measure of α and θ for which more than one of $Z_\alpha(\theta), Z_{\pi-\alpha}(\theta+\frac{\pi}{2}), Z_\alpha(\theta+\pi)$, and $Z_{\pi-\alpha}(\theta+\frac{3\pi}{2})$ must belong to W. Consequently, the odd area of $\mathcal{F}$ is strictly greater than π in such a case.

We conclude that $\mathrm{OA}(\mathcal{F}) = \pi$ if and only if $n = 1$ or $\mathcal{F}$ is a union of pairs of equal unit discs plus another unit disc. This completes the proof of Theorem 2. $\square$

5 Further remarks and open problems

Consider the collection $\mathcal{F}$ of unit discs centered at $(\cos \frac{\pi i}{3}, \sin \frac{\pi i}{3})$ for $i = 0, \ldots, 5$ (see Figure 7).

The reader can easily verify that the partial derivatives $(\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial y_i})$ satisfy the conditions of Theorem 1 in [4] for any $i = 0, \ldots, 5$. That is, the gradient of f is $\vec{0}$, however, this is a saddle point rather than a local extremum.

We are not aware of a similar example with an odd number of unit discs in $\mathcal{F}$ (except for trivial variations like adding another copy of a unit disc not connected to the rest). Moreover, notice that in the example in Figure 7 there are pairs of touching discs. It will be interesting to find similar examples in which no two discs touch.

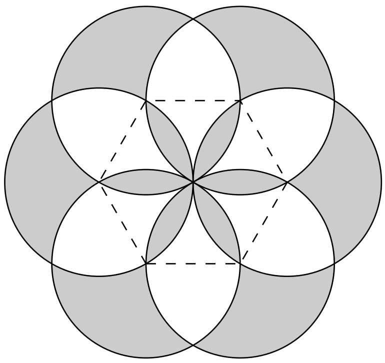

Figure 7: An example of a collection of discs $\mathcal{F}$ such that the function f has a $\vec{0}$ gradient. The discs in $\mathcal{F}$ are centered at the vertices of a hexagon that is enclosed inside a unit circle.

The next example shows the property $f'(r) > 0$, proved in Theorem 1 for the case $0 < r < 1$ is not necessarily true in the case $r > 1$. Hence the result regarding the odd area of discs for the case $r > 1$ cannot be derived in the same manner. In the next example, the discs are located at $(r \cos \frac{2\pi i}{n}, r \sin \frac{2\pi i}{n})$ for $i = 0, \ldots, n-1$. In this example, the number of discs is 23, although this behavior of $f'(r)$ occurs for every $n \geq 9$. As seen in Figure 8, $f'(r)$ fluctuate around 0 for $1 \leq r \leq 3.5$ and we note that a global minimum point is not reached in this example, i.e. $f(r) > \pi$ for every r.

Finally, we would like to mention an independently interesting problem directly related to the study of this paper. Let D be any compact set in the plane. Consider now a family $\mathcal{F}$ of an odd number of rotations of D about the origin O. What can be said about $\mathrm{OA}(\mathcal{F})$? Define the rotational odd area of D to be the infimum of $\mathrm{OA}(\mathcal{F})$ over all such families $\mathcal{F}$. Theorem 1 shows that if D is a unit disc that contains the origin, then it has a rotational odd area that is equal to π. It is interesting to

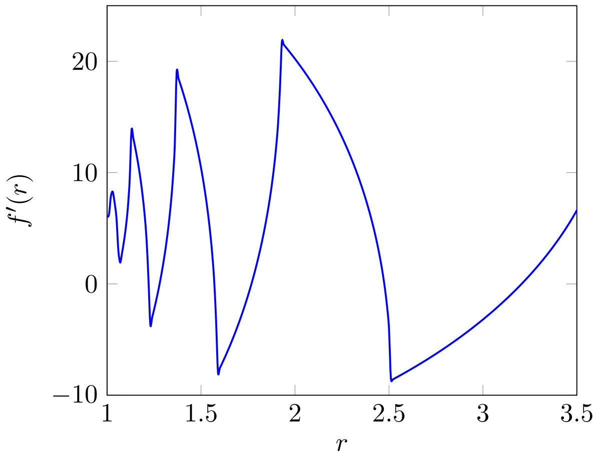

Figure 8: A plot of $f'(r)$ for different r, $1 \leq r \leq 3.5$. The number of discs is 23, where the i'th is located at $(r\cos\frac{2\pi i}{23}, r\sin\frac{2\pi i}{23})$ for $i = 0, ..., 22$.

study the rotational odd area of other sets and in particular of unit discs that do not contain the origin. This is the unresolved case $r > 1$ in Theorem 1.

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