AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **87(1)** (2023), Pages 1–11

Group divisible designs with block size five

A. D. Forbes*

LSBU Business School
London South Bank University
103 Borough Road, London SE1 0AA, U.K.
anthony.d.forbes@gmail.com

Abstract

We report some group divisible designs with block size five, including types 6^{15} and 10^{15}. As a consequence we are able to extend significantly the known spectrum for 5-GDDs of type g^u.

1 Introduction

For the purpose of this paper, a *group divisible design*, K-GDD, of type $g_1^{u_1} g_2^{u_2} \dots g_r^{u_r}$ is an ordered triple $(V, \mathcal{G}, \mathcal{B})$ such that:

(i) V is a base set of cardinality $u_1 g_1 + u_2 g_2 + \dots + u_r g_r$;
(ii) $\mathcal{G}$ is a partition of V into u_i subsets of cardinality g_i, $i = 1, 2, \dots, r$, called *groups*;
(iii) $\mathcal{B}$ is a non-empty collection of subsets of V with cardinalities $k \in K$, called *blocks*; and
(iv) each pair of elements from distinct groups occurs in precisely one block but no pair of elements from the same group occurs in any block.

We abbreviate $\{k\}$-GDD to k-GDD, and a k-GDD of type q^k is also called a *transversal design*, TD(k, q). A *pairwise balanced design*, $(v, K, 1)$-PBD, is a K-GDD of type 1^v.

A *parallel class* in a group divisible design is a subset of the block set that partitions the base set. A k-GDD is called *resolvable*, and is denoted by k-RGDD, if the entire set of blocks can be partitioned into parallel classes. If there exist k mutually orthogonal Latin squares (MOLS) of side q, then there exists a $(k+2)$-GDD of type q^{k+2} and a $(k+1)$-RGDD of type q^{k+1}, [4, Theorem III.3.18]. Furthermore, as is well known, there exist $q-1$ MOLS of side q whenever q is a prime power.

* ORCID: A. D. Forbes https://orcid.org/0000-0003-3805-7056

ISSN: 2202-3518

Because of their widespread use in design theory, especially in the construction of infinite classes of combinatorial designs by means of the technique known as Wilson's Fundamental Construction, [17], [13, Theorem IV.2.5], group divisible designs are useful and important structures. The existence spectrum problem for group divisible designs with constant block sizes, k-GDDs, $k \geq 3$, appears to be a long way from being completely solved. Nevertheless, for $k \in \{3, 4, 5\}$ where all the groups have the same size, considerable progress has been made.

The necessary conditions for the existence of k-GDDs of type g^u, namely

$$\begin{aligned} u &\geq k, \\ g(u-1) &\equiv 0 \pmod{k-1}, \\ g^2u(u-1) &\equiv 0 \pmod{k(k-1)}, \end{aligned} \tag{1}$$

are known to be sufficient for $k = 3$, [14], [9, Theorem IV.4.1], and for $k = 4$ except for types 2^4 and 6^4, [7], [9, Theorem IV.4.6]. For 5-GDDs of type g^u, a partial solution to the design spectrum problem has been achieved, [1, 2, 5, 6, 9, 10, 11, 14, 15, 16, 18], and for future reference, we quote the main result concerning 5-GDDs in the important paper of Wei and Ge, [16], which represents a considerable advance on [9, Theorem IV.4.16] in the Colbourn–Dinitz *Handbook*.

Theorem 1.1 (Wei, Ge, 2014) *The necessary conditions (1) for the existence of a* 5-GDD *of type* g^u *are sufficient except for types* 2^5, 2^{11}, 3^5, 6^5, *and except possibly for:*

$g = 3$ *and* $u \in \{45, 65\}$;
$g = 2$ *and* $u \in \{15, 35, 71, 75, 95, 111, 115, 195, 215\}$;
$g = 6$ *and* $u \in \{15, 35, 75, 95\}$;
$g \in \{14, 18, 22, 26\}$ *and* $u \in \{11, 15, 71, 111, 115\}$;
$g \in \{34, 46, 62\}$ *and* $u \in \{11, 15\}$;
$g \in \{38, 58\}$ *and* $u \in \{11, 15, 71, 111\}$;
$g = 2\alpha$, $\gcd(\alpha, 30) = 1$, $33 \leq \alpha \leq 2443$, *and* $u = 15$;
$g = 10$ *and* $u \in \{5, 7, 15, 23, 27, 33, 35, 39, 47\}$;
$g = 30$ *and* $u = 15$;
$g = 50$ *and* $u \in \{15, 23, 27\}$;
$g = 90$ *and* $u = 23$;
$g = 10\alpha$, $\alpha \in \{7, 11, 13, 17, 35, 55, 77, 85, 91, 119, 143, 187, 221\}$ *and* $u = 23$.

Proof: This is Theorem 2.25 of [16]. □

The objective of this paper is to prove Theorem 1.2, below, which improves Theorem 1.1 by eliminating many possible exceptions.

Theorem 1.2 *The necessary conditions (1) for the existence of a* 5-GDD *of type* g^u *are sufficient except for types* 2^5, 2^{11}, 3^5, 6^5, *and except possibly for:*

$g = 3$ *and* $u = 65$;
$g = 2$ *and* $u \in \{15, 75, 95, 115\}$;
$g = 6$ *and* $u \in \{35, 95\}$;
$g \in \{14, 18, 22, 26, 38, 58\}$ *and* $u \in \{11, 15\}$;
$g \in \{74, 82, 86, 94\}$ *and* $u = 15$;
$g = 10$ *and* $u \in \{5, 7, 27, 39, 47\}$;
$g = 50$ *and* $u = 27$.

2 GDDs with block size 5 and type g^u

We begin with some directly constructed group divisible designs.

Theorem 2.1 *There exist* 5-GDD*s of types* 2^{35}, 2^{71}, 2^{111}, 3^{45}, 6^{15}, 10^{15}, 10^{23} *and* 10^{33}.

Proof: For 2^{35}, 2^{71}, and 10^{23} see [8, Lemma 4.1].

$\mathbf{2^{111}}$ With the point set $\{0, 1, \ldots, 221\}$ partitioned into residue classes modulo 111 for $\{0, 1, \ldots, 221\}$, the design is generated from

$\{137, 73, 211, 182, 50\}$, $\{138, 74, 212, 183, 51\}$, $\{148, 201, 185, 107, 206\}$,
$\{149, 202, 186, 108, 207\}$, $\{202, 148, 11, 152, 191\}$, $\{203, 149, 12, 153, 192\}$,
$\{119, 166, 168, 153, 212\}$, $\{120, 167, 169, 154, 213\}$, $\{123, 106, 46, 71, 188\}$,
$\{124, 107, 47, 72, 189\}$, $\{84, 132, 77, 65, 156\}$, $\{0, 3, 12, 122, 136\}$,
$\{0, 8, 38, 126, 154\}$, $\{0, 7, 83, 156, 219\}$, $\{0, 10, 32, 101, 102\}$,
$\{0, 27, 55, 75, 182\}$, $\{0, 33, 51, 57, 108\}$, $\{0, 1, 107, 121, 204\}$,
$\{0, 79, 119, 151, 189\}$, $\{1, 9, 31, 97, 123\}$, $\{0, 6, 26, 62, 159\}$,
$\{0, 9, 71, 127, 195\}$

by the mapping: $x \mapsto x + 2j \pmod{222}$, $0 \le j < 111$.

$\mathbf{3^{45}}$ With the point set $\{0, 1, \ldots, 134\}$ partitioned into residue classes modulo 44 for $\{0, 1, \ldots, 131\}$, and $\{132, 133, 134\}$, the design is generated from

$\{121, 84, 8, 48, 108\}$, $\{82, 9, 79, 86, 124\}$, $\{133, 30, 56, 57, 35\}$,
$\{131, 80, 60, 9, 37\}$, $\{95, 70, 122, 60, 91\}$, $\{0, 2, 8, 30, 49\}$,
$\{0, 3, 18, 85, 115\}$, $\{0, 12, 75, 77, 86\}$, $\{0, 14, 53, 78, 93\}$,
$\{0, 16, 45, 50, 119\}$, $\{0, 9, 43, 84, 95\}$, $\{0, 7, 23, 83, 131\}$,
$\{1, 7, 19, 33, 97\}$, $\{0, 33, 66, 99, 134\}$

by the mapping: $x \mapsto x + 2j \pmod{132}$ for $x < 132$, $x \mapsto (x + j \pmod 2) + 132$ for $132 \le x < 134$, $134 \mapsto 134$, $0 \le j < 66$ for the first 13 blocks, $0 \le j < 33$ for the last block.

$\mathbf{6^{15}}$ With the point set $\{0, 1, \ldots, 89\}$ partitioned into residue classes modulo 15 for $\{0, 1, \ldots, 89\}$, the design is generated from

$\{80, 41, 45, 18, 25\}$, $\{0, 1, 41, 67, 88\}$, $\{0, 21, 29, 63, 73\}$,

$\{0, 14, 39, 40, 71\}$, $\{0, 5, 34, 81, 83\}$, $\{0, 4, 13, 16, 84\}$,

$\{0, 8, 32, 52, 85\}$, $\{0, 11, 17, 42, 79\}$, $\{0, 18, 36, 54, 72\}$,

$\{1, 19, 37, 55, 73\}$

by the mapping: $x \mapsto x+2j \pmod{90}$, $0 \le j < 45$ for the first eight blocks, $0 \le j < 9$ for the last two blocks.

$\mathbf{10^{15}}$ With the point set $\{0, 1, \ldots, 149\}$ partitioned into residue classes modulo 15 for $\{0, 1, \ldots, 149\}$, the design is generated from

$\{101, 21, 43, 132, 59\}$, $\{12, 85, 61, 88, 129\}$, $\{29, 9, 85, 93, 147\}$,

$\{141, 39, 26, 48, 88\}$, $\{7, 76, 86, 25, 110\}$, $\{0, 1, 12, 108, 137\}$,

$\{0, 14, 32, 111, 145\}$, $\{0, 17, 57, 63, 67\}$, $\{0, 16, 84, 107, 143\}$,

$\{0, 2, 21, 102, 146\}$, $\{0, 8, 86, 95, 112\}$, $\{0, 7, 35, 36, 130\}$,

$\{0, 11, 37, 58, 109\}$, $\{0, 3, 5, 70, 122\}$

by the mapping: $x \mapsto x + 2j \pmod{150}$, $0 \le j < 75$.

$\mathbf{10^{33}}$ With the point set $\{0, 1, \ldots, 329\}$ partitioned into residue classes modulo 33 for $\{0, 1, \ldots, 329\}$, the design is generated from

$\{102, 84, 56, 8, 268\}$, $\{145, 251, 217, 214, 137\}$, $\{57, 303, 73, 97, 184\}$,

$\{304, 149, 216, 134, 104\}$, $\{203, 229, 88, 107, 278\}$, $\{170, 150, 53, 139, 229\}$,

$\{300, 246, 79, 41, 278\}$, $\{108, 129, 65, 133, 48\}$, $\{0, 13, 120, 193, 222\}$,

$\{0, 7, 42, 65, 214\}$, $\{0, 1, 148, 153, 162\}$, $\{0, 27, 63, 110, 201\}$,

$\{0, 10, 62, 136, 197\}$, $\{0, 2, 55, 105, 144\}$, $\{0, 6, 57, 98, 202\}$,

$\{0, 12, 56, 151, 229\}$

by the mapping: $x \mapsto x + j \pmod{330}$, $0 \le j < 330$. □

For our proof of Theorem 1.2, we require some definitions and constructions.

A *double group divisible design*, k-DGDD, is an ordered quadruple $(V, \mathcal{G}, \mathcal{H}, \mathcal{B})$ such that:

(i) V is a base set of points;
(ii) $\mathcal{G}$ is a partition of V, the *groups*;
(iii) $\mathcal{H}$ is another partition of V, the *holes*;
(iv) $\mathcal{B}$ is a non-empty collection of subsets of V of cardinality k, the *blocks*;
(v) for each block $B \in \mathcal{B}$, each group $G \in \mathcal{G}$ and each hole $H \in \mathcal{H}$, we have $|B \cap G| \le 1$ and $|B \cap H| \le 1$;

(vi) each pair of elements of V not in the same group and not in the same hole occurs in precisely one block.

A k-DGDD of type

$$(g_1, h_1^w)^{u_1}(g_2, h_2^w)^{u_2} \dots (g_r, h_r^w)^{u_r}, \quad g_i = wh_i, \quad i = 1, 2, \dots, r,$$

is a double group divisible design where:

(i) there are u_i groups of size g_i, $i = 1, 2, \dots, r$;
(ii) there are w holes;
(iii) for $i = 1, 2, \dots, r$, each group of size g_i intersects each hole in h_i points.

A *modified group divisible design*, k-MGDD, of type g^u is a k-DGDD of type $(g, 1^g)^u$. By interchanging groups and holes we see that a k-MGDD of type g^u exists if and only if a k-MGDD of type u^g exists. See [1] for an extensive treatment of 5-MGDDs.

Lemma 2.1 *Suppose there exists a* 5-GDD *of type* $g_1^{u_1} g_2^{u_2} \dots g_n^{u_n}$. *Then for any positive integer* $h \notin \{2, 3, 6, 10\}$, *there exists a* 5-GDD *of type* $(g_1 h)^{u_1}(g_2 h)^{u_2} \dots (g_n h)^{u_n}$.

Proof: Inflate each point of the 5-GDD by a factor of h and replace the blocks with 5-GDDs of type h^5. By Theorem 1.1, there exists a 5-GDD of type h^5 for $h \geq 1$, $h \notin \{2, 3, 6, 10\}$. □

Lemma 2.2 *Suppose there exists a* K-GDD *of type* $g_1^{u_1} g_2^{u_2} \dots g_r^{u_r}$, *and let* w *be a positive integer. Suppose also that for each* $k \in K$, *there exists a* 5-MGDD *of type* w^k, *and for* $i = 1, 2, \dots, r$, *there exists a* 5-GDD *of type* g_i^w. *Then there exists a* 5-GDD *of type* $(u_1 g_1 + u_2 g_2 + \dots + u_r g_r)^w$.

Proof: This is a combination of Constructions 2.19 and 2.20 in [16], and it also appears (for block size 4) as Constructions 1.8 and 1.10 in [12].

Take the K-GDD and inflate each point by a factor of w. Replace each inflated block by a 5-MGDD of type w^k, $k \in K$ to obtain a 5-DGDD of type

$$(wg_1, g_1^w)^{u_1}(wg_2, g_2^w)^{u_2} \dots (wg_r, g_r^w)^{u_r}.$$

Then overlay the holes of this 5-DGDD with 5-GDDs of types g_i^w, $i = 1, 2, \dots, r$. □

We can now prove our main result.

Proof of Theorem 1.2.

For types 2^{35}, 2^{71}, 2^{111}, 3^{45}, 6^{15}, 10^{15}, 10^{23} and 10^{33}, see Theorem 2.1.

For types 2^{195} and 2^{215}, take a 5-GDD of type $68^5 48^1$ or $68^5 88^1$, [15] (alternatively, see [5, Theorem 2.1] or [9, Theorem IV.4.17]), and adjoin two extra points. Overlay each group together with the new points with a 5-GDD of type 2^{25} or 2^{35} or 2^{45}, as appropriate.

For type 6^{75}, take a 5-GDD of type 90^5 and overlay the groups with 5-GDDs of type 6^{15}.

For type g^t, $g \in \{14, 18, 22, 26, 38, 58\}$, $t \in \{71, 111\}$, use Lemma 2.1 with type 2^{71} or 2^{111} and $h = g/2$.

For type g^{115}, $g \in \{14, 18, 22, 26\}$, construct a 5-GDD of type $(5g)^{23}$ using Lemma 2.1 with a 5-GDD of type 10^{23} and $h = g/2$; then replace each group with a 5-GDD of type g^5.

For types 10^{35}, 30^{15} and 50^{15}, use Lemma 2.1 with a 5-GDD of type 2^{35} or 6^{15} or 10^{15}, as appropriate, and $h = 5$.

For type $(10\alpha)^{23}$, odd $\alpha \geq 5$, use Lemma 2.1 with a 5-GDD of type 10^{23} and $h = \alpha$.

For type g^{11}, $g \in \{34, 46, 62\}$ and g^{15}, $g = 2\alpha$, $\gcd(\alpha, 30) = 1$, $\alpha \geq 33$, let

$$G = \{34, 46, 62\} \cup \left\{\text{even } g \geq 66 : \gcd\left(\frac{g}{2}, 30\right) = 1\right\} \setminus \{74, 82, 86, 94, 98, 106, 118, 178\}.$$

For $g \in G$, there exists a $(g+1, \{5, 7, 9\}, 1)$-PBD, [3, Table IV.3.23]. Take this PBD, remove a point and the blocks containing it to get a $\{5, 7, 9\}$-GDD of type $4^a 6^b 8^c$ for some non-negative integers a, b, c satisfying $4a + 6b + 8c = g$. Now use Lemma 2.2 with this $\{5, 7, 9\}$-GDD and $w = 11$ or 15 to obtain 5-GDDs of types g^{11} and g^{15} for every $g \in G$. For the existence of 5-MGDDs of types w^5, w^7 and w^9, see [1]. For the existence of 5-GDDs of types 4^w, 6^w and 8^w, see Theorems 1.1 and 2.1.

For type 98^{15}, take a TD(9, 11), fill in the groups with blocks of size 11 and remove a point together with the blocks containing it to get a $\{9, 11\}$-GDD of type $8^{11} 10^1$. Now use Lemma 2.2 with this $\{9, 11\}$-GDD and $w = 15$ to obtain a 5-GDD of type 98^{15}. For the existence of 5-MGDDs of types 15^9 and 15^{11}, see [1]. For the existence of 5-GDDs of types 8^{15} and 10^{15}, see Theorems 1.1 and 2.1.

For types 106^{15}, 118^{15} and 178^{15}, we refer the reader to Lemma 3.16 of [11], which proves that there exists a 5-GDD of type h^{11} for $h \equiv 2 \pmod 4$, $h \geq 66$. By [11, Theorem 1.3], there exists a 4-frame of type 6^{15}, i.e. a 4-GDD $(V, \mathcal{G}, \mathcal{B})$ of type 6^{15} in which the block set can be partitioned into into 30 partial parallel classes of size 21 each of which partitions $V \setminus G$ for some $G \in \mathcal{G}$. Also we have the 5-GDD of type 6^{15} from Theorem 2.1 as well as 5-GDDs of type h^{15} for $h \equiv 0 \pmod 4$ from Theorem 1.1. Then, by a straightforward adaptation of the proof of [11, Lemma 3.16], we obtain 5-GDDs of type g^{15} for $g \in \{6n, 6n + 4, \ldots, 8n - 2\}$ whenever there exists a TD$(15, n)$ with odd n. This interval contains 106 and 118 when $n = 17$, and 178 when $n = 23$. □

3 GDDs with block size 5 and type $g^u m^1$

Assuming they might be of some use for future research, we collect together an assortment of directly constructed 5-GDDs of type $g^u m^1$ that we have found during our investigations.

Theorem 3.1 *There exist* 5-GDD*s of types* $2^{36}10^1$, $6^{12}2^1$, $7^{20}19^1$, $8^{10}4^1$, $8^{12}16^1$, $8^{13}12^1$, $8^{18}12^1$, $8^{20}4^1$, $8^{20}24^1$, $12^5 8^1$, $16^8 24^1$ *and* $24^7 8^1$.

Proof: $\mathbf{2^{36}10^1}$ With the point set $\{0, 1, \ldots, 81\}$ partitioned into residue classes modulo 36 for $\{0, 1, \ldots, 71\}$, and $\{72, 73, \ldots, 81\}$, the design is generated from

$\{21, 76, 30, 35, 0\}$, $\{38, 9, 33, 7, 30\}$, $\{65, 23, 8, 15, 4\}$,

$\{32, 79, 55, 30, 61\}$, $\{72, 63, 9, 64, 54\}$, $\{1, 35, 80, 60, 34\}$,

$\{9, 61, 28, 21, 65\}$, $\{6, 12, 28, 40, 60\}$, $\{0, 14, 59, 69, 73\}$

by the mapping: $x \mapsto x + 2j \pmod{72}$ for $x < 72$, $x \mapsto (x - 72 + 5j \pmod{10}) + 72$ for $x \geq 72$, $0 \leq j < 36$.

$\mathbf{6^{12}2^1}$ With the point set $\{0, 1, \ldots, 73\}$ partitioned into residue classes modulo 12 for $\{0, 1, \ldots, 71\}$, and $\{72, 73\}$, the design is generated from

$\{32, 70, 25, 41, 21\}$, $\{14, 31, 46, 56, 0\}$, $\{9, 11, 48, 39, 70\}$,

$\{64, 58, 60, 41, 63\}$, $\{26, 55, 21, 34, 54\}$, $\{57, 72, 32, 47, 50\}$,

$\{0, 19, 37, 45, 51\}$

by the mapping: $x \mapsto x + 2j \pmod{72}$ for $x < 72$, $x \mapsto (x + j \pmod 2) + 72$ for $x \geq 72$, $0 \leq j < 36$.

$\mathbf{7^{20}19^1}$ With the point set $\{0, 1, \ldots, 158\}$ partitioned into residue classes modulo 19 for $\{0, 1, \ldots, 132\}$, $\{133, 134, \ldots, 139\}$, and $\{140, 141, \ldots, 158\}$, the design is generated from

$\{64, 48, 14, 54, 115\}$, $\{39, 2, 156, 51, 94\}$, $\{39, 101, 24, 128, 21\}$,

$\{0, 4, 91, 98, 145\}$, $\{0, 1, 14, 22, 147\}$, $\{0, 2, 25, 30, 88\}$,

$\{0, 17, 48, 81, 133\}$, $\{0, 9, 68, 122, 140\}$, $\{0, 24, 97, 138, 158\}$

by the mapping: $x \mapsto x + j \pmod{133}$ for $x < 133$, $x \mapsto (x + j \pmod 7) + 133$ for $133 \leq x < 140$, $x \mapsto (x - 140 + j \pmod{19}) + 140$ for $x \geq 140$, $0 \leq j < 133$.

$\mathbf{8^{10}4^1}$ With the point set $\{0, 1, \ldots, 83\}$ partitioned into residue classes modulo 10 for $\{0, 1, \ldots, 79\}$, and $\{80, 81, 82, 83\}$, the design is generated from

$\{56, 2, 24, 70, 3\}$, $\{80, 42, 19, 60, 57\}$, $\{14, 49, 6, 30, 77\}$,

$\{0, 2, 6, 31, 75\}$

by the mapping: $x \mapsto x + j \pmod{80}$ for $x < 80$, $x \mapsto (x + j \pmod 4) + 80$ for $x \geq 80$, $0 \leq j < 80$.

$\mathbf{8^{12}16^1}$ With the point set $\{0, 1, \ldots, 111\}$ partitioned into residue classes modulo 12 for $\{0, 1, \ldots, 95\}$, and $\{96, 97, \ldots, 111\}$, the design is generated from

$\{34, 42, 100, 36, 59\}$, $\{92, 89, 55, 85, 36\}$, $\{88, 3, 12, 66, 103\}$,

$\{111, 28, 66, 56, 1\}$, $\{43, 4, 22, 48, 108\}$, $\{0, 1, 14, 46, 81\}$

by the mapping: $x \mapsto x + j \pmod{96}$ for $x < 96$, $x \mapsto (x + j \pmod{16}) + 96$ for $x \geq 96$, $0 \leq j < 96$.

$\mathbf{8^{13}12^{1}}$ With the point set $\{0, 1, \ldots, 115\}$ partitioned into residue classes modulo 13 for $\{0, 1, \ldots, 103\}$, and $\{104, 105, \ldots, 115\}$, the design is generated from

$\{52, 16, 14, 24, 64\}$, $\{38, 99, 70, 95, 79\}$, $\{90, 5, 0, 109, 87\}$,

$\{41, 103, 10, 113, 68\}$, $\{35, 2, 17, 72, 105\}$, $\{0, 1, 7, 60, 81\}$

by the mapping: $x \mapsto x+j \pmod{104}$ for $x < 104$, $x \mapsto (x-104+3j \pmod{12})+104$ for $x \geq 104$, $0 \leq j < 104$.

$\mathbf{8^{18}12^{1}}$ With the point set $\{0, 1, \ldots, 155\}$ partitioned into residue classes modulo 18 for $\{0, 1, \ldots, 143\}$, and $\{144, 145, \ldots, 155\}$, the design is generated from

$\{49, 57, 14, 17, 15\}$, $\{137, 122, 77, 61, 55\}$, $\{52, 21, 14, 65, 150\}$,

$\{56, 79, 60, 23, 32\}$, $\{6, 84, 32, 11, 59\}$, $\{53, 12, 92, 152, 142\}$,

$\{2, 71, 13, 83, 100\}$, $\{0, 10, 30, 95, 149\}$

by the mapping: $x \mapsto x + j \pmod{144}$ for $x < 144$, $x \mapsto (x + j \pmod{12}) + 144$ for $x \geq 144$, $0 \leq j < 144$.

$\mathbf{8^{20}4^{1}}$ With the point set $\{0, 1, \ldots, 163\}$ partitioned into residue classes modulo 20 for $\{0, 1, \ldots, 159\}$, and $\{160, 161, 162, 163\}$, the design is generated from

$\{70, 95, 117, 58, 51\}$, $\{9, 133, 124, 148, 61\}$, $\{88, 99, 57, 3, 89\}$,

$\{67, 144, 10, 136, 14\}$, $\{13, 117, 94, 123, 156\}$, $\{15, 66, 80, 64, 148\}$,

$\{56, 99, 10, 38, 51\}$, $\{0, 3, 58, 93, 160\}$

by the mapping: $x \mapsto x + j \pmod{160}$ for $x < 160$, $x \mapsto (x + j \pmod{4}) + 160$ for $x \geq 160$, $0 \leq j < 160$.

$\mathbf{8^{20}24^{1}}$ With the point set $\{0, 1, \ldots, 183\}$ partitioned into residue classes modulo 20 for $\{0, 1, \ldots, 159\}$, and $\{160, 161, \ldots, 183\}$, the design is generated from

$\{142, 54, 150, 133, 40\}$, $\{172, 8, 137, 115, 2\}$, $\{112, 17, 6, 69, 153\}$,

$\{72, 114, 39, 175, 129\}$, $\{78, 137, 177, 114, 116\}$, $\{46, 19, 145, 170, 108\}$,

$\{89, 40, 179, 43, 134\}$, $\{125, 52, 120, 42, 174\}$, $\{35, 54, 6, 36, 140\}$, $\{0, 4, 16, 125, 132\}$

by the mapping: $x \mapsto x+j \pmod{160}$ for $x < 160$, $x \mapsto (x-160+9j \pmod{24})+160$ for $x \geq 160$, $0 \leq j < 160$.

$\mathbf{12^{5}8^{1}}$ With the point set $\{0, 1, \ldots, 67\}$ partitioned into residue classes modulo 5 for $\{0, 1, \ldots, 59\}$, and $\{60, 61, \ldots, 67\}$, the design is generated from

$\{0, 2, 49, 51, 64\}$, $\{0, 1, 7, 33, 59\}$, $\{0, 4, 38, 41, 57\}$, $\{0, 18, 19, 26, 32\}$,

$\{0, 9, 13, 31, 62\}$, $\{0, 3, 6, 17, 65\}$, $\{0, 8, 22, 29, 60\}$, $\{0, 11, 42, 58, 61\}$,

$\{1, 18, 22, 39, 55\}$, $\{1, 2, 30, 53, 66\}$, $\{0, 16, 39, 43, 67\}$, $\{1, 15, 34, 43, 61\}$,

$\{0, 12, 24, 36, 48\}$

by the mapping: $x \mapsto x + 4j \pmod{60}$ for $x < 60$, $x \mapsto (x + j \pmod 5) + 60$ for $60 \le x < 65$, $x \mapsto (x - 65 + j \pmod 3) + 65$ for $x \ge 65$, $0 \le j < 15$ for the first 12 blocks; $x \mapsto x + j \pmod{60}$ for $x < 60$, $x \mapsto (x + j \pmod 5) + 60$ for $60 \le x < 65$, $x \mapsto (x - 65 + j \pmod 3) + 65$ for $x \ge 65$, $0 \le j < 12$ for the last block.

$\mathbf{16^8 24^1}$ With the point set $\{0, 1, \dots, 151\}$ partitioned into residue classes modulo 8 for $\{0, 1, \dots, 127\}$, and $\{128, 129, \dots, 151\}$, the design is generated from

$\{62, 129, 95, 9, 19\}$, $\{94, 11, 93, 55, 146\}$, $\{18, 115, 0, 15, 148\}$,

$\{30, 77, 9, 23, 96\}$, $\{143, 31, 22, 81, 101\}$, $\{3, 80, 106, 102, 135\}$,

$\{40, 70, 3, 5, 97\}$, $\{0, 5, 11, 116, 139\}$

by the mapping: $x \mapsto x + j \pmod{128}$ for $x < 128$, $x \mapsto (x - 128 + 9j \pmod{24}) + 128$ for $x \ge 128$, $0 \le j < 128$.

$\mathbf{24^7 8^1}$ With the point set $\{0, 1, \dots, 175\}$ partitioned into residue classes modulo 7 for $\{0, 1, \dots, 167\}$, and $\{168, 169, \dots, 175\}$, the design is generated from

$\{135, 1, 159, 70, 81\}$, $\{13, 63, 15, 54, 32\}$, $\{159, 28, 29, 3, 114\}$,

$\{107, 162, 91, 87, 55\}$, $\{127, 17, 12, 173, 104\}$, $\{115, 161, 55, 88, 155\}$,

$\{90, 16, 24, 120, 133\}$, $\{0, 3, 18, 47, 170\}$

by the mapping: $x \mapsto x + j \pmod{168}$ for $x < 168$, $x \mapsto (x + j \pmod 8) + 168$ for $x \ge 168$, $0 \le j < 168$. □

The existence of type $12^5 8^1$ means that we can give the following update of [16, Theorem 2.27] (also [6, Theorem 5] or [9, Theorem IV.4.17]).

Theorem 3.2 *A* 5-GDD *of type* $g^5 m^1$ *exists whenever* $g \equiv m \equiv 0 \pmod 4$ *and* $m \le 4g/3$ *except possibly when* $(g, m) = (12, 4)$.

During the time this paper has been under review, direct constructions for many more small 5-GDDs have been obtained, the majority of them of type $g^u m^1$ for various $g \le 48$. The results are recorded in Theorem 3.3, below. We save space here by placing the details of the constructions in a separate supplement, which is available online at `http://arxiv.org/abs/2211.14124`. Although the seven types $8^{15}4^1$, $12^{10}8^1$, $12^{10}16^1$, $12^{12}4^1$, $12^{13}8^1$, $16^7 12^1$ and $16^8 4^1$ are listed in [9, Remark IV.4.19] as known, the only existence proofs we are aware of appear in an unpublished manuscript of J. Wang and H. Shen, Embeddings of Near Resolvable Designs with Block Size Four; therefore we include these 5-GDDs in Theorem 3.3 and, with our constructions, in the supplement. Types $4^u m^1$ are covered by [5]; delete a point from the block of size $m + 1$ of a $(4u + m + 1, \{5, (m + 1)^*\}, 1)$-PBD.

Theorem 3.3 *There exist* 5-GDD*s of types*
$1^{48}9^1$, $1^{60}9^1$, $1^{60}13^1$, $1^{68}9^1$, $1^{72}17^1$, $1^{80}9^1$, $1^{80}13^1$, $1^{80}25^1$, $1^{84}21^1$, $1^{88}9^1$, $1^{92}17^1$, $1^{96}25^1$, $1^{100}9^1$, $1^{100}13^1$, $1^{100}25^1$, $1^{104}21^1$, $1^{108}9^1$, $1^{108}29^1$, $1^{112}17^1$, $1^{124}21^1$, $1^{128}9^1$, $1^{128}29^1$, $1^{132}17^1$, $1^{132}37^1$, $1^{136}25^1$, $1^{144}21^1$, $1^{144}41^1$, $1^{148}29^1$, $1^{152}17^1$, $1^{152}37^1$, $1^{156}25^1$, $1^{156}45^1$,

$1^{160}33^1$, $1^{164}21^1$, $1^{164}41^1$, $1^{168}29^1$, $1^{168}49^1$, $1^{172}17^1$, $1^{176}25^1$, $1^{176}45^1$, $1^{184}21^1$, $1^{184}41^1$, $1^{188}29^1$, $1^{192}17^1$, $1^{192}37^1$, $1^{192}57^1$, $1^{196}25^1$, $1^{196}45^1$, $2^{32}14^1$, $2^{40}6^1$, $2^{48}18^1$, $2^{52}14^1$, $2^{56}10^1$, $2^{60}6^1$, $3^{20}11^1$, $3^{28}7^1$, $3^{32}11^1$, $3^{36}15^1$, $5^{24}25^1$, $5^{28}25^1$, $5^{32}25^1$, $5^{32}45^1$, $5^{36}45^1$, $5^{40}45^1$, $5^{44}25^1$, $5^{44}45^1$, $6^{16}10^1$, $8^{8}12^1$, $8^{10}16^1$, $8^{10}20^1$, $8^{14}28^1$, $8^{15}4^1$, $8^{15}16^1$, $8^{15}24^1$, $8^{15}36^1$, $8^{16}20^1$, $8^{17}16^1$, $8^{18}32^1$, $8^{20}44^1$, $8^{21}20^1$, $8^{21}40^1$, $8^{22}16^1$, $8^{22}36^1$, $8^{23}32^1$, $8^{24}28^1$, $8^{25}4^1$, $8^{25}24^1$, $8^{26}20^1$, $8^{27}16^1$, $8^{28}12^1$, $9^{8}1^1$, $9^{12}13^1$, $9^{16}5^1$, $9^{16}25^1$, $9^{20}17^1$, $9^{20}29^1$, $9^{20}37^1$, $9^{20}49^1$, $9^{28}1^1$, $10^{10}18^1$, $10^{20}38^1$, $11^{20}19^1$, $12^{5}8^1$, $12^{10}8^1$, $12^{10}16^1$, $12^{10}28^1$, $12^{11}20^1$, $12^{12}4^1$, $12^{12}24^1$, $12^{13}8^1$, $12^{13}28^1$, $12^{14}32^1$, $12^{15}8^1$, $12^{15}16^1$, $12^{15}28^1$, $12^{15}36^1$, $12^{15}48^1$, $12^{16}20^1$, $12^{16}40^1$, $12^{17}4^1$, $12^{17}24^1$, $12^{17}44^1$, $12^{18}8^1$, $12^{18}28^1$, $12^{18}48^1$, $12^{19}32^1$, $12^{19}52^1$, $12^{20}8^1$, $12^{21}20^1$, $12^{21}40^1$, $12^{21}60^1$, $12^{22}4^1$, $12^{22}24^1$, $12^{22}44^1$, $12^{22}64^1$, $12^{23}28^1$, $12^{23}48^1$, $12^{23}68^1$, $12^{24}32^1$, $12^{24}52^1$, $12^{27}4^1$, $13^{8}17^1$, $13^{12}1^1$, $13^{12}21^1$, $13^{12}41^1$, $13^{16}5^1$, $13^{16}25^1$, $13^{20}1^1$, $14^{8}6^1$, $16^{6}20^1$, $16^{7}12^1$, $16^{8}4^1$, $16^{9}36^1$, $16^{10}8^1$, $16^{10}20^1$, $16^{10}28^1$, $16^{10}36^1$, $16^{10}40^1$, $16^{11}20^1$, $16^{11}40^1$, $16^{12}12^1$, $16^{12}52^1$, $16^{13}4^1$, $16^{13}24^1$, $16^{13}44^1$, $16^{14}36^1$, $16^{15}8^1$, $16^{15}28^1$, $16^{16}20^1$, $16^{16}40^1$, $16^{17}12^1$, $17^{8}13^1$, $17^{8}33^1$, $17^{12}9^1$, $17^{12}29^1$, $17^{12}49^1$, $17^{16}5^1$, $20^{8}40^1$, $20^{9}40^1$, $20^{10}36^1$, $20^{10}40^1$, $20^{11}40^1$, $21^{8}9^1$, $21^{8}29^1$, $21^{12}17^1$, $23^{8}7^1$, $24^{6}20^1$, $24^{7}8^1$, $24^{7}28^1$, $24^{8}16^1$, $24^{8}36^1$, $24^{9}4^1$, $24^{9}44^1$, $24^{10}4^1$, $24^{10}12^1$, $24^{10}32^1$, $24^{11}20^1$, $25^{8}5^1$, $25^{8}45^1$, $28^{6}20^1$, $28^{6}40^1$, $28^{7}16^1$, $28^{7}36^1$, $28^{8}12^1$, $28^{8}32^1$, $28^{8}52^1$, $28^{9}8^1$, $28^{9}48^1$, $28^{10}4^1$, $28^{10}24^1$, $29^{8}1^1$, $29^{8}21^1$, $32^{6}20^1$, $32^{6}40^1$, $32^{7}4^1$, $32^{7}24^1$, $32^{7}44^1$, $32^{8}8^1$, $32^{8}28^1$, $32^{9}12^1$, $36^{6}20^1$, $36^{6}40^1$, $36^{7}12^1$, $36^{7}32^1$, $36^{8}24^1$, $40^{6}20^1$, $44^{6}20^1$, $44^{6}40^1$, $44^{7}8^1$, $48^{6}20^1$, $1^{16}9^5$, $1^{46}5^3$ *and* $4^{5}8^{5}$.

Proof: See `http://arxiv.org/abs/2211.14124`. □

Acknowledgement

I would like to thank the reviewers for helpful comments from which the paper has benefitted.

References

[1] R. J. R. Abel and A. M. Assaf, Modified group divisible designs with block size 5 and $\lambda = 1$, *Discrete Math.* **256** (2002), 1–22.

[2] R. J. R. Abel, A. M. Assaf, I. Bluskov, M. Greig and N. Shalaby, New results on GDDs, covering, packing and directable designs with block Size 5, *J. Combin. Des* **18** (2010), 337–368.

[3] R. J. R. Abel, F. E. Bennett and M. Greig, PBD-Closure, *Handbook of Combinatorial Designs*, 2nd edition (Eds.: C.J. Colbourn and J.H. Dinitz), Chapman & Hall/CRC Press, Boca Raton, 2007, 255–260.

[4] R. J. R. Abel, C. J. Colbourn and J. H. Dinitz, Mutually Orthogonal Latin Squares (MOLS), *Handbook of Combinatorial Designs*, 2nd edition (Eds.: C.J. Colbourn and J.H. Dinitz), Chapman & Hall/CRC Press, Boca Raton, 2007, 160–193.

[5] R. J. R. Abel, G. Ge, M. Greig and A. C. H. Ling, Further results on $(v, 5, w^*, 1)$-PBDs, *Discrete Math.* **309** (2009), 2323–2339.

[6] F. E. Bennett, Y. Chang, G. Ge and M. Greig, Existence of $(v, 5, w^*, 1)$-PBDs, *Discrete Math.* **279**, 2004, 61–105.

[7] A. E. Brouwer, A. Schrijver and H. Hanani, Group divisible designs with block size four, *Discrete Math.* **20** (1977), 1–10.

[8] A. D. Forbes, Pentagonal geometries with block sizes 3, 4 and 5, *J. Combin. Des.* **29** (2021), 307–330; preprint: `https://arxiv.org/abs/2006.15734v3`.

[9] G. Ge, Group Divisible Designs, *Handbook of Combinatorial Designs*, 2nd edition (Eds.: C.J. Colbourn and J.H. Dinitz), Chapman & Hall/CRC Press, Boca Raton, 2007, 255–260.

[10] G. Ge and A. C. H. Ling, Some More 5-GDDs and Optimal $(v, 5, 1)$-Packings, *J. Combin. Des.* **12** (2004), 132–141).

[11] G. Ge and A. C. H. Ling, Asymptotic results on the existence of 4-RGDDs and uniform 5-GDDs, *J. Combin. Des.* **13** (2005), 222–237.

[12] G. Ge, R. S. Rees and L. Zhu, Group-divisible designs with block size four and group-type $g^u m^1$ with m as large or as small as possible, *J. Combin. Theory, Series A* **98** (2002), 357–376.

[13] M. Greig and R. C. Mullin, PBDs: Recursive Constructions, *Handbook of Combinatorial Designs*, 2nd edition (Eds.: C.J. Colbourn and J.H. Dinitz), Chapman & Hall/CRC Press, Boca Raton, 2007, 236–246.

[14] H. Hanani, Balanced incomplete block designs and related designs, *Discrete Math.* **11** (1975), 255–369.

[15] R. S. Rees, Group-divisible designs with block size k having $k + 1$ groups for $k = 4, 5$, *J. Combin. Des.* **8** (2000), 363–386.

[16] H. Wei and G. Ge, Some more 5-GDDs, 4-frames and 4-RGDDs, *Discrete Mathematics* **336** (2014), 7–21.

[17] R. M. Wilson, An existence theory for pairwise balanced designs I: Composition theorems and morphisms, *J. Combin. Theory Ser. A* **13** (1972), 220–245.

[18] J. Yin, A. C. H. Ling, C. J. Colbourn and R. J. R. Abel, The existence of uniform 5-GDDs, *J. Combin. Des.* **5** (1997), 275–299.

(Received 14 Mar 2022; revised 29 Nov 2022, 28 July 2023)