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Every planar graph without 5-cycles adjacent to 6-cycles is DP-4-colorable

Xiangwen Li* Mao Zhang

School of Mathematics and Statistics
Central China Normal University
Wuhan 430079, China

Abstract

DP-coloring of a graph was introduced by Dvořák and Postle [*J. Combin. Theory Ser. B* 129 (2018), 38–54] as a generalization of a list coloring. Kim and Ozeki [*Discrete Math.* 341 (2018), 1983–1986] proved that planar graphs without k-cycles where $k \in \{3, 4, 5, 6\}$ are DP-4-colorable. Kim and Yu [*Graphs Combin.* 35 (2019), 707–718] proved that every planar graph without 3-cycles adjacent to 4-cycles is DP-4-colorable. So it was natural to ask whether every planar graph without i-cycles adjacent to j-cycles is DP-4-colorable for $i, j \in \{3, 4, 5, 6\}$ and $i \neq j$. For each $k \in \{5, 6\}$, Liu, Li, Nakprast, Sittitrai and Yu [*Discrete Appl. Math.* 277 (2020), 245–251] proved that every planar graph without 3-cycles adjacent to k-cycles is DP-4-colorable; Chen, Liu, Yu, Zhao and Zhou [*Discrete Math.* 341 (2019), 2984–2993] proved that every planar graph without 4-cycles adjacent to k-cycles is DP-4-colorable. In this paper, we answer the last case of this question and prove that every planar graph G without 5-cycles adjacent to 6-cycles is DP-4-colorable. This result also improves a result of Kim and Ozeki in the 2018 paper mentioned above.

1 Introduction

Coloring is one of the most popular topics in graph theory. Let G be a simple graph. A *proper coloring* of G is a function $c : V(G) \to [k] = \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ for any edge $uv \in E(G)$. A graph G is *k-colorable* if it has a k-coloring. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer k such that G is k-colorable. A *list assignment* L of a graph G is a mapping that assigns a set of colors to each vertex. An *L-coloring* of G is a function $f : V(G) \to \cup_{v\in V(G)} L(v)$ such that $f(v) \in L(v)$ for any $v \in V(G)$ and $f(u) \neq f(v)$ for any edge $uv \in E$. A graph G is *k-choosable* if G has an L-coloring for every assignment L with $|L(v)| \geq k$ for

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each $v \in V(G)$. The *choice number* of G, denoted by $\chi_l(G)$, is the smallest integer k such that G is k-choosable.

As a generalization of list coloring, DP-coloring (or corresponding-coloring) was first introduced by Dvořák and Postle [7]. The following equivalent definition is given by Bernsheteyn, Kostochka and Pron [3].

Definition 1.1 Let G be a simple graph, and L be a list assignment of G. Definite $L_v = \{u\} \times L(v)$ for any vertex $v \in V(G)$, and let M_{uv} be a matching (may be empty) between sets of L_v and L_u. Let $\mathcal{M}_L = \{M_{uv} : uv \in E(G)\}$, which is called the *matching assignment* over L. Let G_L be a graph, called an $\mathcal{M}_L$-*cover* of G, which satisfies the following conditions.

- The vertex set of G_L is $\cup_{v\in V(G)} L_v$.
- $G_L[L_v]$ is a clique for any vertex $v \in V(G)$.
- If $uv \in E(G)$, then the edges between L_u and L_v form a matching in M_{uv}.
- If $uv \notin E(G)$, then there is no any edge between L_u and L_v.

Definition 1.2 If G_L contains an independent set of size $|V(G)|$, then we say that G has an $\mathcal{M}_L$-*coloring*. If G has an $\mathcal{M}_L$-coloring for any k-list assignment L, and any matching assignment $\mathcal{M}_L$ over L, then G is *DP-k-colorable*. The *DP-chromatic number*, denoted by $\chi_{DP}(G)$, is the minimum positive integer k such that G is *DP-k*-colorable.

If for each $uv \in E(G)$, we define edges on G_L to match exactly the same colors between $L(u)$ and $L(v)$, then this $\mathcal{M}_L$-coloring is the ordinary list coloring. So list coloring is a special case of DP-coloring and $\chi_{DP}(G) \geq \chi_l(G)$ for each graph G.

DP-coloring has proved attractive recently. Dvořák and Postle [7] proved that $\chi_{DP}(G) \leq 5$ if G is a planar graph, and $\chi_{DP}(G) \leq 3$ if G is a planar graph with girth at least 5. Meanwhile, DP-coloring and list coloring are quite different. Bernshteyn [2] showed that the DP-chromatic number of every graph with average degree d is $\Omega(d/\log d)$, while Alon [1] proved that $\chi_l(G) = \Omega(\log d)$ and the gap is large. More results about DP-coloring can be found in [2, 3, 4, 5, 8, 11, 10, 12, 14, 15] and others.

A k-cycle is a cycle of length k. Kim and Ozeki [8] proved that planar graphs without k-cycles where $k \in \{3, 4, 5, 6\}$ are DP-4-colorable. Kim and Yu [9] proved that every planar graph without 3-cycles adjacent to 4-cycles is DP-4-colorable. One naturally asked the following question.

Question 1.3 *Is every planar graph without i-cycles adjacent to j-cycles DP-4-colorable for $i, j \in \{3, 4, 5, 6\}$ and $i \neq j$?*

For each $k \in \{5, 6\}$, Liu, Li, Nakprast, Sittitrai, Yu [13] proved that every planar graph without 3-cycles adjacent to k-cycles is DP-4-colorable; Chen, Liu, Yu, Zhao and Zhou [6] proved that every planar graph without 4-cycles adjacent to k-cycles is DP-4-colorable. In this paper, we answer the last case of Question 1.3 and prove the following result.

Theorem 1.4 *Every planar graph G without 5-cycles adjacent to 6-cycles is DP-4-colorable.*

A *cluster* in a plane graph G is a subgraph of G that consists of a minimal set of 3-faces such that no other 3-face is adjacent to any 3-face in this set. It is called a *k-cluster* if it contains k 3-faces. We present four clusters here (see Figure 1).

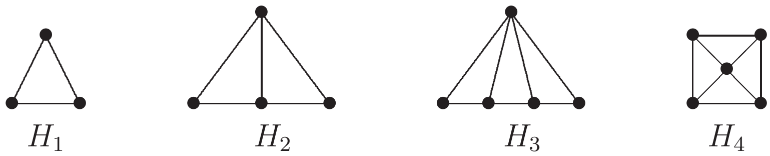

Figure 1: Four clusters

A face f in H_3 is *semi-poor, poor face* if it is adjacent to exactly one or two 3-faces, respectively. So, there are two semi-poor faces and exactly one poor face in any H_3. A 4-vertex in H_4 is called a *hub*.

Finally we introduce some notation and terminology used in this paper. Let G be a simple plane graph. We use F or $F(G)$ to denote the face set of G. For $f \in F(G)$, we write $f = [u_1u_2 \dots u_n]$ if $u_1, u_2, \dots, u_n$ are the boundary vertices of f in a cyclic order. A face of G is said to be incident with all edges and vertices in its boundary. The degree of a face f, denoted by $d_G(f)$, is the number of edges incident with it, where a cut edge is counted twice. A k-vertex (k^+-vertex, k^--vertex) is a vertex of degree k (at least k, at most k). A k-face (k^--face or k^+-face) is defined similarly. For convenience, a k-face $f = [v_1v_2 \dots v_k]$ is often said to be a $(d(v_1), d(v_2), \dots, d(v_k))$-face. Let C be a cycle of a plane graph G. We use $int(C)$ and $ext(C)$ to denote the sets of vertices located inside and outside C, respectively. The cycle C is called a *separating cycle* if $int(C) \neq \emptyset \neq ext(C)$.

2 Proof of Theorem 1.4

This section is devoted to proof of Theorem 1.4.

Let G_L be a cover of a graph G with a list assignment L. Let $G' = G - H$ where H is an induced subgraph of G. A list assignment L' is a restriction of L on G' if $L'(u) = L(u)$ for each vertex u in G'. A graph $G_{L'}$ is a restriction of G_L on G' if $G_{L'} = G_L[v \times L(v) : v \in V(G')]$. Assume that $G_{L'}$ has an $\mathcal{M}_{L'}$-coloring. Then $G_{L'}$ has an independent set I' of size $|I'| = |V(G)| - |V(H)|$. Define $L^*_x = L_x - \cup_{ux \in E(G)}\{(x, c') \in L_x : (u, c)(x, c') \in E(G_L), c' \in L(x), (u, c) \in I'\}$ for each $x \in V(H)$, and define $G_{L^*} = G_L[x \times L^*(x) : x \in V(H)]$. If H has an $\mathcal{M}_{L^*}$-coloring, then G_{L^*} has an independent set I^* of size $|I^*| = |V(H)|$. Since there are no edges between I' and I^*, $I' \cup I^*$ is an independent set in G_L of size $|I'| + |I^*| = |V(G)|$. Thus, G_L has an $\mathcal{M}_L$-coloring.

Lemma 2.1 ([8]) *For each $k \in \{3, 4, 5, 6\}$, every planar graph without k-cycles is DP-4-colorable.*

We now introduce *extendability*. Let G be a graph and C be a subgraph of G. Then (G, C) is DP-4-colorable if every DP-4-coloring of C can be extended to G.

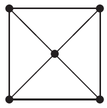

Figure 2. A bad 4-cycle

A 4-cycle is *bad* if it is the outer 4-cycle in the subgraph isomorphic to the graph in Figure 2 and *good* otherwise. For convenience, we say that every 3-cycle is a good cycle. In order to prove Theorem 1.4, we prove a stronger result as follows.

Theorem 2.2 *If G is a planar graph without 5-cycles adjacent to 6-cycles, then every precoloring of a induced good k-cycle can be extended to a DP-4-coloring of G, where $k = 3, 4$.*

Proof of Theorem 1.4 via Theorem 2.2. By Lemma 2.1, we may assume that G contains a k-cycle C, where $k = 3, 4$. By Theorem 2.2, every precoloring of C can be extended to G, so G is also DP-4-colorable. □

Let (G, C_0) be a minimal counterexample to Theorem 2.2 with $|V(G)| + |E(G)|$ minimized, where C_0 is a precolored k-cycle in G, where $k = 3, 4$. We claim that C_0 has no chord. Suppose otherwise that C_0 has a chord e_0 and two vertices of e_0 have colored different colors. Let $G' = G - e_0$. By the minimality of G, any DP-4-coloring of C_0 can be extended to a DP-4-coloring of G'. Thus, G has a DP-4-coloring, a contradiction. If C_0 is a separating cycle, then any precoloring of C_0 can be extended to $int(C_0)$ and $ext(C_0)$, respectively. Then we get a DP-4-coloring of G, a contradiction. So we may assume that C_0 is the boundary of the outer face of G in the rest of this paper. A vertex v is an *internal vertex* if $v \notin C_0$. For an internal 4^+-vertex v is in a cluster H, where $H \in \{H_1, H_2, H_3, H_4\}$, v is called *i-type* to H if v is incident with exactly i edges in H.

Lemma 2.3 *Each internal vertex is a 4^+-vertex.*

Proof. Suppose to the contrary that x is an internal 3^--vertex. By the minimality of G, $G' = G - x$ admits an $\mathcal{M}_{L'}$-coloring where L' is a restriction of L in G'. Thus $G_{L'}$ has an independent set I' of size $|I'| = |V(G')|$. Consider a list assignment L^* on x. Since $|L(x)| = 4$ and $d(x) \leq 3$, we obtain $|L^*_x| \geq 1$. Clearly, $(x, c) \in L^*_x$ is an independent set in $G[\{x\}]$. Then $I' \cup \{(x, c)\}$ is an independent set of G_L and hence G has an $\mathcal{M}_L$-coloring, a contradiction. □

By Lemma 2.3, since G has no 5-cycles adjacent to 6-cycles, G has four clusters depicted in Figure 1.

Lemma 2.4 *G contains no separating good k-cycle, where $k = 3, 4$.*

Proof. Let C be a separating good k-cycle in G. By the minimality of (G, C_0), any precoloring of C_0 can be extended to $G - int(C)$. After that, C is precolored, then again the coloring of C can be extended to $int(C)$. Thus, G has a DP-4-coloring, a contradiction. □

Lemma 2.5 *(a) Assume that g is a 4-cycle which is not bad and f is a 3-face which is not C_0. If a 4-face g is adjacent to f, then f cannot be adjacent to any 3-face and g cannot be adjacent to any 3- or 4-face h, where $h \neq f$.*

(b) If v is a 5^+-vertex incident with three consecutive 3-faces, then none of the 3-faces can be adjacent to any other 3-faces.

(c) A 3-face f is not adjacent to a 5-face g.

(d) For $k \geq 5$, a k-vertex is incident to at most $k - 2$ triangles.

Proof. (a) Let $f = [uvw]$ and $g = [uwxy]$. Since f is not C_0, x and y are outside f and v is outside g by Lemma 2.4.

We first show that f cannot be adjacent to a 3-face. Suppose to the contrary that f is adjacent to a 3-face $h = [vzw]$ by symmetry. Since x and y are outside f, by Lemma 2.4, z is outside of both f and g. Let $S = \{u, v, w, x, y\}$. If $z \notin S$, then $uvwxyu$ is a 5-cycle adjacent to a 6-cycle $uvzwxyu$, a contradiction. Thus, assume that $z \in S$. Then $z = x$ or $z = y$. If $z = x$, then u and y are either inside or outside $vwxw$. In the former case, $vuyxv$ is a 4-cycle. By Lemma 2.4, such a 4-cycle is a 4-face and hence $d(y) = 2$, contrary to Lemma 2.3. In the latter case, $vwxv$ is a 3-face by assumption and hence $d(w) = 3$, contrary to Lemma 2.3. If $z = y$, then u and x are either inside or outside $vwyv$ of G. In each case, $uwyu$ is a separating 3-cycle, contrary to Lemma 2.4.

Next we show that g cannot be adjacent to any other 3-face. Suppose to the contrary that g is adjacent to a 3-face $h \neq f$. By symmetry h shares exactly one edge xw or xy with g.

We first assume that $h = [xwz]$. If $z \notin S$, then $uvwxyu$ is a 5-cycle adjacent to a 6-cycle $uvwzxyu$, a contradiction. Thus, assume that $z \in S$. Since G is a simple graph, $z = v$ or $z = y$. Since x and y are outside of f, by Lemma 2.4, z is outside g. If $z = v$, this is the case that $x = z$ in above proof and we are done. If $z = y$, then $wuyw$ is a separating 3-cycle, contrary to Lemma 2.4.

Now let $h = [zyx]$. If $z \notin S$, then $uvwxyu$ is a 5-cycle adjacent to a 6-cycle $uvwxzyu$, a contradiction. Thus, assume that $z \in S$. In this case, assume that $z = v$ or $z = u$ by symmetry. Since x and y are outside of f, by Lemma 2.4, z is outside g. If $z = u$, then $uwxu$ is a separating 3-cycle, contrary to Lemma 2.4. If $v = z$, then u and w are either inside or outside $xyzx$. In the former case, vuy is a 3-cycle, by Lemma 2.4, $d(u) = 3$, contrary to Lemma 2.3. In the later case, either $vuyv$ or $vwxv$ is a 3-cycle, by Lemma 2.4, such a 3-cycle is a 3-face and hence $d(u) = 3$ (or $d(w) = 3$), contrary to Lemma 2.3.

Finally, we show that g cannot be adjacent to a 4-face. Suppose to the contrary that g is adjacent to a 4-face h. By symmetry h shares exactly one edge xw or xy with g. Assume first that $h = [xwzt]$. If $\{z, t\} \cap S = \emptyset$, then $uvwxyu$ is a 5-cycle adjacent to a 6-cycle $uwztxyu$, a contradiction. Thus, $\{z, t\} \cap S \neq \emptyset$. Since x and

y are outside f, by Lemma 2.4, z, t are outside g. Assume first that $z \in S$ and $t \notin S$. Since G is planar, $z \neq u$. Then $z = v$ or $z = y$. If $z = v$, then $vwxtv$ is a 4-cycle. By Lemma 2.4, such a 4-cycle is a 4-face and hence $d(w) = 3$, contrary to Lemma 2.3. If $z = y$, then G has a separating 3-cycle $uywu$, contrary to Lemma 2.4. Then assume that $z \notin S$ and $t \in S$. If $t = y$, then $wzyuw$ is a separating 4-cycle, contrary to Lemma 2.4. If $t = u$, then $xwux$ is a separating 3-cycle, contrary to Lemma 2.4. If $t = v$, then G has a separating 3-cycle $vwxv$, contrary to Lemma 2.4. Thus, $\{z, t\} \subset S$. Since G is planar, $z = v$ and $t = u$. Then $uyxu$ is a 3-cycle. By Lemma 2.4, such a 3-cycle is a 3-face and hence $d(y) = 2$, contrary to Lemma 2.3.

Thus, assume that $h = [yxzt]$. If $\{z, t\} \cap S = \emptyset$, then $uvwxyu$ is a 5-cycle adjacent to a 6-cycle $uwxztyu$, a contradiction. Thus, assume that $\{s, t\} \cap S \neq \emptyset$. Since x and y are both outside of f, by Lemma 2.4, z and t are outside of g. Assume first that one of s and t is in S. By symmetry, assume that $t \notin S$ and $z \in S$. If $z = v$, then G has a separating 4-cycle $vuyxv$, contrary to Lemma 2.4. If $z = w$, then G has a separating 4-cycle $wuytw$, contrary to Lemma 2.4. If $z = u$, then G has a separating 3-cycle $uwxu$, contrary to Lemma 2.4. Thus, assume that z and t are both in S. Since G is simple, $\{z, t\} \cap \{x, y\} = \emptyset$ and $\{z, t\} \cap \{u, w\} = \emptyset$. By symmetry, assume that $z = v$. Since G is planar, $t = u$. In this case, $xwvx$ is a 3-cycle. By Lemma 2.4, such a 3-cycle is a 3-face and hence $d(w) = 3$, contrary to Lemma 2.3.

(b) Assume that v is a 5^+-vertex incident with three consecutive 3-faces $f_1 = [uvw]$, $f_2 = [wvx]$ and $f_3 = [xvy]$. Let $S = \{u, v, w, x, y\}$. Suppose to the contrary that at least one of the three 3-faces is adjacent to another 3-face f_4. By Lemma 2.4, f_4 shares exactly one edge with one of f_1, f_2 and f_3. By symmetry we may assume that $f_4 = [uzw]$ or $[uvz]$ or $[wzx]$. If $z \notin S$, then there exists a 5-cycle adjacent to a 6-cycle, a contradiction. So, assume that $z \in S$. If $f_4 = [wxv]$, then $v \neq z$ since $f_4 \neq f_2$. Thus, let $z = u$ by symmetry. In this case, $xwux$ is a 3-cycle. By Lemma 2.4, $d(w) = 3$, contrary to Lemma 2.3. By symmetry, assume that $f_4 = [uwz]$. If $z = x$, then $xwux$ is a 3-cycle. By Lemma 2.4, such a 3-cycle is a 3-face and hence $d(w) = 3$, contrary to Lemma 2.3. Thus, $f_4 = [uzw]$ and $z = y$. In this case, $vwyv$ is a separating 3-cycle, contrary to Lemma 2.4.

(c) Suppose to the contrary that $f = [xyz]$ and $g = [uvwxy]$. If $z \notin S$, then $uvwxyu$ is a 5-cycle adjacent to a 6-cycle $uvwxzyu$, a contradiction. If $z \in S$, then we assume $z = u$ or $z = v$ by symmetry. In the former case, $xyux$ is a 3-cycle, by Lemma 2.4, $d(y) = 2$ (or $d(w) = 2$), contrary to Lemma 2.3. In the later case, $uvyu$ (or $xwvx$) is a 3-cycle. By Lemma 2.4, such a 3-cycle is a 3-face and hence $d(u) = 2$ (or $d(w) = 2$), contrary to Lemma 2.3.

(d) It follows that G has no 5-cycles adjacent to 6-cycles. □

Lemma 2.6 *Two* $(4, 4, 4)$*-faces in* $int(C_0)$ *cannot share exactly one common edge in* G.

Proof. Suppose to the contrary that $T_1 = [uvx]$ and $T_2 = [uvy]$ share a common edge uv. Let $S = \{u, v, x, y\}$ and $G' = G - S$. By the minimality of G, $G_{L'}$ admits an $\mathcal{M}_L$-coloring where L' (and $G_{L'}$) is a restriction of L (and G_L, respectively). Thus $G_{L'}$ has an independent set I' of size $|V(G')| = |V(G)| - 4$.

We claim that $xy \notin E(G)$. Suppose otherwise. Then G has either a 3-cycle $D_1 = xvyx$ such that u is in $int(D_1)$ or a 3-cycle $D_2 = xuyx$ such that v is in $int(D_2)$. In the former case, since both x and y are 4-vertices, D_1 is a separating 3-cycle, contrary to Lemma 2.4. In the latter case, similarly, D_2 is a separating 3-cycle, contrary to Lemma 2.4. Consider a list assignment L^* on S. Since $|L(v)| \geq 4$ for all $v \in V(G)$, we have

$$|L_u^*| \geq 3,\ |L_v^*| \geq 3,\ |L_x^*| \geq 2,\ |L_y^*| \geq 2.$$

Since $|L_v^*| > |L_x^*|$, we can choose a vertex (v, c) in

$$L_v^* - \{(v, c') : (x, c'') \in L_x^*, (v, c')(x, c'') \in M_{vx}\}.$$

Then L_x^* has at least two available colors. We color y, u, x in order, we can find an independent set I^* with $|I^*| = 4$. So $I' \cup I^*$ is an independent set of G_L with $|I' \cup I^*| = |V(G)|$. Then G has an $\mathcal{M}_L$-coloring, a contradiction. □

We are now ready to present a discharging procedure that will complete the proof of the Theorem 1.4. For each $x \in V \cup F$, we define the initial charge $ch(x) = d(x) - 4$ if $x \in V \cup (F \setminus \{C_0\})$ and $ch(C_0) = |C_0| + 4$. By Euler's Formula,

$$\sum_{x \in V} ch(x) + \sum_{x \in F \setminus \{C_0\}} ch(x) + ch(C_0) = \sum_{x \in V}(d(x) - 4) + \sum_{x \in F}(d(x) - 4) + 8 = 0.$$

We define suitable discharging rules such that, for every $x \in V \cup (F \setminus \{C_0\})$, the final charge of x, denoted $ch'(x)$, is non-negative and $ch'(C_0) > 0$. So, we get $0 < \sum_{x \in V \cup F} ch'(x) = \sum_{x \in V \cup F} ch(x) = 0$. This contradiction proves our result.

A 5-vertex v is *special* if v is 3-type to H_4 and 2-type to one of H_2 and H_3. Denote by $w(v \to f)$ to transfer the charge from a vertex v to a face f. We define the discharging rules as follows.

(R1) Let v be an internal vertex in a 3-face f.

(a) If v is a 4-vertex, then

$$w(v \to f) = \begin{cases} \frac{1}{6}, & \text{if } v \text{ is 3-type to } H_2; \\ \frac{1}{7}, & \text{if } v \text{ is 3-type to } H_3 \text{ and } f \text{ is a semi-poor face}; \\ \frac{2}{7}, & \text{if } v \text{ is 3-type to } H_3 \text{ and } f \text{ is a poor face}; \\ \frac{1}{7}, & \text{if } v \text{ is 3-type to } H_4 \text{ and } f \text{ is a } (4,4,5^+)\text{-face}; \\ \frac{2}{7}, & \text{if } v \text{ is 3-type to } H_4 \text{ and } f \text{ is a } (4,4,4)\text{-face}. \end{cases}$$

(b) If v is a 5-vertex, then

$$w(v \to f) = \begin{cases} \frac{3}{7}, & \text{if } v \text{ is 3-type to } H_4; \\ \frac{1}{7}, & \text{if } v \text{ is 2-type to } H_2 \text{ or } H_3 \text{ and } v \text{ is special}; \\ \frac{1}{3}, & \text{otherwise}. \end{cases}$$

(c) If v is a 6^+-vertex, then

$$w(v \to f) = \begin{cases} \frac{3}{7}, & \text{if } v \text{ is 3-type to } H_4; \\ \frac{1}{3}, & \text{otherwise}. \end{cases}$$

(R2) Every 4-face sends $\frac{2}{5}$ to each adjacent 3-face; every k-face sends $\frac{k-4}{k}$ to each adjacent 3-face, where $k \geq 6$.

(R3) Every 5-face sends $\frac{1}{5}$ to each adjacent 4-face.

(R4) Let v be an internal 4-vertex. If v is incident with two adjacent 6^+-faces, then each such a 6-face gives $\frac{1}{6}$ to v and each such a 7^+-face gives $\frac{3}{14}$ to v.

(R5) The outercycle C_0 gets $ch(v)$ from each incident vertex and sends 1 to any 3-face sharing at least one vertex with C_0.

It suffices to check that each $x \in V(G) \cup F(G)$ has nonnegative final charge and C_0 has positive final charge. By (R4), we have $ch'(v) = 0$ for each $v \in V(C_0)$. Thus, we need to check $ch'(v) \geq 0$ for each internal 4^+-vertex v by Lemma 2.3.

(1) Let v be a 4-vertex. If v is incident with at most one 3-face, then v is 2-type to one of H_1, H_2 and H_3. By (R1)(a), $ch'(v) = ch(v) = 0$. If v is incident with two nonadjacent 3-faces, then v is 2-type to one of H_1, H_2 and H_3 and also 2-type to the other of H_1, H_2 and H_3. By (R1)(a), $ch'(v) = ch(v) = 0$. If v is incident with three 3-faces, then v is 4-type to H_3. Similarly by (R1)(a), $ch'(v) = ch(v) = 0$. If v is incident with four 3-faces, then v is 4-type to H_4. Thus, $ch'(v) = ch(v) = 0$ by (R1) (a). Thus, assume that v is incident with two adjacent 3-faces. Then v is 3-type to one of H_2, H_3 and H_4. If v is 3-type to H_2, by Lemma 2.5(a) and (c), v is incident with two 6^+-faces. By (R1)(a) and (R4), $ch'(v) = ch(v) + \frac{1}{6} \times 2 - \frac{1}{6} \times 2 = 0$. If v is 3-type to H_3, by Lemma 2.5(a) and (c), v is incident with two 7^+-faces since G has no 5-cycles adjacent to 6-cycles. By (R1)(a) and (R4), $ch'(v) = ch(v) + \frac{3}{14} \times 2 - (\frac{1}{7} + \frac{2}{7}) = 0$. If v is 3-type to H_4, by Lemma 2.5(b), v is incident with two 7^+-faces since G has no 5-cycles adjacent to 6-cycles. By Lemma 2.6, v is incident with at least one $(4, 4, 5^+)$-face in H_4. By (R1)(a) and (R4), $ch'(v) = ch(v) + \frac{3}{14} \times 2 - (\frac{1}{7} + \frac{2}{7}) = 0$.

(2) Let v be a 5-vertex. By Lemma 2.5(b), v is incident with at most three consecutive 3-faces. If v is not incident with any 3-face, then $ch'(v) = ch(v) = 1 \geq 0$ by (R1)(b). If v is incident with exactly one 3-face, then v is 2-type to one of H_1, H_2 and H_3. Thus, $ch'(v) = ch(v) - \frac{1}{3} = \frac{2}{3} > 0$ by (R1)(b). If v is incident with two nonadjacent 3-faces, then v is 2-type to one of H_1, H_2 and H_3 and also 2-type to the other one of H_1, H_2 and H_3. By (R1)(b), $ch'(v) = ch(v) - 2 \times \frac{1}{3} = \frac{1}{3} > 0$. If v is incident with two adjacent 3-faces, then v is 3-type to one of H_2, H_3 and H_4. If v is 3-type to H_2 or H_3, then v sends $\frac{1}{3}$ to each 3-face. By (R1)(b), $ch'(v) = ch(v) - 2 \times \frac{1}{3} = \frac{1}{3} > 0$. If v is 3 type to H_4, then $ch'(v) = ch(v) - 2 \times \frac{3}{7} = \frac{1}{7} > 0$ by (R1)(b). We now assume that v is incident with three 3-faces. If v is incident with consecutive three 3-faces, then v is 4-type to H_3. By (R1)(b), $ch'(v) = ch(v) - 3 \times \frac{1}{3} = 0$. Thus, v is incident two adjacent 3-faces and the other 3-face. Then v is 3-type to one of H_2, H_3 and H_4 and 2-type to one of H_1, H_2 and H_3. If v is 3-type to one of H_2 and H_3 and 2-type of H_1, H_2 and H_3. Then $ch'(v) = ch(v) - \frac{1}{3} \times 3 = 0$ by (R1)(b). If v is 3-type of H_4 and 2-type to one of

H_2 and H_3, then v sends $\frac{3}{7}$ to each 3-face in the H_4 and $\frac{1}{7}$ to the other 3-face. Thus, $ch'(v) = ch(v) - (\frac{3}{7} \times 2 + \frac{1}{7}) \geq 0$ by (R1)(b).

(3) Let v be 6^+-vertex. If v is not incident with 3-faces, then $ch'(v) = ch(v) = d(v) - 4 \geq 2 > 0$. By Lemma 2.5(d), v is incident with at most $(d(v) - 2)$ 3-faces. Then $ch'(v) \geq (d(v)-4) - \frac{3}{7} \times (d(v)-2) = \frac{4}{7}d(v) - \frac{22}{7} \geq \frac{24}{7} - \frac{22}{7} = \frac{2}{7} > 0$ by (R1)(c).

We now check that $ch'(f) \geq 0$ for each $f \in F$. For simplicity, we also use f to denote the set of vertices of f for a face f. Let $f_1, f_2, \ldots, f_l$ be 3-faces of a l-cluster H_l. Define $ch(H_l) = ch(f_1) + \cdots + ch(f_l)$ and $ch'(H_l) = ch'(f_1) + \cdots + ch'(f_l)$.

We first check that $f \cap C_0 \neq \emptyset$.

(1) Let f be a 3-face in G. If f is not adjacent with any other 3-face, then by (R5) f gets 1 from C_0. So $ch'(f) \geq 3 - 4 + 1 = 0$.

Assume that f is in H_2. Let g be the 3-face in H_2 adjacent to f. If $C_0 \cap g \neq \emptyset$, then C_0 sends 1 to both f and g. Thus, $ch'(H_2) = -2 + 1 + 1 = 0$ by (R5). Thus, assume that $C_0 \cap g = \emptyset$. In this case, C_0 sends charge 1 to f. By Lemma 2.5, there are four 6^+-faces adjacent to this H_2. By (R2) and (R5), $ch'(H_2) \geq -2 + 1 + \frac{1}{3} \times 4 > 0$.

Assume that f is in H_3. Assume that the H_3 is induced by three 3-faces f, g and h. If $g \cap C_0 \neq \emptyset$ and $h \cap C_0 \neq \emptyset$, then $ch'(H_3) = -3 + 1 + 1 + 1 = 0$ by (R5). Thus, assume that one of $g \cap C_0$ and $h \cap C_0$ is not empty. By Lemma 2.5, there are five 7^+-faces adjacent to this H_3. Thus, $ch'(H_3) \geq -3 + 1 + \frac{3}{7} \times 5 > 0$ by (R2) and (R5).

Assume that f is in H_4. Let $V(H_4) = \{u, v, w, x, y\}$, where x, y, u, v are 3-type vertices to H_4. Assume that $x \in V(H_4) \cap C_0$ and $x \in f$. If $|V(H_4) \cap C_0| \geq 2$, then G has a 5-cycle adjacent to a 6-cycle, a contradiction. Thus, $V(H_4) \cap C_0 = \{x\}$. By Lemma 2.5, there are four 7^+-faces adjacent to this H_4, and the C_0 is incident with two 3-faces in H_4. Applying Lemma 2.6 to the subgraph induced by $\{y, u, v, w\}$, there are at least one 5^+-vertex in $int(C_0)$ in H_4. Thus, $ch'(H_4) \geq -4 + 1 + 1 + \frac{3}{7} \times 4 + \frac{3}{7} \times 2 > 0$ by (R1)(b), (R2) and (R5).

(2) Let $d(f) = 4$. If $|f \cap C_0| = 2$, by Lemma 2.4, then f cannot be adjacent to any 3-face rather than C_0 (if C_0 is a 3-face) since G has no 5-cycle adjacent to 6-cycle. Thus, $ch'(f) = ch(f) = 0$. Thus, assume that $|f \cap C_0| = 1$. By Lemma 2.5(a), f is adjacent to at most one 3-face. If f is not adjacent to any 3-face, then $ch'(f) = ch(f) = 0$. If f is adjacent to one 3-face, then f is not adjacent to any 4-face by Lemma 2.5(a). Thus, f is adjacent to three 5^+-faces. By (R2) and (R3), $ch'(f) \geq ch(f) + \frac{1}{5} \times 3 - \frac{2}{5} = \frac{1}{5} > 0$.

(3) Let $d(f) \geq 5$. If f is a 5-face, then f is not adjacent to any 3-face and adjacent to at most five 4-faces. Thus, $ch'(f) \geq ch(f) - \frac{1}{5} \times 5 = 0$ by (R3). Thus, f is a 6^+-face. By (R2), $ch'(f) \geq (k - 4) - k \times \frac{k-4}{k} = 0$.

From now on we may assume that $f \cap C_0 = \emptyset$.

(1) Let f be a 3-face. If f is H_1, then f is adjacent to at most one 4-face. If f is adjacent to one 4-face, by Lemma 2.5, the other faces incident with f are 6^+-faces. By (R1), $ch'(f) \geq d(f) - 4 + \frac{2}{5} + 2 \times \frac{k-4}{k} \geq -1 + \frac{2}{5} + 2 \times \frac{1}{3} > 0$. Thus, assume that f is not adjacent to any 4-face. Since G had no 5-cycle adjacent to any 6-cycle, f cannot adjacent to any 5-face. Thus, f is adjacent to three 6^+-faces. By (R1)(c), $ch'(f) \geq d(f) - 4 + \frac{1}{3} \times 3 = 0$.

Assume first that f is in H_2. Then $ch(H_2) = -2$ and let f_1 and f_2 be two 3-faces in H_2. Let $V(H_2) = \{u, v, x, y\}$, where x, y are 2-type to H_2 and u, v are 3-type to H_2. Since G has no separating 3-cycle, x is not adjacent to y. Since G has no 5-cycles adjacent to 6-cycles, each face adjacent to an internal face of H_2 is a 6^+-face by Lemma 2.5. In this case, there are four 6^+-faces adjacent to this H_2. By (R1), each such face sends $\frac{1}{3}$ to f_1 or f_2 in H_2. If one of u and v is a 5^+-vertex, then it sends $\frac{1}{3}$ to each of the two adjacent 3-face by (R1)(b), (R1)(c) and (R1)(e). Thus, $ch'(H_2) \geq -2 + \frac{1}{3} \times 4 + \frac{1}{3} \times 2 = 0$. We now assume that each of u and v is a 4-vertex. By Lemma 2.6, at least one of x and y is a 5^+-vertex. Assume that x is a 5^+-vertex. If x is a 6^+-vertex or a non-special 5-vertex, then x sends $\frac{1}{3}$ to the 3-face in H_3 by (R1)(c). By (R1)(a), (R1)(b) and (R2), each of u and v sends $\frac{1}{6}$ to each adjacent 3-faces. Thus, $ch'(H_2) \geq -2 + \frac{1}{3} \times 4 + \frac{1}{3} + \frac{1}{6} \times 4 > 0$. Thus, assume that x is a special 5-vertex. In this case, x is incident with two 7^+-faces. So, there are two 6^+-faces and two 7^+-faces adjacent to this H_2. In this case x sends $\frac{1}{7}$ to the 3-face in H_2. By (R1)(a), (R1)(b) and (R2), $ch'(H_2) \geq -2 + \frac{1}{3} \times 2 + \frac{3}{7} \times 2 + \frac{1}{7} + \frac{1}{6} \times 4 > 0$.

Next, assume that f is in H_3. Then $ch(H_3) = -3$ where f_1, f_2, and f_3 are 3-faces in H_3. Let $V(H_3) = \{u, v, w, x, y\}$, where v is 4-type to H_3, u, w are 3-type to H_3, and x and y are 2-type to H_3. By Lemma 2.4, x is not adjacent to w and y is not adjacent to u. Since G has no 5-cycles adjacent to 6-cycles, each face adjacent to an internal face of H_3 is a 7^+-face by Lemma 2.5. By (R2), each such 7^+-face sends at least $\frac{3}{7}$ to the H_3. If v is a 5^+-vertex, then v sends $\frac{1}{3}$ to each of three 3-faces in the H_3 by (R1)(b). Thus, $ch'(H_3) \geq -3 + \frac{3}{7} \times 5 + \frac{1}{3} \times 3 > 0$. If one of u and w, say u, is a 5^+-vertex, then w is a 4^+-vertex by Lemma 2.4. In this case, u sends $\frac{1}{3}$ to two 3-faces incident with u in the H_3 by (R1)(b), w sends $\frac{1}{7}$ and $\frac{2}{7}$ to two 3-faces incident with w in the H_3 by (R1)(a). So, $ch'(H_3) \geq -3 + \frac{3}{7} \times 5 + \frac{1}{3} \times 2 + \frac{2}{7} + \frac{1}{7} > 0$. Now we assume that each of u, v, w is a 4-vertex. By Lemma 2.6, x and y are 5^+-vertices. If x and y are 5-vertices, then they may be special 5-vertices. By (R1)(b) and by (R1)(c), each of x and y sends at least $\frac{1}{7}$ to the 3-face in the H_3. Each of u and w sends $\frac{2}{7}$ and $\frac{1}{7}$ to the two 3-faces in the H_3 by (R1)(a). Thus, $ch'(H_3) = -3 + (\frac{3}{7} \times 5) + (\frac{1}{7} \times 2) + (\frac{2}{7} \times 2) + (\frac{1}{7} \times 2) > 0$.

Finally, assume that f is in H_4. Let x be a hub and u, v, w, y be 3-type to H_4. Similarly, $ch(H_4) = -4$. where $f_1 = [xuv]$, $f_2 = [xvw]$, $f_3 = [xwy]$ and $f_4 = [xyu]$ are 3-faces in H_4. By Lemma 2.4, u is not adjacent to w, and v is not adjacent to y. By Lemma 2.5, each 3-face in H_4 is adjacent to a 7^+-face. By (R2), each such 7^+-face sends at least $\frac{3}{7}$ to the adjacent 3-face in the H_4.

By Lemma 2.6, at least two 3-type vertices to H_4 are 5^+-vertices. By (R1)(a), (R1)(b) and (R1)(c), each 3-type 5^+-vertex sends $\frac{3}{7}$ to each incident 3-face in H_4 and the other 3-type vertices to H_4 are 4-vertices, each of which sends at least $\frac{1}{7}$ to each incident 3-faces in H_4. If H_4 contains exactly two 3-type 5^+-vertices, then $ch'(H_4) \geq -4 + \frac{3}{7} \times 4 + \frac{3}{7} \times 4 + \frac{1}{7} \times 4 = 0$. If H_4 contains at least three 3-type 5^+-vertices, then $ch'(H_4) \geq -4 + \frac{3}{7} \times 4 + \frac{3}{7} \times 6 + \frac{1}{7} \times 2 > 0$.

(2) Let f be a 4-face. Let $f = [v_1v_2v_3v_4]$. By Lemma 2.5(a), f is adjacent to at most one 3-face. If f is adjacent to a 3-face, then the other faces adjacent to f are 5^+-faces by Lemma 2.5(a). By (R1)(d) and (R2), $ch(f) \geq d(f) - 4 + \frac{1}{5} \times 3 - \frac{2}{5} > 0$. If f is not adjacent to any 3-face, then f is adjacent to at most four 4-faces. Thus, $ch(f) \geq d(f) - 4 = 0$.

(3) Let f be a 5^+-face. If f is a 5-face, then f is adjacent at most five 4-faces. Since G has no 5-cycles adjacent to 6-cycles, f is not adjacent to any 3-face. By (R2), f sends $\frac{1}{5}$ to each adjacent 4-face. Thus, $ch'(f) \geq d(f) - 4 - 5 \times \frac{1}{5} = 0$. Assume that f is a k-face where $k \geq 6$. Then f sends at most $\frac{k-4}{k}$ to 3-faces or 4-faces by (R2). This yields $ch'(f) \geq (k-4) - k \times \frac{k-4}{k} = 0$.

We now consider the final charge of the outer face C_0.

Let $F_3' = \{f : f \text{ is a 3-face and } |b(f) \cap C_0| = 1\}$ and $F_3'' = \{f : f \text{ is a } k\text{-face and } |b(f) \cap C_0| = 2\}$, and $f_3' = |F_3'|, f_3'' = |F_3''|$. Let $E(C_0, V(G) - C_0)$ be the set of edges between C_0 and $V(G) - C_0$ and let $e(C_0, V(G) - C_0)$ be its size. Then by (R4),

$$\begin{aligned} ch'(C_0) &= |C_0| + 4 + \sum_{v \in C_0} (d(v) - 4) - f_3' - f_3'' \\ &= |C_0| + 4 + \sum_{v \in C_0} (d(v) - 2) - 2|C_0| - f_3' - f_3'' \\ &= -|C_0| + 4 + (e(C_0, V(G) - C_0) - f_3' - f_3''). \end{aligned}$$

So we may think that each edge $e \in E(C_0, V(G) - C_0)$ contributes 1 to $e(C_0, V(G) - C_0)$. Note that each 3-face contains two edges in $E(C_0, V(G) - C_0)$. Since C_0 is not a bad 4-cycle, any vertex $v \in int(C_0)$ is adjacent at most three vertices on C_0. Thus, if $F_3'' \neq \emptyset$, then all the 3-faces in F_3'' contributes at least $f_3'' + 1$ to $e(C_0, V(G) - C_0)$ while get at most f_3'' from C_0. Similarly, if $F_3' \neq \emptyset$, then all the 3-faces in F_3' contribute at least $f_3' + 1$ to $e(C_0, V(G) - C_0)$ while get at most f_3' from C_0. Thus, if $f_3' \neq 0$ or $f_3'' \neq 0$, then $e(C_0, V(G) - C_0) - f_3' - f_3'' > 0$ and so $ch'(C_0) > 0$. Thus, $f_3' = f_3'' = 0$ and $e(C_0, V(G) - C_0) - f_3' - f_3'' \geq 0$. If $|C_0| = 3$, then $ch'(C_0) > 0$. Let $|C_0| = 4$. If $e(C_0, V(G) - C_0) = 0$, then G is a 4-cycle, a contradiction. If $e(C_0, V(G) - C_0) \neq 0$, then $ch'(C_0) > 0$.

This completes the proof.

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