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Erdős-Ginzburg-Ziv type generalizations for linear equations and linear inequalities in three variables

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Abstract

For any linear inequality in three variables $\mathcal{L}$, we determine (if it exists) the smallest integer $R(\mathcal{L},\mathbb{Z}/3\mathbb{Z})$ such that: for every mapping $\chi:[1,n]\to\{0,1,2\}$, with $n\geq R(\mathcal{L},\mathbb{Z}/3\mathbb{Z})$, there is a solution $(x_1,x_2,x_3)\in[1,n]^3$ of $\mathcal{L}$ with $\chi(x_1)+\chi(x_2)+\chi(x_3)\equiv 0 \pmod 3$. Moreover, we prove that $R(\mathcal{L},\mathbb{Z}/3\mathbb{Z})=R(\mathcal{L},2)$, where $R(\mathcal{L},2)$ denotes the classical 2-color Rado number, that is, the smallest integer (provided it exists) such that for every 2-coloring of $[1,n]$, with $n\geq R(\mathcal{L},2)$, there is a monochromatic solution of $\mathcal{L}$. Thus, we get an Erdős-Ginzburg-Ziv type generalization for all linear Diophantine inequalities in three variables having a solution in the positive integers. We also show a number of families of linear Diophantine equations in three variables $\mathcal{L}$ which do not admit such Erdős-Ginzburg-Ziv type generalization, named $R(\mathcal{L},\mathbb{Z}/3\mathbb{Z})\neq R(\mathcal{L},2)$. At the end of this paper some questions are proposed.

1 Introduction

Ever since Erdős, Ginzburg, and Ziv proved their famous zero-sum theorem [11], EGZ-type results have been widely used in mathematics. The techniques employed in their proofs have proven to be quite useful in solving various problems. In recent years, it has become evident that EGZ-type results continue to provide new and interesting applications, see for instance [4, 15, 19]. In combinatorics, EGZ-type results are used in Ramsey theory and graph theory, as evidenced by [6, 7, 8, 9]. In

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number theory, the use of zero-sum results has been fruitful and has generated its own set of problems and research topics, exemplified by [3, 12, 13, 14, 16, 17]. In discrete geometry, EGZ-type results are used to solve geometrical problems in finite fields, as well as to study the minimum number of distinct distances generated by a set of points in the Euclidean plane, see [10, 20].

In this paper we investigate colorings of sets of natural numbers. We denote by $[a,b]$ the interval of natural numbers $\{x \in \mathbb{N} : a \leq x \leq b\}$, and by $[a,b]^k$ the set of vectors $(x_1, x_2, ..., x_k)$ where $x_i \in [a,b]$ for each $1 \leq i \leq k$. An r-coloring of $[1,n]$ is a function $\chi : [1,n] \to [0, r-1]$. Given an r-coloring of $[1,n]$, a vector $(x_1, x_2, ..., x_k) \in [1,n]^k$ is called *monochromatic* if all its entries received the same color, *rainbow* if all its entries received pairwise distinct colors, and *zero-sum* if $\sum_{i=1}^{k} \chi(x_i) \equiv 0 \pmod{r}$.

For a Diophantine system of equalities (or inequalities) in k variables $\mathcal{L}$, we denote by $R(\mathcal{L}, r)$ the classical *r-color Rado number*, that is, the smallest integer, provided it exists, such that for every r-coloring of $[1,n]$, with $n \geq R(\mathcal{L}, r)$, there exists $(x_1, x_2, ..., x_k) \in [1,n]^k$ a solution of $\mathcal{L}$ which is monochromatic. Rado numbers have been widely studied for many years (see for instance [18]). When studying the existence of zero-sum solutions, it is common to refer to an r-coloring as a $(\mathbb{Z}/r\mathbb{Z})$-coloring. In this setting, Bialostocki, Bialostocki and Schaal [5] started the study of the parameter $R(\mathcal{L}, \mathbb{Z}/r\mathbb{Z})$ defined as the smallest integer, provided it exists, such that for every $(\mathbb{Z}/r\mathbb{Z})$-coloring of $[1, R(\mathcal{L}, \mathbb{Z}/r\mathbb{Z})]$ there exists a zero-sum solution of $\mathcal{L}$. Recently, Robertson and other authors studied the same parameter concerning different equations or systems of equations, [21, 22, 23, 24].

We shall note that, if $\mathcal{L}$ is a system of equalities (or inequalities) in k variables, then

$$R(\mathcal{L}, 2) \leq R(\mathcal{L}, \mathbb{Z}/k\mathbb{Z}) \leq R(\mathcal{L}, k), \tag{1}$$

where the first inequality follows since, in particular, a $(\mathbb{Z}/k\mathbb{Z})$-coloring that uses only colors 0 and 1 is a 2-coloring where a zero-sum solution is a monochromatic solution; the second inequality of (1) follows since any monochromatic solution of $\mathcal{L}$ in a k-coloring of $[1, R(\mathcal{L}, k)]$ is a zero-sum solution too. In view of the Erdős-Ginzburg-Ziv theorem, the authors of [5] state that a system $\mathcal{L}$ admits an EGZ-*generalization* if $R(\mathcal{L}, 2) = R(\mathcal{L}, \mathbb{Z}/k\mathbb{Z})$. For example, it is not hard to see that the system $AP(3) : x + y = 2z,\ x < y$, admits an EGZ-generalization while the Schur equation, $x + y = z$, does not. More precisely, we have that

$$9 = R(AP(3), 2) = R(AP(3), \mathbb{Z}/3\mathbb{Z}) = 9,$$

and

$$5 = R(x + y = z, 2) < R(x + y = z, \mathbb{Z}/3\mathbb{Z}) = 10,$$

where $R(AP(3), 2)$ and $R(x + y = z, 2)$ are respectively the well-known van der Waerden number for 3-term arithmetic progressions concerning two colors and the Schur number concerning two colors, while $R(x + y = z, \mathbb{Z}/3\mathbb{Z}) = 10$ can be found in [21] and $R(AP(3), \mathbb{Z}/3\mathbb{Z}) = 9$ can be found in [22]. In [5] the authors consider the systems of inequalities

$$\mathcal{L}_1 : \sum_{i=1}^{k-1} x_i < x_k,$$

and

$$\mathcal{L}_2 : \sum_{i=1}^{k-1} x_i < x_k, \quad x_1 < x_2 < \cdots < x_k,$$

proving that $\mathcal{L}_2$ admits an EGZ-generalization for k prime, and $\mathcal{L}_1$ admits an EGZ-generalization for any k, particularly, $R(\mathcal{L}_1, 2) = R(\mathcal{L}_1, \mathbb{Z}/k\mathbb{Z}) = k^2 - k + 1$ (see [5]). In this paper we provide analogous results concerning any linear inequality in three variables. More precisely, let $a, b, c, d \in \mathbb{Z}$, such that $abc \neq 0$. Then we consider,

$$\mathcal{L}_3 : ax + by + cz + d < 0.$$

We prove that $\mathcal{L}_3$ admits an EGZ-generalization for every set of integers $\{a, b, c, d\}$ such that the corresponding 2-color Rado number exists. Moreover, we determine, in each case, such Rado numbers (see Theorem 2.2).

Note that, as we investigate linear systems, $\mathcal{L}$, in three variables, to have an EGZ-generalization means that

$$R(\mathcal{L}, 2) = R(\mathcal{L}, \mathbb{Z}/3\mathbb{Z}),$$

and the parameter $R(\mathcal{L}, \mathbb{Z}/3\mathbb{Z})$ is defined as the smallest integer, provided it exists, such that for every $f : [1, R(\mathcal{L}, \mathbb{Z}/3\mathbb{Z})] \to \{0, 1, 2\}$ there exists a zero-sum (mod 3) solution of $\mathcal{L}$ which, in this case, is either a monochromatic or a rainbow solution of $\mathcal{L}$. Therefore, the study of $R(\mathcal{L}, \mathbb{Z}/3\mathbb{Z})$ is considered as a canonical Ramsey problem.

The paper is organized as follows. In Section 2, we find the explicit values of $R(\mathcal{L}_3, \mathbb{Z}/3\mathbb{Z})$ and $R(\mathcal{L}_3, 2)$ (whenever $\mathcal{L}_3$ has solutions in the positive integers) in terms of the coefficients of $\mathcal{L}_3$. As a corollary, we get that $\mathcal{L}_3$ admits an EGZ-generalization in this case. In Section 3 we provide some negative results; that is, we exhibit families of linear equations in three variables which admit no EGZ-generalization. In Section 4, we talk about r-regular linear equations and the families $\mathcal{F}_k$ and $\mathcal{F}_{\mathbb{Z}/k\mathbb{Z}}$. At the end of this section, we give some problems related with these families.

2 The 2-color Rado numbers for $\mathcal{L}_3$

In this section we prove that any linear inequality on three variables, $\mathcal{L}_3$, for which the corresponding 2-color Rado number exists, admits an EGZ-generalization. We also determine the value of such Rado numbers depending on the coefficients of $\mathcal{L}_3$.

We will repeatedly use the following fact.

Remark 2.1. *Let A and B be integers such that $A < 0$. Then*

$$A\left(\left\lfloor \frac{B}{-A} \right\rfloor + 1\right) + B < 0 \leq A \left\lfloor \frac{B}{-A} \right\rfloor + B.$$

Recall that, for integers a, b, c and d, such that $abc \neq 0$,

$$\mathcal{L}_3 : ax + by + cz + d < 0.$$

Theorem 2.2. *Let $a, b, c, d \in \mathbb{Z}$, such that $abc \neq 0$, $a \leq b \leq c$, and define $\sigma = a + b + c + d$. If $\mathcal{L}_3$ has a solution in the positive integers, then*

$$R(\mathcal{L}_3, 2) = R(\mathcal{L}_3, \mathbb{Z}/3\mathbb{Z}) = \begin{cases} 1 & \text{if } \sigma < 0, \\ \left\lfloor \frac{d}{-a-b-c} \right\rfloor + 1 & \text{if } \sigma \geq 0 \text{ and } a \leq b \leq c < 0, \\ \left\lfloor \frac{c\left(\left\lfloor \frac{c+d}{-a-b} \right\rfloor + 1\right) + d}{-a-b} \right\rfloor + 1 & \text{if } \sigma \geq 0 \text{ and } a \leq b < 0 < c, \\ \left\lfloor \frac{(b+c)\left(\left\lfloor \frac{b+c+d}{-a} \right\rfloor + 1\right) + d}{-a} \right\rfloor + 1 & \text{if } \sigma \geq 0 \text{ and } a < 0 < b \leq c. \end{cases}$$

Proof. Let $\{a, b, c, d\}$ be a set of integers such that $\mathcal{L}_3$ has some (integer) positive solution. If $a + b + c + d < 0$ then $(1, 1, 1)$ is a monochromatic solution of $\mathcal{L}_3$ and so $R(\mathcal{L}_3, 2) = R(\mathcal{L}_3, \mathbb{Z}/3\mathbb{Z}) = 1$. If $a + b + c + d \geq 0$, then necessarily some of the coefficients, a, b or c, must be negative (otherwise, for all x, y, z positive integers, $ax + by + cz + d \geq a + b + c + d \geq 0$ and $\mathcal{L}_3$ would have no solution in the positive integers). Thus, assuming that $a + b + c + d \geq 0$, we consider three cases.

Case 1. Assume that $a \leq b \leq c < 0$. Define $k_0 = \left\lfloor \frac{d}{-a-b-c} \right\rfloor + 1$. First note that, since $a + b + c + d \geq 0$ and $-a - b - c > 0$, then $k_0 > 1$. Observe now that, for any $x, y, z \in [1, k_0 - 1]$,

$$\begin{aligned} ax + by + cz + d &\geq a(k_0 - 1) + b(k_0 - 1) + c(k_0 - 1) + d \\ &= (a + b + c) \left\lfloor \frac{d}{-a - b - c} \right\rfloor + d \geq 0, \end{aligned}$$

where the last inequality follows by taking $A = a+b+c < 0$ and $B = d$ in Remark 2.1. Then we conclude that $\mathcal{L}_3$ has no solution in $[1, k_0 - 1]$. On the other hand,

$$ak_0 + bk_0 + ck_0 + d = (a + b + c) \left(\left\lfloor \frac{d}{-a - b - c} \right\rfloor + 1 \right) + d < 0, \tag{2}$$

where the inequality follows by Remark 2.1 (taking again $A = a + b + c < 0$ and $B = d$). From (2), we conclude that (k_0, k_0, k_0) is a solution of $\mathcal{L}_3$, and so any coloring of $[1, k_0]$ will contain a monochromatic (zero-sum) solution of $\mathcal{L}_3$. Hence, $R(\mathcal{L}_3, 2) = R(\mathcal{L}_3, \mathbb{Z}/3\mathbb{Z}) = k_0$.

Case 2. Assume that $a \leq b < 0 < c$. Define the function

$$\psi : \mathbb{Z} \longrightarrow \mathbb{Z}, \qquad \psi(x) = \left\lfloor \frac{cx + d}{-a - b} \right\rfloor + 1,$$

and set $k_1 = \psi(1)$ and $k_2 = \psi(k_1)$. First note that, since $a+b+c+d \geq 0$ and $-a-b > 0$ then $k_1 > 1$ and, as ψ is a nondecreasing function, then $1 < k_1 \leq k_2$. From (1), it suffices to show that

$$k_2 \leq R(\mathcal{L}_3, 2) \tag{3}$$

and

$$R(\mathcal{L}_3, \mathbb{Z}/3\mathbb{Z}) \leq k_2. \tag{4}$$

To show (3), we exhibit a 2-coloring of $[1, k_2 - 1]$ without a monochromatic solution to $\mathcal{L}_3$. Define $\chi_1 : [1, k_2 - 1] \to \{0, 1\}$ as

$$\chi_1(x) = \begin{cases} 0 & \text{if } 1 \leq x \leq k_1 - 1, \\ 1 & \text{if } k_1 \leq x \leq k_2 - 1. \end{cases}$$

Note that if $x, y, z \in [1, k_1 - 1]$, then

$$\begin{aligned} ax + by + cz + d &\geq a(k_1 - 1) + b(k_1 - 1) + c + d \\ &= (a+b)\left\lfloor \frac{c+d}{-a-b} \right\rfloor + c + d \geq 0, \end{aligned} \tag{5}$$

where the last inequality follows by taking $A = a+b < 0$ and $B = c+d$ in Remark 2.1. Also, if $x, y, z \in [k_1, k_2 - 1]$, then

$$\begin{aligned} ax + by + cz + d &\geq a(k_2 - 1) + b(k_2 - 1) + ck_1 + d \\ &= (a+b)\left\lfloor \frac{ck_1+d}{-a-b} \right\rfloor + ck_1 + d \geq 0, \end{aligned} \tag{6}$$

where the last inequality follows by taking $A = a+b < 0$ and $B = ck_1 + d$ in Remark 2.1. From (5) and (6), we conclude that there is no monochromatic solution to $\mathcal{L}_3$ with respect to χ_1, which completes the proof of (3).

Now we prove (4). Let $\chi : [1, k_2] \to \{0, 1, 2\}$ be an arbitrary coloring, and assume that χ contains no zero-sum solutions of $\mathcal{L}_3$. We will use two times the first inequality of Remark 2.1. First take $A = a+b < 0$ and $B = c+d$ to obtain

$$ak_1 + bk_1 + c + d = (a+b)\left(\left\lfloor \frac{c+d}{-a-b} \right\rfloor + 1\right) + c + d < 0. \tag{7}$$

Now, take $A = a+b < 0$ and $B = ck_1 + d$ to obtain

$$ak_2 + bk_2 + ck_1 + d = (a+b)\left(\left\lfloor \frac{ck_1+d}{-a-b} \right\rfloor + 1\right) + ck_1 + d < 0. \tag{8}$$

By (7) we know that $(k_1, k_1, 1)$ is a solution of $\mathcal{L}_3$ which, by assumption, cannot be zero-sum. Suppose, since any zero-sum solution must be either rainbow or monochromatic, that $\chi(1) = 0$ and $\chi(k_1) = 1$. Next, we prove that $\chi(k_2)$ cannot be 0, 1 or 2.

- By (8), we know that (k_2, k_2, k_1) is a solution of $\mathcal{L}_3$, and so $\chi(k_2) \neq \chi(k_1) = 1$.

- Since $c > 0$ and $k_1 > 1$ then $ak_2+bk_2+c+d \leq ak_2+bk_2+ck_1+d$, which together with (8) implies that $(k_2, k_2, 1)$ is a solution of $\mathcal{L}_3$. Thus, $\chi(k_2) \neq \chi(1) = 0$.
- Since $a < 0$ and $k_2 > k_1$ then $ak_2 + bk_1 + c + d \leq ak_1 + bk_1 + c + d$, which together with (7) implies that $(k_2, k_1, 1)$ is a solution of $\mathcal{L}_3$. Thus, $\chi(k_2) \neq 2$.

This contradiction implies the existence of a zero-sum solution in any $(\mathbb{Z}/3\mathbb{Z})$-coloring of $[1, k_2]$, and we have completed the proof of (4).

Case 3. Assume that $a < 0 < b \leq c$. Define the function

$$\phi : \mathbb{Z} \longrightarrow \mathbb{Z}, \qquad \phi(x) = \left\lfloor \frac{(b+c)x+d}{-a} \right\rfloor + 1,$$

and set $k_3 = \phi(1)$ and $k_4 = \phi(k_3)$. First note that, since $a+b+c+d \geq 0$, then $k_3 > 1$ and, as ϕ is a nondecreasing function, then $1 < k_3 \leq k_4$. From (1), it is enough to show that

$$k_4 \leq R(\mathcal{L}_3, 2) \tag{9}$$

and

$$R(\mathcal{L}_3, \mathbb{Z}/3\mathbb{Z}) \leq k_4. \tag{10}$$

To show (9), we exhibit a 2-coloring of $[1, k_4 - 1]$ without a monochromatic solution to $\mathcal{L}_3$. Define $\chi_2 : [1, k_4 - 1] \to \{0, 1\}$ as

$$\chi_2(x) = \begin{cases} 0 & \text{if } 1 \leq x \leq k_3 - 1, \\ 1 & \text{if } k_3 \leq x \leq k_4 - 1. \end{cases}$$

Note that if $x, y, z \in [1, k_3 - 1]$, then

$$\begin{aligned} ax + by + cz + d &\geq a(k_3 - 1) + b + c + d \\ &= a \left\lfloor \frac{b+c+d}{-a} \right\rfloor + b + c + d \geq 0, \end{aligned} \tag{11}$$

where the last inequality follows by taking $A = a < 0$ and $B = b+c+d$ in Remark 2.1. Also, if $x, y, z \in [k_3, k_4 - 1]$, then

$$\begin{aligned} ax + by + cz + d &\geq a(k_4 - 1) + bk_3 + ck_3 + d \\ &= a \left\lfloor \frac{(b+c)k_3+d}{-a} \right\rfloor + bk_3 + ck_3 + d \geq 0, \end{aligned} \tag{12}$$

where the last inequality follows by taking $A = a < 0$ and $B = bk_3 + ck_3 + d$ in Remark 2.1. From (11) and (12), we conclude that there is no monochromatic solution to $\mathcal{L}_3$ with respect to χ_2, which completes the proof of (9).

Now we prove (10). Let $\chi : [1, k_4] \to \{0, 1, 2\}$ be an arbitrary coloring, and assume that χ contains no zero-sum solution of $\mathcal{L}_3$. We will use two times the first inequality of Remark 2.1. First take $A = a < 0$ and $B = b + c + d$ to obtain

$$ak_3 + b + c + d = a \left(\left\lfloor \frac{b+c+d}{-a} \right\rfloor + 1 \right) + b + c + d < 0. \tag{13}$$

Now take $A = a < 0$ and $B = (b+c)k_3 + d$ to obtain

$$ak_4 + bk_3 + ck_3 + d = a\left(\left\lfloor \frac{(b+c)k_3+d}{-a} \right\rfloor + 1\right) + (b+c)k_3 + d < 0. \qquad (14)$$

By (13) we know that $(k_3, 1, 1)$ is a solution of $\mathcal{L}_3$ which, by assumption, cannot be zero-sum. Suppose, without loss of generality, that $\chi(1) = 0$ and $\chi(k_3) = 1$. Next, we prove that $\chi(k_4)$ cannot be 0, 1 or 2.

- By (14), we know that (k_4, k_3, k_3) is a solution of $\mathcal{L}_3$, and so $\chi(k_4) \neq \chi(k_3) = 1$.
- Since $c \geq b > 0$ and $k_3 > 1$ then $ak_4 + b + c + d \leq ak_4 + bk_3 + ck_3 + d$, which together with (14) implies that $(k_4, 1, 1)$ is a solution of $\mathcal{L}_3$. Thus, $\chi(k_4) \neq \chi(1) = 0$.
- Since $c > 0$ and $k_3 > 1$ then $ak_4 + bk_3 + c + d \leq ak_4 + bk_3 + ck_3 + d$, which together with (14) implies that $(k_4, k_3, 1)$ is a solution of $\mathcal{L}_3$. Thus, $\chi(k_4) \neq 2$.

This contradiction implies the existence of a zero-sum solution in any $(\mathbb{Z}/3\mathbb{Z})$-coloring of $[1, k_4]$, and we have completed the proof of (10). □

As an immediate consequence of Theorem 2.2 we conclude the following.

Corollary 2.3. *Let $a, b, c, d \in \mathbb{Z}$, such that $abc \neq 0$. If $\mathcal{L}_3 : ax + by + cz + d < 0$ has a solution in the positive integers, then $\mathcal{L}_3$ admits an EGZ-generalization.*

3 Negative results

In this section we exhibit different families of linear equations in three variables that admit no EGZ-generalization. In other words, we study equations, $\mathcal{L}$, where $R(\mathcal{L}, 2) \neq R(\mathcal{L}, \mathbb{Z}/3\mathbb{Z})$. Naturally, we focus our attention on equations such that both $R(\mathcal{L}, 2)$ and $R(\mathcal{L}, \mathbb{Z}/3\mathbb{Z})$ exist. Although Rado's Theorem characterizes the equations $\mathcal{L}$ such that $R(\mathcal{L}, 2)$ exists, there is a small number of families of equations where the value $R(\mathcal{L}, 2)$ is explicitly known; see [18], [1]. In this section we develop some ideas to compare $R(\mathcal{L}, 2)$ and $R(\mathcal{L}, \mathbb{Z}/3\mathbb{Z})$ for some equations, and then we get some applications to show that $\mathcal{L}$ does not admit an EGZ-generalization.

Theorem 3.1. *Let a, b, c, d be integers where a, b, c are odd and d is even, such that both $R\left(ax + by + cz = \frac{d}{2}, 2\right)$ and $R(ax + by + cz = d, \mathbb{Z}/3\mathbb{Z})$ exist. Then*

$$2R\left(ax + by + cz = \frac{d}{2}, 2\right) \leq R(ax + by + cz = d, \mathbb{Z}/3\mathbb{Z}).$$

Proof. Abbreviate writing $R := R\left(ax + by + cz = \frac{d}{2}, 2\right)$. Let $\chi_0 : [1, R-1] \to \{0, 1\}$ be a coloring such that $ax + by + cz = \frac{d}{2}$ has no monochromatic solution with respect to χ_0. Define

$$\chi : [1, 2R-1] \longrightarrow \{0, 1, 2\}, \qquad \chi(n) = \begin{cases} \chi_0\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

To prove the claim of the theorem, it is enough to show that $ax+by+cz=d$ has no zero-sum solution with respect to χ. Let (x_0,y_0,z_0) be a solution of $ax+by+cz=d$. Since d is even and a,b,c are odd, we have that either the three entries of (x_0,y_0,z_0) are even or exactly one of the entries is even.

First assume that the three entries of (x,y,z) are even. Then $\chi(x_0)=\chi_0\left(\frac{x_0}{2}\right)$, $\chi(y_0)=\chi_0\left(\frac{y_0}{2}\right)$ and $\chi(z_0)=\chi_0\left(\frac{z_0}{2}\right)$; since $ax+by+cz=\frac{d}{2}$ has no monochromatic solution with respect to χ_0, $\left(\frac{x_0}{2},\frac{y_0}{2},\frac{z_0}{2}\right)$ is not monochromatic and does not contain the color 2. Therefore (x_0,y_0,z_0) is no zero-sum solution with respect to χ.

Now assume that exactly one of the entries of (x,y,z) is even; without loss of generality assume that x_0 is even. Then $\chi(x_0)=\chi_0\left(\frac{x_0}{2}\right)$, $\chi(y_0)=2$ and $\chi(z_0)=2$. This means that (x_0,y_0,z_0) is not a zero-sum solutions with respect to χ. □

The next result is an immediate corollary of Theorem 3.1.

Corollary 3.2. *Let $\mathcal{L}$ be the equation $ax+by+cz=0$, and assume that $R(\mathcal{L},\mathbb{Z}/3\mathbb{Z})$ exists. Then $\mathcal{L}$ admits no EGZ-generalization if a, b and c are odd integers.*

Also Theorem 3.1 provides some applications for non-homogeneous linear equations.

Corollary 3.3. *Let d be a negative even integer. Then the equation $x+y-z=d$ admits no EGZ-generalization.*

Proof. From [18, Thm. 9.14], we have that

$$R(x+y-z=d,2)=5-4d,$$

and

$$R\left(x+y-z=\frac{d}{2},2\right)=5-2d.$$

On the other hand, Theorem 3.1 leads to

$$2R\left(x+y-z=\frac{d}{2},2\right)\leq R(x+y-z=d,\mathbb{Z}/3\mathbb{Z}).$$

Hence

$$\begin{aligned}R(x+y-z=d,\mathbb{Z}/3\mathbb{Z})&\geq 2R\left(x+y-z=\frac{d}{2},2\right)\\&=2(5-2d)\\&>5-4d\\&=R(x+y-z=d,2).\end{aligned}$$

□

Corollary 3.4. *Let d be a positive integer congruent to $6,8$ or 0 modulo 10. Then the equation $x+y-z=d$ admits no EGZ-generalization.*

Proof. On the one hand, we have from [18, Thm. 9.15] that

$$R(x+y-z=d,2)=d-\left\lceil\frac{d}{5}\right\rceil+1,$$

and

$$R\left(x+y-z=\frac{d}{2},2\right)=\frac{d}{2}-\left\lceil\frac{\frac{d}{2}}{5}\right\rceil+1.$$

On the other hand, Theorem 3.1 leads to

$$2R\left(x+y-z=\frac{d}{2},2\right)\leq R(x+y-z=d,\mathbb{Z}/3\mathbb{Z}).$$

Thus, since d is congruent to $6, 8$ or 0 modulo 10, we get that

$$\begin{aligned}R(x+y-z=d,\mathbb{Z}/3\mathbb{Z})&\geq 2R\left(x+y-z=\frac{d}{2},2\right)\\&=2\left(\frac{d}{2}-\left\lceil\frac{\frac{d}{2}}{5}\right\rceil+1\right)\\&>d-\left\lceil\frac{d}{5}\right\rceil+1\\&=R(x+y-z=d,2).\end{aligned}$$

□

4 Other directions

A linear homogenous equation is called *r-regular* if every r-coloring of $\mathbb{N}$ contains a monochromatic solution of it (equivalently, an equation $\mathcal{L}$ is called r-regular if $R(\mathcal{L},r)$ exist). A linear homogenous equation is called *regular* if it is r-regular for all positive integers r. Denote by $\mathcal{F}_r$ the family of linear homogenous equations which are r-regular. For equations on $k\geq 3$ variables, Rado completely determined $\mathcal{F}_2$: it is the set of equations, $\sum_{i=1}^{k}c_ix_i=0$ for which there exist $i,j\in\{1,\ldots,k\}$ such that $c_i<0$ and $c_j>0$ (see, for instance [18]). For other values of $r\in\mathbb{Z}^+$, the family $\mathcal{F}_r$ is not characterized. Rado's Single Equation Theorem states that a linear homogenous equation on $k\geq 2$ variables, $\sum_{i=1}^{k}c_ix_i=0$ (c_i's are non-zero integers), is regular if and only if there exists a non-empty $D\subseteq\{1,...,k\}$ such that $\sum_{d\in D}c_d=0$. Naturally, $\mathcal{F}_{r+1}\subseteq\mathcal{F}_r$ for all $r\in\mathbb{Z}^+$. In his Ph.D. dissertation, Rado conjectured that, for all $r\in\mathbb{Z}^+$, there are equations that are r-regular but not $(r+1)$-regular. This conjecture was solved by Alexeev and Tsimerman in 2010 [2], where they confirm that $\mathcal{F}_{r+1}\subsetneq\mathcal{F}_r$. For any $k\in\mathbb{N}$, define $\mathcal{F}_{\mathbb{Z}/k\mathbb{Z}}$ to be the family of linear homogeneous equations in k variables, $\mathcal{L}$, for which $R(\mathcal{L},\mathbb{Z}/k\mathbb{Z})$ exist. By (1) we know that

$$\mathcal{F}_3\subseteq\mathcal{F}_{\mathbb{Z}/3\mathbb{Z}}\subseteq\mathcal{F}_2.$$

We will show that

$$\mathcal{F}_3\subsetneq\mathcal{F}_{\mathbb{Z}/3\mathbb{Z}}. \tag{15}$$

For all $n \in \mathbb{N}$, denote by $\mathrm{ord}_2(n)$ the maximum $m \in \mathbb{Z}$ such that 2^m divides n. First note that the equation $x + 2y - 4z = 0$ is not in $\mathcal{F}_3$ since the coloring

$$\chi : \mathbb{N} \to \{0, 1, 2\}, \qquad \chi(n) = \begin{cases} 0 & \text{if } \mathrm{ord}_2(n) \equiv 0 \mod 3, \\ 1 & \text{if } \mathrm{ord}_2(n) \equiv 1 \mod 3, \\ 2 & \text{if } \mathrm{ord}_2(n) \equiv 2 \mod 3. \end{cases}$$

has no monochromatic solution of it. The next proposition implies that $x+2y-4z = 0$ is in $\mathcal{F}_{\mathbb{Z}/3\mathbb{Z}}$ and therefore (15) holds.

Proposition 4.1. *The equation* $x+2y-4z = 0$ *satisfies* $R(x+2y-4z = 0, \mathbb{Z}/3\mathbb{Z}) = 8$.

Proof. Let $\mathcal{L}$ be the equation $x + 2y - 4z = 0$. First we show that $R(\mathcal{L}, \mathbb{Z}/3\mathbb{Z}) \geq 8$. Let $\chi : [1, 8] \to \{0, 1, 2\}$ be a coloring. We assume that there is no zero-sum solution of $\mathcal{L}$ with respect to χ, and we will get a contradiction. Assume without loss of generality that $\chi(1) = 0$. Since $(2, 1, 1)$ is a solution of $\mathcal{L}$, $\chi(2) \neq 0$; assume without loss of generality that $\chi(2) = 1$. Note that $(2, 3, 2)$, $(4, 2, 2)$ and $(4, 4, 3)$ are solutions of $\mathcal{L}$. Thus either $\chi(3) = 0$ and $\chi(4) = 2$, or $\chi(3) = 2$ and $\chi(4) = 0$. Notice that $(6, 3, 3)$ and $(4, 6, 4)$ are solutions of $\mathcal{L}$ so $\chi(6) = 1$. Since $(6, 5, 4)$ and $(2, 5, 3)$ are solutions of $\mathcal{L}$, we get $\chi(5) = 1$. For any value of $\chi(8)$, we obtain a zero-sum solution inasmuch as $(8, 2, 3)$, $(4, 8, 5)$ and $(8, 6, 5)$ are solutions of $\mathcal{L}$ and this is the desired contradiction.

On the other hand, $R(\mathcal{L}, \mathbb{Z}/3\mathbb{Z}) > 7$ since there is no zero-sum solution with respect to the coloring

$$\chi : [1, 7] \longrightarrow \{0, 1, 2\}, \qquad \chi(n) = \begin{cases} 0 & \text{if } n \in \{1, 4, 7\} \\ 1 & \text{if } n \in \{2, 5, 6\} \\ 2 & \text{if } n = 3, \end{cases}$$

and this completes the proof. $\square$

From the previous discussion, we know that $\mathcal{F}_3 \subsetneq \mathcal{F}_{\mathbb{Z}/3\mathbb{Z}} \subseteq \mathcal{F}_2$. A natural question arises from this chain.

Problem 1. *Is it true that* $\mathcal{F}_{\mathbb{Z}/3\mathbb{Z}} \subsetneq \mathcal{F}_2$*?*

From (1) we know that $\mathcal{F}_k \subseteq \mathcal{F}_{\mathbb{Z}/k\mathbb{Z}}$ for all $k \geq 3$. Thus it would be interesting to know if there are $k \in \mathbb{N}$ such that equality is achieved.

Problem 2. *For all* $k \geq 3$, $\mathcal{F}_k \subsetneq \mathcal{F}_{\mathbb{Z}/k\mathbb{Z}}$*?*

Finally we know that $\mathcal{F}_k \subseteq \mathcal{F}_{\mathbb{Z}/k\mathbb{Z}}$ and $\mathcal{F}_k \subsetneq \mathcal{F}_{k-1}$ for all $k \geq 3$. However we do not know whether there is a relation between $\mathcal{F}_{k-1}$ and $\mathcal{F}_{\mathbb{Z}/k\mathbb{Z}}$.

Problem 3. *For all* $k \geq 3$, $\mathcal{F}_{\mathbb{Z}/k\mathbb{Z}} \subsetneq \mathcal{F}_{k-1}$*?*

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