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Some properties of minimal arbitrarily partitionable graphs

Julien Bensmail

Université Côte d'Azur
CNRS, Inria, I3S
France

Abstract

A graph G on n vertices is arbitrarily partitionable (AP for short) if for every partition $(\lambda_1, \dots, \lambda_p)$ of n (that is, $\lambda_1 + \cdots + \lambda_p = n$), the vertex set $V(G)$ can be partitioned into p parts $V_1, \dots, V_p$ such that $G[V_i]$ has order λ_i and is connected for every $i \in \{1, \dots, p\}$. We investigate minimal AP graphs, which are those AP graphs that are not spanned by any proper AP subgraph. We pursue previous investigations by Ravaux and Baudon, Przybyło, and Woźniak, who established that minimal AP graphs are not all trees, but conjectured that they should all be somewhat sparse. We investigate several aspects of minimal AP graphs, including their minimum degree, their maximum degree, and their clique number. Some of the results we establish arise from an exhaustive list we give, of all minimal AP graphs of order at most 10. We also address new questions on the structure of minimal AP graphs.

1 Introduction

This work deals mainly with **arbitrarily partitionable graphs**, which are defined accordingly to the following notions. Let G be an *n-graph*, i.e., a graph with order (number of vertices) n. Let also $\pi = (\lambda_1, \dots, \lambda_p)$ be an *n-partition*, i.e., a sequence of integers forming a partition of n (that is, $\lambda_1 + \cdots + \lambda_p = n$). Now, a *realisation* $\mathcal{R} = (V_1, \dots, V_p)$ of π in G is a partition of $V(G)$ into p parts $V_1, \dots, V_p$ such that $G[V_i]$ is a connected graph of order λ_i for every $i \in \{1, \dots, p\}$. In other words, we aim at partitioning G into disjoint connected subgraphs, the number of desired subgraphs and their orders being indicated by the size $|\pi|$ of π and its elements $\lambda_1, \dots, \lambda_p$. We now say that G is *arbitrarily partitionable* (AP for short) if every n-partition is realisable in G, or in other words, if G can be partitioned into arbitrarily many connected subgraphs with arbitrary orders.

AP graphs were introduced independently by Barth, Baudon, and Puech [1], and Horňák and Woźniak [18], in the early 2000s, but they also relate to much older,

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fundamental results on graph partitions into connected subgraphs, the most influential of which is surely the Győri-Lovász Theorem [17, 23] dating back to the 1970s. Practical applications of AP graphs apart, an interesting point behind APness lies in the fact that this notion stands somewhere in between factorability and Hamiltonicity, two fundamental fields of graph theory. Note indeed that any realisation of the n-partition $(2, \dots, 2)$ (if n is even) or $(2, \dots, 2, 1)$ (otherwise, if n is odd) in an n-graph forms a perfect matching or quasi-perfect matching. We remark also that a graph is AP whenever it is spanned by an AP graph; from this and the fact that paths and cycles are obviously AP, we deduce that any Hamiltonian or traceable graph (i.e., having a Hamiltonian cycle or path) is AP.

Quite a few aspects of AP graphs have been investigated in the literature to date. We survey a few of them in what follows and later in Section 2; for a deeper insight into the topic, we refer the interested reader to, e.g., [9].

- Most of the very first works dedicated to AP graphs, such as [1, 2, 3, 18], focused on AP trees. Among notable properties of interest, it was proved by Barth and Fournier that AP trees are of maximum degree at most 4 (see [2]). These investigations on AP trees were, later on, the starting point of more general investigations on the structure of AP graphs with given connectivity. In particular, it was proved by Baudon, Foucaud, Przybyło, and Woźniak in [6] that removing a cut vertex from an AP graph results in at most four connected components, while removing the vertices of a k-cutset for any $k \geq 2$ can result in arbitrarily many connected components (but their orders must follow an exponential growth).

- From the algorithmic point of view, determining whether an n-partition π is realisable in a given n-graph G is notoriously NP-complete, even under various strong restrictions on π and/or G. In particular, the problem remains NP-complete even when $\pi = (3, \dots, 3)$ and when $|\pi| = 2$ (see [10, 15]), and even when G is a tree of maximum degree 3, a subdivided star, or a split graph (see [2, 10, 13]). Regarding the question of determining whether a given graph G is AP, the problem is suspected to lie in NP (which was raised by Barth and Fournier in [2]), and that was proved to be true when G is a subdivided star, a split graph, or a graph with various properties (see [1, 11, 13]). Even stronger, determining whether a given split graph is AP can be done in polynomial time [13].

- As mentioned earlier, APness has been regarded as a notion being weaker than traceability, and thus than Hamiltonicity. As such, an interesting line of research consists in investigating, for APness, weaker versions of known sufficient conditions for traceability or Hamitonicity. This was initiated by Marczyk in [24], in which Ore-type conditions for APness (which were later improved upon in [19, 25]) were investigated. In [21], Kalinowski, Pilśniak, Schiermeyer, and Woźniak investigated size (number of edges) conditions for a graph to be AP. In [12], Bensmail and Li considered other classical results from the realm of Hamiltonian graphs, showing that some of them can be weakened to APness while some others can not.

- A number of more or less involved variants of AP graphs have also been investigated, one point being that, in various situations, showing that a graph is "more than AP" is easier than showing it is AP only. In particular, *OL-AP graphs*, investigated for instance in [20], stand as an online variant of AP graphs that can be partitioned into connected subgraphs on the fly, as parts are requested one after the other, one at a time. *R-AP graphs*, investigated notably in [7], stand as a recursive variant of AP graphs that can be partitioned into connected subgraphs that are arbitrarily partionable themselves. In [5] a variant of AP graphs, called *AP+k graphs*, was also investigated; in these graphs, realisations of partitions are also required to have certain of their parts (precisely k of them) to contain some given vertices.

In this work, we focus on another interesting aspect of AP graphs, being the notion of **minimality**. As mentioned earlier, a graph is AP as soon as it is spanned by an AP graph. From this prospect, a fundamental notion, towards understanding AP graphs fully, is that of those AP graphs that are not spanned by any AP graph with fewer edges. More precisely, an AP graph G is said to be *minimal* if $G - e$ is not AP for every edge e of G. In other words, a minimal AP graph, when being removed any edge, loses the property of being AP.

Note that being connected is an obvious necessary condition for an n-graph to be AP (because of the n-partition (n)). From this, we deduce immediately that AP trees are minimal AP. Also, note that if an n-partition π is realisable in an n-graph G, then G contains a spanning tree in which π is realisable. Back in the days where AP trees were the main focus in most investigations dedicated to AP graphs, a legitimate presumption was that, perhaps, all AP graphs are actually spanned by an AP tree (or, in other words, that only AP trees are minimal AP). This was refuted by Ravaux, who was the first to investigate minimal AP graphs as such in [28], as he exhibited the very first example of a minimal AP graph that is not a tree. To be more precise, this graph, which has order and size 20, is obtained from a 19-path $v_1 \dots v_{19}$ by adding a new vertex v_{20} joined to both v_5 and v_8.

In his seminal work on the topic [28], Ravaux also initiated the study of structural properties of minimal AP graphs from a more general perspective. In particular, he proved that, provided $n \geq 4$, a minimal AP n-graph G must have maximum degree $\Delta(G) \leq n - 2$. He also suspected that minimal AP graphs should have linear size, which prompted him to pose the following:

Conjecture 1.1 (Ravaux [28]). *If G is a minimal AP n-graph, then $|E(G)|$ is $\mathcal{O}(n)$.*

Following Ravaux's work, other investigations on structural properties of minimal AP graphs were led by Baudon, Przybyło, and Woźniak in [8]. In particular, they provided an infinite family of minimal AP graphs containing arbitrarily many disjoint cycles (which can be of arbitrarily large length), thereby showing that a minimal AP graph can have size arbitrarily larger than that of a tree (on the same number of vertices). From their construction, they also considered the maximum density (being $|E(G)|/|V(G)|$ for a given graph G) of a minimal AP graph, and showed that some of the minimal AP graphs they constructed have density tending asymptotically to 31/30.

To the best of our knowledge, this is pretty much everything known on minimal AP graphs. It has to be emphasised that the structure of AP graphs is hard to comprehend, and thus even harder to comprehend is that of minimal AP graphs. The aim of the current paper, following the previous works of Ravaux and Baudon, Przybyło, and Woźniak, is to establish more properties of minimal AP graphs, and to raise directions to guide future investigations on this interesting topic.

This work is organised as follows. We start by recalling preliminary contents (such as known results on AP graphs) in Section 2. We then investigate a few aspects of minimal AP graphs through Sections 3 to 6. In Section 3, we first provide an exhaustive list of all minimal AP n-graphs with $n \leq 10$, from which we deduce new possible properties of minimal AP graphs. In Section 4, reusing ideas of Marczyk from [24], we then establish an upper bound on the minimum degree of minimal AP graphs. In Section 5, we improve the upper bound on the maximum degree of minimal AP graphs established by Ravaux in [28]. In Section 6, we then investigate the size of maximum cliques in minimal AP graphs. We conclude this work in Section 7, in which, following our investigations, we raise numerous questions on the structure of minimal AP graphs.

2 Preliminaries

Throughout this work, it is assumed that the reader is familiar with the most standard notions of graph theory. In case anything is unclear, we refer the reader to any monograph on the topic.

We start by recalling simple graph structures which arose in several works on AP graphs. For $n \geq 1$, the *n-path* P_n is the path of order n. Other types of trees of interest include subdivided stars, which were often referred to as *multipodes* in previous works on AP graphs. That is, the multipode (or *k-pode*) $\mathrm{P}_k(a_1, \dots, a_k)$, where $k \geq 3$ and $a_1, \dots, a_k \geq 1$, is obtained from a star with k edges $uv_1, \dots, uv_k$ by subdividing uv_i exactly $a_i - 1$ times for every $i \in \{1, \dots, k\}$. In other words, $\mathrm{P}_k(a_1, \dots, a_k)$ is obtained by identifying one end of each of an $(a_1 + 1)$-path, of an $(a_2 + 1)$-path, etc., to a single vertex u of degree k. A 3-pode is sometimes called a *tripode*. In the very particular case of a tripode $\mathrm{P}_3(1, a, b)$, we sometimes speak of a *caterpillar*, denoted by $\mathrm{Cat}(a + 1, b + 1)$ (the point for this notation being that the order of any $\mathrm{Cat}(x, y)$ is $x + y$).

Paths apart, caterpillars form the simplest structure for an AP tree, and this is why they received considerable attention in several works dedicated to trees that are (sometimes more than) AP. The characterisation of the AP ones can actually be retrieved from the following more general result of Ravaux on graphs with long paths. In that result and further, for a given partition π we denote by $\mathrm{sp}(\pi)$ the *spectrum* of π, being the set of the distinct element values that appear in π.

Theorem 2.1 (Ravaux [29]). *If G is an n-graph with a path of length $n - \alpha$, then every n-partition π with $|\mathrm{sp}(\pi)| \geq \alpha$ is realisable in G.*

In particular, the longest path of a caterpillar $\mathrm{Cat}(a, b)$ with order $a + b$ having length $a + b - 2$, Theorem 2.1 implies the following (the last part of the statement being easy to check):

Corollary 2.2. *A caterpillar $C = \mathrm{Cat}(a, b)$ with order $n = a + b$ is AP if and only if all n-partitions π with $|\mathrm{sp}(\pi)| = 1$ are realisable in C. Consequently, C is AP if and only if a and b are coprime.*

Regarding AP trees with maximum degree at least 4, let us recall the following:

Theorem 2.3 (Barth, Fournier [2]). *AP trees have maximum degree at most 4. Furthermore, there exist arbitrarily large AP trees with maximum degree 4. For instance, every 4-pode $\mathrm{P}_4(1, 1, 2t - 2, 2t)$ with $t \geq 3$ and $t \not\equiv 2 \bmod 3$ is AP.*

Another class of graphs of interest in the study of AP graphs, see, e.g., [4], is that of *balloons*, which are particular series-parallel graphs defined as follows. Let $k \geq 3$, and $b_1, \dots, b_k \geq 0$ be integers, at most one of which is 0. The balloon (or *k-balloon*) $\mathrm{B}_k(b_1, \dots, b_k)$ is obtained from k paths of order $b_1+2, \dots, b_k+2$ by identifying any one of their ends to a single vertex r_1 of degree k, and similarly identifying their second ends to a single vertex r_2 of degree k. Another way to describe this construction is by saying that $\mathrm{B}_k(b_1, \dots, b_k)$ is obtained from the multigraph with two vertices r_1 and r_2 joined by k parallel edges, by subdividing every ith edge exactly b_i times. Note that $\mathrm{B}_k(b_1, \dots, b_k)$ is simple provided at most one of the b_i's is 0.

A result of interest on AP balloons reads as follows:

Theorem 2.4 (Baudon, Foucaud, Przybyło, Woźniak [6]). *For every $k \geq 3$, there exist AP k-balloons $\mathrm{B}_k(b_1, \dots, b_k)$. Furthermore, assuming $b_1 \leq \cdots \leq b_k$, the b_i's grow exponentially.*

3 Exhaustive search of small minimal AP graphs

We here report observations on (minimal) AP graphs with small order, found through exhaustive search via computer programs. Namely, we were able to investigate all graphs on up to 10 vertices.

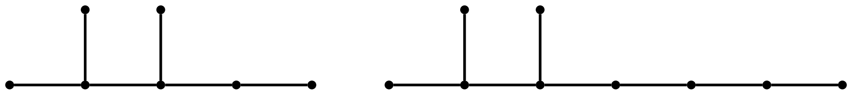
3.1 Methodology

Let n be a fixed integer. The computer programs we designed to find all (minimal) AP n-graphs proceed according to the following rules.

- Throughout what follows, we maintain a set $\mathcal{M}$ of minimal AP n-graphs, which is updated as soon as a new minimal AP n-graph is found.
- We initialise $\mathcal{M}$ so that it contains the minimal AP n-graphs with the simplest structure we know of, being the n-path P_n and the AP caterpillars on n vertices (given by Corollary 2.2). These graphs being trees, are indeed minimal AP.
- We then go through all n-graphs one by one, in increasing order over their size. For every such n-graph G, we proceed as follows:
  - We first check whether G is spanned by one of the graphs in $\mathcal{M}$. If it is, then G is AP but not minimal AP. In that case, we just proceed with the next n-graph G.
  - If G is not spanned by any of the graphs in $\mathcal{M}$, we then determine whether G is AP.
    * If G is not AP, then we just proceed with the next n-graph G.
    * Otherwise, then, due to the fact that we are considering n-graphs in increasing order over their size, the fact that no graph of $\mathcal{M}$ spans G means that G is not only AP but also minimal AP. We then add G to $\mathcal{M}$, and proceed with the next n-graph G.

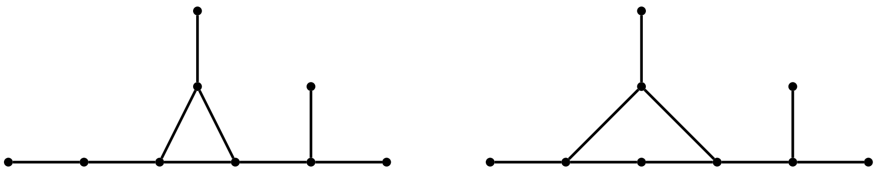
Once the algorithm above ends, it should be clear that the eventual $\mathcal{M}$ contains all minimal AP n-graphs. The crucial and tricky point in the algorithm above, however, is determining whether a given graph is AP. Indeed, while determining whether a given graph on up to 10 vertices is AP can be performed in a greedy way, it is worth recalling that the number of pairwise non-isomorphic 10-graphs is $11,716,571$. Thus, repeating even the simplest tasks that many times requires some optimisation to be achieved within a reasonable amount of time. Below, we summarise some optimisations we performed to allow our programs to go through all 10-graphs:

- For small n-graphs, one of the most common obstructions to APness is the fact of having no large matchings, and thus, in particular, no realisation of the n-partition $(2,\dots,2)$ (if n is even) or $(2,\dots,2,1)$ (otherwise, if n is odd). One first optimisation, when checking whether an n-graph is AP, is then to first check whether it admits a matching of size $\lfloor n/2 \rfloor$, which, fortunately, can be done efficiently (recall, for instance, Edmonds' Blossom Algorithm [16]).
- In any n-graph, any realisation of an n-partition $\pi = (\lambda_1,\dots,\lambda_p)$ can be seen as a p-vertex-colouring $(V_1,\dots,V_p)$ where $|V_i| = \lambda_i$ and $G[V_i]$ is connected for every $i \in \{1,\dots,p\}$. A naive way to check whether there is a realisation of π in G is thus to go through all p-vertex-colourings of G and check whether any of them fulfils the required properties. While this is quite demanding when n is large, for small values of n this is manageable in reasonable time.



(a) (b)

Figure 1: Two AP trees.



(a) (b)

Figure 2: The only two minimal AP 9-graphs that are not trees.

- To check whether an n-graph is AP, by definition we have to consider all n-partitions and check whether they are all realisable. An obvious observation (also made in earlier works, see e.g. [22]) is that we do not have to consider n-partitions $(\lambda_1, \dots, \lambda_p, 1)$ containing 1 as an element, as its realisation would follow, for instance, from any realisation of the n-partition $(\lambda_1 + 1, \dots, \lambda_p)$. As an illustration, for 10-graphs, when checking APness we only have to consider the eleven 10-partitions $(8, 2)$, $(7, 3)$, $(6, 4)$, $(6, 2, 2)$, $(5, 5)$, $(5, 3, 2)$, $(4, 4, 2)$, $(4, 3, 3)$, $(4, 2, 2, 2)$, $(3, 3, 2, 2)$, and $(2, 2, 2, 2, 2)$, while, without restrictions, there exist 42 pairwise distinct 10-partitions.

3.2 Results and observations

Through our previous approach and the described optimisations, we were able to go through all graphs on up to 10 vertices to check which ones are minimal AP. Our conclusions are the following:

- For $n \in \{3, 4\}$, the only minimal AP n-graph is the n-path P_n. In other words, for these values of n, every AP n-graph is spanned by the n-path (and is thus traceable).
- For $n = 5$, the only two minimal AP n-graphs are the caterpillar $\mathrm{Cat}(2, 3)$ and the 5-path P_5. In particular, every minimal AP 5-graph is a tree.
- For $n = 6$, the only minimal AP n-graph is the 6-path P_6. Thus, every AP 6-graph is traceable.
- For $n = 7$, there are four minimal AP n-graphs, being the caterpillar $\mathrm{Cat}(2, 5)$, the caterpillar $\mathrm{Cat}(3, 4)$, the 7-path P_7, and the tree in Figure 1(a). In particular, every minimal AP 7-graph is a tree.
- For $n = 8$, the only two minimal AP n-graphs are the caterpillar $\mathrm{Cat}(3, 5)$ and the 8-path P_8. In particular, every minimal AP 8-graph is a tree.

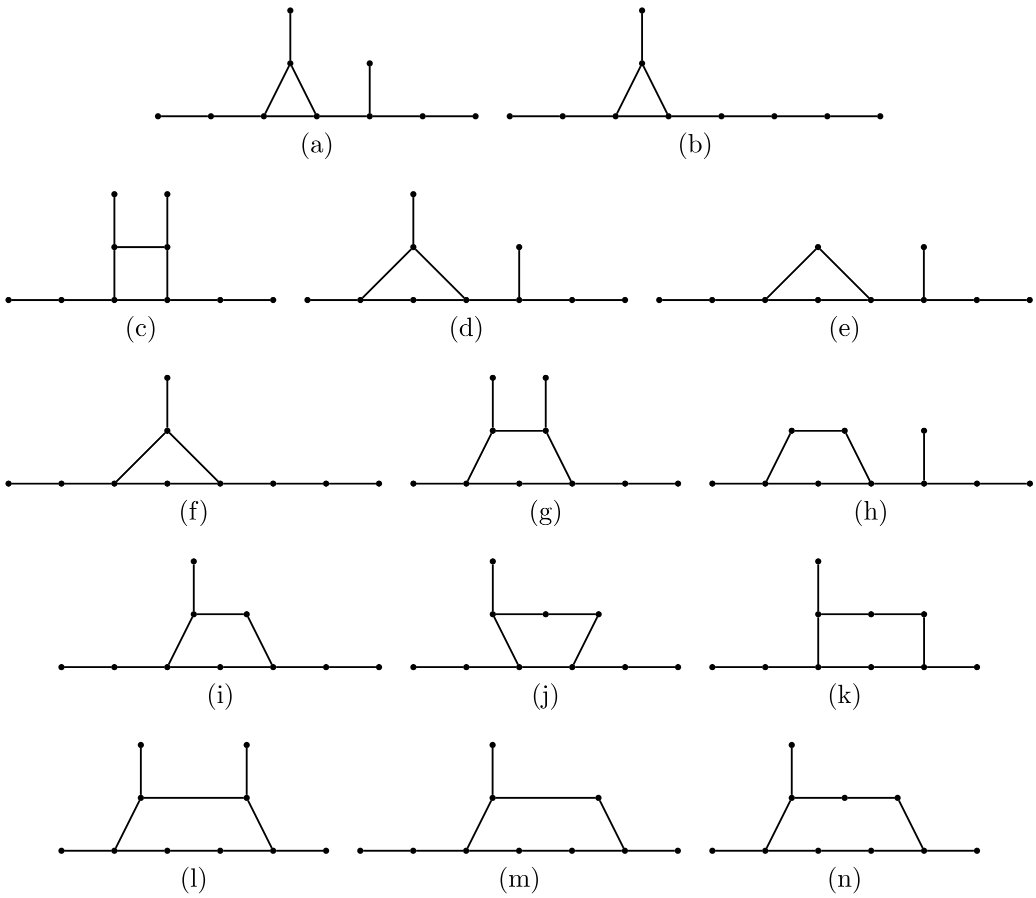


Figure 3: The fourteen minimal AP 10-graphs that are not trees.

- For $n = 9$, there are six minimal AP n-graphs, being the caterpillar $\mathrm{Cat}(2, 7)$, the caterpillar $\mathrm{Cat}(4, 5)$, the 9-path P_9, the tree depicted in Figure 1(b), and the two 9-graphs depicted in Figure 2. These two graphs are thus the smallest two minimal AP graphs that are not trees. Also, their size is 9.

- For $n = 10$, there are sixteen minimal AP n-graphs, being the caterpillar $\mathrm{Cat}(3, 7)$, the 10-path P_{10}, and the fourteen 10-graphs depicted in Figure 3. Note that all these fourteen 10-graphs, which are not trees, have size 10.

Regarding more general properties of interest, note that minimal AP n-graphs with $n \leq 10$ that are not trees have size exactly n, thus only one cycle (which can be basically of any length), are of maximum degree at most 3, and are of minimum degree 1 (and, thus, have cut vertices). Of course, given how limited this sample is, these properties are surely far from being representative of the properties shared by most minimal AP graphs that are not trees.

Note that we did not prove that the new minimal AP graphs (in particular those from Figure 3) we have exhibited are indeed minimal AP. It is however an easy exercise to check that they are indeed minimal AP. In case this is unclear, one can refer to the proof of later Theorem 6.1, in which we prove formally that the graph in Figure 3(b) (and generalisations on it) is indeed minimal AP. In particular, this proof shows different types of arguments that can be invoked.

4 On the minimum degree of minimal AP graphs

Let us mention as a starting point that all minimal AP graphs known at this point, including both those from the literature and those exhibited in Section 3, have minimum degree 1. In particular, we do not know whether minimal AP graphs with minimum degree at least 2 exist.

Of course, graphs with very large minimum degree tend to be dense, and thus likely to be AP. Conversely, the more dense an AP graph is, the more likely it is that some edges are useless for partitioning it into connected subgraphs. We investigate this aspect by showing that if the minimum degree of an AP graph is large enough, then it cannot be minimal AP.

The next result is through some terminology that is borrowed from the realm of Hamiltonian graphs. For a graph G, we denote by $\sigma_2(G)$ the smallest value of $d(u) + d(v)$ taken over all pairs of non-adjacent vertices u and v. By a well-known theorem of Ore [26], an n-graph G is Hamiltonian whenever $\sigma_2(G) \geq n$, which implies another well-known theorem of Dirac [14] which states that G is Hamiltonian whenever $\delta(G) \geq n/2$. The result of Ore was also derived to traceability, as satisfying $\sigma_2(G) \geq n-1$ is a sufficient condition for G to be traceable, and to APness, as having $\sigma_2(G)$ being lower than $n-1$ is sometimes sufficient for G to be AP. In particular, the best result in that line, from [19], states that an n-graph G is AP provided $n \geq 20$, $\sigma_2(G) \geq n - 5$, and G admits a perfect matching (or quasi-perfect matching if n is odd).

Through the next result, we show how one such result can be adapted to minimal APness. In particular, better results could probably be obtained by refining proofs from [19] with our approach. Namely, the result we prove reads as follows:

Theorem 4.1. *If G is an AP n-graph with $\sigma_2(G) \geq n-2$, then G is not minimal AP.*

Proof. The proof follows the lines of the proof given by Marczyk in [24], which we enhanced with new arguments and tools such as Theorem 2.1. As a first remark, we observe that the condition $\sigma_2(G) \geq n - 2$ implies that $|E(G)| \geq n$, as, if G is a tree (i.e., $|E(G)| = n - 1$), then by considering two leaves u and v we would get that $d(u) + d(v) = 2$, which is strictly less than $n - 2$ unless we are dealing with a small graph that can be treated separately. In particular, this implies that, below, whenever exposing that G is traceable (thus spanned by the n-path), then G is not minimal AP.

By a classical result of Pósa [27], because $\sigma_2(G) \geq n - 2$, the graph G has a path $P = v_1 \dots v_p$ for some $p \geq n - 1$. If $p = n$, then G is traceable, and, thus, is spanned by the n-path. So, assume now that $p = n - 1$, and let u denote the sole vertex in $V(G) \setminus V(P)$. Since G must be connected, u is a neighbour of some vertices of P, say $v_{i_1}, \dots, v_{i_d}$, where $i_1 < \dots < i_d$ for $d = d(u) \geq 1$. Because G is not traceable, note that $v_1v_p \notin E(G)$, and $uv_1, uv_p \notin E(G)$. Thus, $1 < i_1$ and $i_d < p$. Also, because G is not traceable, note that there is no $j \in \{1, \dots, d-1\}$ such that $i_{j+1} = i_j + 1$. In other words, no two neighbours of u are consecutive along P.

Due to P and any edge uv_{i_j}, we have that G is spanned by a caterpillar $\mathrm{Cat}(a, b)$, that is, by a tree with a path of length $n-2$, and Theorem 2.1 implies that all n-partitions π with $|\mathrm{sp}(\pi)| \geq 2$ are realisable in G. Note that this remains true for any subgraph of G restricted to only the edges of P and only one edge uv_{i_j} incident to u. Now, consider any n-partition $\pi = (\lambda, \dots, \lambda)$ with $|\mathrm{sp}(\pi)| = 1$. Note that if there exists a $j \in \{1, \dots, d\}$ such that $i_j \not\equiv 0 \bmod \lambda$, then π is realisable in G (just pick parts of size λ along P as going from v_1 to v_p, and, when reaching a v_{i_j} with $i_j \not\equiv 0 \bmod \lambda$, just add u to the current part before going on). This implies that if there are two coprime values in $\{i_1, \dots, i_d\}$, then G is spanned by a subgraph H, containing the edges of P and two edges incident to u, which is AP. Note that H has less edges than G, since $d_H(v_1) = d_H(v_p) = 1$ while $d_G(v_1) + d_G(v_p) \geq n-2$, and thus one of $d_G(v_1)$ or $d_G(v_p)$ must be at least 2 (unless $n-2=2$, and thus $n=4$, which case can easily be treated separately). The existence of H then implies that G is not minimal AP. Thus, in what follows, we can assume that no two values in $\{i_1, \dots, i_d\}$ are coprime. So there is a λ such that all values in $\{i_1, \dots, i_d\}$ are multiples of λ.

Because u and v_1 are not adjacent, and $\sigma_2(G) \geq n-2$, we have $d(u)+d(v_1) \geq n-2$, and, thus, $d(v_1) \geq n-d-2$. Note that we cannot have $v_1v_{i_j+1} \in E(G)$ for any $j \in \{1, \dots, d\}$, as otherwise it can be observed that G would be traceable. Hence, $d(v_1) \leq n-d-2$, and we thus have $d(v_1) = n-d-2$. This means that v_1 is adjacent to every v_i with $i \in \{2, \dots, p-1\} \setminus \{i_1+1, \dots, i_d+1\}$. Now, because $v_1v_p \notin E(G)$, we have $i_d = p-1$. Since all this reasoning applies the same way to v_p, we have $d(v_p) = n-d-2$, and $i_1 = 2$. From this, we deduce that $\lambda = 2$.

Now note that for every $j \in \{2, \dots, d\}$, we have $v_1v_{i_j-1} \notin E(G)$ as otherwise G would be traceable. Since $d(u)+d(v_1) \geq n-2$, this implies that $i_{j+1} = i_j+2$ for every $j \in \{1, \dots, d-1\}$. Applying the same arguments for v_p, we deduce that $N(u) = \{v_2, v_4, \dots, v_{p-1}\} = N(v_1) = N(v_p)$. Now note that having any edge of the form $v_{2i-1}v_{2j-1}$ for some $i \neq j$ would imply that G is traceable. Then $\{u, v_1, v_3, \dots, v_p\}$ is a stable set of cardinality $(n+2)/2$ (note that n is indeed even, since $\lambda = 2$), which implies that G admits no perfect matching, thus no realisation of the n-partition $(2, \dots, 2)$. This contradicts that G is AP. $\square$

Corollary 4.2. *If G is a minimal AP n-graph, then $\delta(G) < \frac{n-2}{2}$.*

5 On the maximum degree of minimal AP graphs

Our investigations in this section stem from a starting point similar to that described at the beginning of Section 4, being that all minimal AP graphs we know of are of small maximum degree. More precisely, AP trees were shown to have maximum degree at most 4 (but infinitely many of them exist, recall Theorem 2.3), while all known minimal AP graphs (from Section 3 and the literature, and in particular from [8, 28]) that are not trees have maximum degree at most 3.

We start by pointing out the non-obvious fact that minimal AP graphs with arbitrarily large maximum degree exist; more precisely:

Observation 5.1. *There is no k such that all minimal AP graphs have maximum degree less than k.*

Proof. By Theorem 2.4, there exist AP Δ-balloons $B = \mathrm{B}(b_1, \dots, b_\Delta)$ for any arbitrarily large integer Δ, and, by arguments from [7], we can assume the b_i's are at least 1. Recall that, in B, there are two vertices r_1 and r_2 joined by Δ paths, being thus both of degree Δ. Since B is AP, it is spanned by a minimal AP graph B'. Since B' must be connected, it must be that B' was obtained from B by considering each of the Δ paths joining r_1 and r_2, and removing at most one edge of that path. In particular, for every such path, the edge that was possibly removed when going from B to B' is incident to at most one of r_1 and r_2. This means that, in B', one of r_1 and r_2 must have degree at least about $\Delta/2$, which is the maximum degree of B'. Thus, taking Δ large enough shows that any k is not an upper bound on the maximum degree of all minimal AP graphs. □

In light of Observation 5.1 (which might sound counterintuitive, as having large maximum degree for a graph does not guarantee anything on its density), we now consider the opposite direction, showing that the maximum degree of a minimal AP graph cannot be too large. Precisely, our main result stands in the line of a previous one established by Ravaux in [28], stating that a minimal AP n-graph G with $n \geq 4$ must satisfy $\Delta(G) \leq n - 2$. Namely, we prove:

Theorem 5.2. *If G is a minimal AP n-graph with $n \geq 6$, then $\Delta(G) \leq n - 3$.*

Proof. We show that an AP n-graph G with $n \geq 6$ cannot be minimal AP whenever $\Delta(G) \geq n - 2$. Since G cannot be minimal AP if $\Delta(G) \geq n - 1$ by a previous result of Ravaux [28], we can assume that $\Delta(G) = n - 2$. So there is a vertex u of G that is adjacent to all other vertices but one. We denote by $v_0, \dots, v_{n-3}$ the neighbours of u, and by w the sole vertex of G not adjacent to u. Since G has to be connected to be AP, this vertex w must be adjacent to at least one of the v_i's. Without loss of generality, we may assume that w is adjacent to $v = v_0$.

The proof reads as follows. We first show that if certain edges from a set F are present in G, then we can modify any realisation $\mathcal{R}$ of any n-partition π in G to get another realisation $\mathcal{R}'$ of π such that some edges of G belong to none of the induced connected subgraphs. In other words, we prove that if at least one edge of F is present in G, then G cannot be minimal AP. The contradiction will eventually arise from the fact that F has so many edges that G cannot be AP.

In the arguments below, for a vertex x of G and a realisation $\mathcal{R}$ of some n-partition in G, we denote by $p(x)$ the part of $\mathcal{R}$ that contains x. We consider a few successive cases; in each case, it is assumed that none of the previous cases applies.

Claim 1. *G has no edge v_iw for some $i \in \{1, \dots, n-3\}$.*

Proof of the claim. Assume, without loss of generality, that v_1w is an edge of G. We claim that uv_1 can be removed from G without breaking APness. Set $G' = G - uv_1$; note that G' is connected. Now consider π, any n-partition, and $\mathcal{R}$ a realisation of π in G. Our goal is to prove that $\mathcal{R}$ can always be derived to a realisation of π in G'.

If $p(u) \neq p(v_1)$, then note that $\mathcal{R}$ is also a realisation of π in G'. So, from now on, assume $p(u) = p(v_1)$. If $p(u) = p(v_1) = p(v) = p(w)$, then note that the subgraph $G[p(u) \setminus \{uv_1\}]$ is connected, and so $\mathcal{R}$ is a realisation of π in G'. So, we may assume that not all of u, v_1, v, and w belong to the same part of $\mathcal{R}$. We treat a few different cases in what follows.

- **First case:** $p(u) = p(v_1) = p(w) \neq p(v)$.

  If $|p(v)| = 1$, then, by replacing the parts $p(u)$ and $p(v)$ of $\mathcal{R}$ with $p(u) \setminus \{v_1\} \cup \{v\}$ and $\{v_1\}$, respectively, we obtain a realisation of π in G'. Similarly, if $|p(v)| = 2$, then, by replacing the parts $p(u)$ and $p(v)$ of $\mathcal{R}$ with $p(u) \setminus \{w, v_1\} \cup p(v)$ and $\{w, v_1\}$, respectively, we obtain a realisation of π in G'. In particular, note that $p(u) \setminus \{w, v_1\} \cup p(v)$ induces a connected subgraph since u is a neighbour of all vertices of G but w.

  Now assume that $|p(v)| \geq 3$. Let x_1 and x_2 be two distinct vertices of $p(v) \setminus \{v\}$ such that $G[p(v) \setminus \{x_1, x_2\}]$ is connected (x_1 and x_2 can be deduced by considering, e.g., two "successive leaves" of a spanning tree of $G[p(v)]$ rooted at v). In particular, these two vertices x_1 and x_2 are neighbours of u. Then, by replacing the parts $p(u)$ and $p(v)$ of $\mathcal{R}$ with $p(u) \setminus \{w, v_1\} \cup \{x_1, x_2\}$ and $p(v) \setminus \{x_1, x_2\} \cup \{w, v_1\}$, respectively, we obtain a realisation of π in G'.

- **Second case:** $p(u) = p(v_1) = p(v) \neq p(w)$.

  As previously, if $|p(w)| = 1$, then by replacing the parts $p(u)$ and $p(w)$ of $\mathcal{R}$ with $p(u) \setminus \{v_1\} \cup \{w\}$ and $\{v_1\}$, respectively, we obtain a realisation of π in G'. Now, for the general case, i.e., when $|p(w)| \geq 2$, consider a vertex $x \in p(w) \setminus \{w\}$ such that $G[p(w) \setminus \{x\}]$ is connected. Since x is clearly a neighbour of u, then, by replacing the parts $p(u)$ and $p(w)$ of $\mathcal{R}$ with $p(u) \setminus \{v_1\} \cup \{x\}$ and $p(w) \setminus \{x\} \cup \{v_1\}$, respectively, we obtain a realisation of π in G'.

- **Third case:** $p(u) = p(v_1) \neq p(v)$ and $p(u) = p(v_1) \neq p(w)$.

  If $p(v) = p(w)$, then consider a vertex $x \in p(w) \setminus \{w\}$ such that $G[p(w) \setminus \{x\}]$ is connected (in particular $x = v$ when $|p(w)| = 2$). By this choice of x, note that u and x are adjacent. Then, by replacing the parts $p(u)$ and $p(w)$ of $\mathcal{R}$ with $p(u) \setminus \{v_1\} \cup \{x\}$ and $p(w) \setminus \{x\} \cup \{v_1\}$, respectively, we obtain a realisation of π in G'.

  Now assume $p(v) \neq p(w)$. If $|p(w)| \geq 2$, then we can proceed as in an earlier case. If $|p(v)| = 1$, then by replacing the parts $p(u)$ and $p(v)$ of $\mathcal{R}$ with $p(u) \setminus \{v_1\} \cup \{v\}$ and $\{v_1\}$, respectively, we obtain a realisation of π in G'. So the last case is when $|p(w)| = 1$ and $|p(v)| \geq 2$. Here, let x be a vertex of $p(v) \setminus \{v\}$ such that $G[p(v) \setminus \{x\}]$ is connected. Clearly, we have $ux \in E(G)$.

Then, by replacing the parts $p(u)$, $p(v)$, and $p(w)$ of $\mathcal{R}$ with $p(u) \setminus \{v_1\} \cup \{x\}$, $p(v) \setminus \{x\} \cup \{w\}$, and $\{v_1\}$, respectively, we obtain a realisation of π in G'.

Thus, if $v_i w \in E(G)$ for any $i \in \{1, \dots, n-3\}$, then G' is AP, and G is not minimal AP. $\diamond$

Claim 2. *G has no edge vv_i with $i \in \{1, \dots, n-3\}$.*

Proof of the claim. According to Claim 1, we can assume that $N(w) = \{v\}$ from now on. Assume vv_1 belongs to G without loss of generality. We prove below that, under that assumption, $G' = G - uv$ is AP. Note that G' is connected. Consider any n-partition π, and any realisation $\mathcal{R}$ of π in G. Again, we aim at proving that $\mathcal{R}$ can be derived to a realisation of π in G'.

Clearly, if $p(u) \neq p(v)$, then $\mathcal{R}$ holds directly as a realisation of π in G'. The same clearly holds if $p(u) = p(v) = p(v_1)$. So, assume that $p(u) = p(v) \neq p(v_1)$. We split the analysis into three cases.

- **First case:** $|p(v_1)| = 1$.

  By our assumptions, recall that either $|p(w)| = 1$ or $p(w) = p(v)$ holds throughout. In the first case, by replacing the parts $p(u)$ and $p(v_1)$ of $\mathcal{R}$ with $p(u) \setminus \{v\} \cup \{v_1\}$ and $\{v\}$, respectively, we obtain a realisation of π in G'. In the second case, by replacing $p(u)$ and $p(v_1)$ with $p(u) \setminus \{w\} \cup \{v_1\}$ and $\{w\}$, respectively, we fall into one of the easy cases we have treated at the very beginning of the proof of this claim.

- **Second case:** $|p(v_1)| = 2$.

  Assume $p(v_1) = \{v_1, v_2\}$ without loss of generality. If $|p(w)| = 1$, then by replacing the parts $p(u)$, $p(v_1)$, and $p(w)$ of $\mathcal{R}$ with $p(u) \setminus \{v\} \cup \{v_2\}$, $\{v, w\}$, and $\{v_1\}$, respectively, we obtain a realisation of π in G'. Now, if $p(w) = p(v)$, then by replacing $p(u)$ and $p(v_1)$ with $p(u) \setminus \{v, w\} \cup \{v_1, v_2\}$ and $\{v, w\}$, respectively, we obtain a realisation of π in G'.

- **Third case:** $|p(v_1)| \geq 3$.

  If we have $|p(w)| = 1$, then let x be a vertex from $p(v_1) \setminus \{v_1\}$ such that $G[p(v_1) \setminus \{x\}]$ is connected. Then, by replacing the parts $p(u)$ and $p(v_1)$ of $\mathcal{R}$ with $p(u) \setminus \{v\} \cup \{x\}$ and $p(v_1) \setminus \{x\} \cup \{v\}$, respectively, we obtain a realisation of π in G'. In particular, $p(u) \setminus \{v\} \cup \{x\}$ induces a connected subgraph in G since u and x are adjacent. Now assume $p(w) = p(v)$. Then let x_1 and x_2 be two vertices of $p(v_1) \setminus \{v_1\}$ such that $G[p(v_1) \setminus \{x_1, x_2\}]$ is connected. By now replacing the parts $p(u)$ and $p(v_1)$ of $\mathcal{R}$ with $p(u) \setminus \{v, w\} \cup \{x_1, x_2\}$ and $p(v_1) \setminus \{x_1, x_2\} \cup \{v, w\}$, respectively, we obtain a realisation of π in G'.

Thus, G' is AP, meaning that G is not minimal AP. $\diamond$

Claim 3. *G has no edge $v_i v_j$ with $i, j \in \{1, \dots, n-3\}$.*

Proof of the claim. Since Claims 1 and 2 do not apply, we have $N(w) = \{v\}$ and $N(v) = \{u, w\}$. Assume v_1v_2 is an edge of G without loss of generality, and set $G' = G - uv_1$. We prove that G', which is connected, is AP. For that, let π be any n-partition and $\mathcal{R}$ be any realisation of π in G. Our goal is to prove that, from $\mathcal{R}$, we can deduce a realisation of π in G'.

For similar reasons as previously, if $p(u) \neq p(v_1)$, then $\mathcal{R}$ also forms a realisation of π in G'. Similarly, if $p(u) = p(v_1) = p(v_2)$, then the subgraph $G[p(u) \setminus \{uv_1\}]$ is connected, and so, again, $\mathcal{R}$ is a realisation of π in G'. So, now, assume $p(u) = p(v_1) \neq p(v_2)$. We start by choosing a vertex $x \in p(v_2)$ as follows:

- if $|p(v_2)| = 1$, then set $x = v_2$;
- otherwise, choose, as x, a vertex in $p(v_2) \setminus \{v_2\}$ such that $G[p(v_2) \setminus \{x\}]$ is connected.

By our assumptions, we have $ux \in E(G)$. Now, since $N(v_2) \cup \{v_2\} \subset N(u)$, then note that by replacing the parts $p(u)$ and $p(v_2)$ of $\mathcal{R}$ with $p(u) \setminus \{v_1\} \cup \{x\}$ and $p(v_2) \setminus \{x\} \cup \{v_1\}$, respectively, we obtain a realisation of π in G'.

From these arguments, we deduce that G' is AP, and thus that G is not minimal AP. $\diamond$

We now conclude the proof. Due to Claims 1 to 3, we have $E(G) = \{vw, uv_0, \dots, uv_{n-3}\}$. Thus, G is isomorphic to a k-pode $\mathrm{P}_k(1, \dots, 1, 2)$ with $k \geq 4$ since $n \geq 6$. But such a tree is not AP since it admits no perfect matching (if n is even) or no quasi-perfect matching (otherwise). Thus, G is not AP, a contradiction. So G cannot be minimal AP when $\Delta(G) = n - 2$. □

We note that the requirement on n in the statement of Theorem 5.2 cannot be lowered, as the tripode $\mathrm{P}_3(1, 1, 2)$ has order 5, maximum degree 3, and is (minimal) AP.

6 On the clique number of minimal AP graphs

As all known minimal AP graphs are quite sparse, one can legitimately wonder whether they can contain dense structures, such as large cliques. Prior to the current work, it was actually unknown whether minimal AP graphs can contain triangles. Some of the small minimal AP graphs exhibited in Section 3 are thus the very first examples of minimal AP graphs with clique number at least 3. Actually, we can generalise some of these graphs to show they are not that exceptional.

Theorem 6.1. *There exist arbitrarily large minimal AP graphs with clique number* 3.

Proof. We generalise the minimal AP 10-graph Q depicted in Figure 3(b). Note that Q can be described as having a main triangle $uvwu$ such that a pending path

$P_u = uu_1u_2$ of length 2 is attached at u, a pending path $P_v = vv_1$ of length 1 is attached at v, and a pending path $P_w = ww_1w_2w_3w_4$ of length 4 is attached at w. The generalisation of Q we consider, denoted by Q_k for any $k \geq 4$, is obtained when replacing P_w with a pending path of length k. Thus, $Q = Q_4$. Note that for any $k \geq 4$, we have $|V(Q_k)| = 6 + k$.

The claim follows from the fact that Q_k is minimal AP for particular values of k. Precisely, note first that Q_k is AP whenever $|V(Q_k)| \not\equiv 0 \bmod 4$. Indeed, note that $Q_k - v_1$, $Q_k - \{u_1, u_2\}$, and $Q_k - \{u_1, u_2, u\}$ are traceable, while $Q_k - \{u_1, u_2, v_1, u, v\}$ has a Hamiltonian path with one end (w), being a neighbour of u and v. From this, we deduce that any $(6+k)$-partition containing an element different from 4 is realisable in Q_k. Actually, it is easily seen that when $6 + k \equiv 0 \bmod 4$, the $(6 + k)$-partition $(4, \dots, 4)$ is not realisable in Q_k.

Now consider any $k \geq 4$ fulfilling the following:

1. $6 + k \equiv 0 \bmod 2$;
2. $6 + k \equiv 0 \bmod 5$;
3. $6 + k \equiv 1 \bmod 3$; and
4. $6 + k \not\equiv 0 \bmod 4$.

Note that suitable values for $6 + k$ are $10, 70, 130$, and so on. For any corresponding value of k, previous arguments imply that Q_k is AP (since the fourth item above applies). To see now that Q_k is minimal AP for such values of k, consider the following arguments:

1. Since the first item above applies, $\pi = (2, \dots, 2)$ is a $(6 + k)$-partition. Note that there is a unique way to partition Q_k following π. In particular, $\{u_1, u_2\}$, $\{v_1, v\}$, and $\{u, w\}$ must be parts, which shows that removing uw from Q_k results in a graph that is not AP.
2. Since the second item above applies, $\pi = (5, \dots, 5)$ is a $(6 + k)$-partition. Note that there is a unique way to partition Q_k following π. In particular, $\{u_1, u_2, u, v, v_1\}$ must be a part, which shows that removing uv from Q_k results in a graph that is not AP.
3. Since the third item above applies, $\pi = (4, 3, \dots, 3)$ is a $(6+k)$-partition. Note that in any realisation of π in Q_k, the set $\{u_1, u_2, u\}$ must be a part. From this, we deduce that another part must contain both v and w, the edge vw being necessary for that part to induce a connected subgraph. Thus, removing vw from Q_k results in a graph that is not AP.

Since removing any other edge from Q_k results in a non-connected graph, we deduce that Q_k is minimal AP. The claim is thus proved. $\square$

Conversely, in the next result we prove that minimal AP graphs cannot have their clique number being too large. Recall that, for a graph G, we denote by $\omega(G)$ its clique number.

Theorem 6.2. *If G is a minimal AP n-graph with $n \geq 9$, then $\omega(G) \leq n-4$.*

Proof. Let us denote by K the largest clique of G. Because G must be connected to be AP, there must be an edge from a vertex in K to a vertex in $V(G) \setminus K$. If $|K| \geq n-2$, then note that this implies that $\Delta(G) \geq n-2$, and thus G cannot be minimal AP by Theorem 5.2. So assume $|K| = n-3$, and let $V(G) \setminus K = \{u, v, w\}$. Again, if a vertex in K is a neighbour of at least two vertices in $\{u, v, w\}$, then $\Delta(G) \geq n-2$ and, once more, G is not minimal AP by Theorem 5.2. Hence, we can assume every vertex in K has at most one neighbour in $\{u, v, w\}$. Also $|K| \geq 6$ since $n \geq 9$.

Consider the longest path in $G[\{u, v, w\}]$. Assume first that, say, uvw is a path. Note that if u or w has a neighbour in K, then G is traceable, and thus G is not minimal AP. Note that the same conclusion can be reached if $uw \in E(G)$. So v must be the only vertex in $\{u, v, w\}$ with neighbours in K, and v is the only neighbour of both u and w. Note then that G is spanned by a caterpillar $\text{Cat}(2, n-2)$ where v is the unique degree-3 vertex. If n is odd, then, by Corollary 2.2, this caterpillar is AP and thus G is not minimal AP. Now, if n is even, then it can be noted that G admits no perfect matching, thus no realisation of the n-partition $(2, \dots, 2)$, and thus G is not AP, a contradiction. More precisely, this is because v is the sole neighbour of both u and w.

If, say, uv is the longest path of $G[\{u, v, w\}]$, then w has all its neighbours in K. Also, since G is connected, one of u and v must have neighbours in K. Assume v has neighbours in K. Since v and w cannot have common neighbours in K (by the maximum degree assumption), we get that G is traceable (in particular, there is a Hamiltonian path starting from uv, then going through all vertices of K, and finishing in w), and thus G is not minimal AP.

The last case is when $\{u, v, w\}$ is a stable set. Recall that, again, any vertex in $\{u, v, w\}$ must have neighbours in K so that G is connected (and thus AP), and no two of these vertices can share neighbours in K (due to the maximum degree assumption). We can thus choose three pairwise distinct vertices $u', v', w' \in K$ such that $uu', vv', ww' \in E(G)$. Since $n \geq 9$, there are also two vertices $x, y \in K \setminus \{u', v', w'\}$. Set $G' = G - xy$. We claim that G' is AP, a contradiction to the fact that G is minimal AP. To prove that, consider any realisation $\mathcal{R}$ of any n-partition π in G. Our goal is to show that $\mathcal{R}$ can, if needed, be modified to reach a realisation of π that holds in G'. As in the proof of Theorem 5.2, for a vertex z we denote by $p(z)$ the part of $\mathcal{R}$ that contains z.

If $p(x) \neq p(y)$, then $\mathcal{R}$ also holds in G'. So now assume that $p(x) = p(y)$. We claim first that we can modify $\mathcal{R}$, if needed, so that if $|p(u)| > 1$ then $p(u) = p(u')$, and similarly if $|p(v)| > 1$ then $p(v) = p(v')$, and if $|p(w)| > 1$ then $p(w) = p(w')$. Let us show this for u and u', that is, assume $|p(u)| > 1$ but $p(u) \neq p(u')$. Since u' cannot be a neighbour of any of v and w, either $|p(u')| = 1$ or $p(u')$ contains another vertex $z \in K$. In the first case, we can replace the parts $p(u')$ and $p(u)$ with $\{u\}$ and $p(u) \setminus \{u\} \cup \{u'\}$ (which induces a connected subgraph, since every neighbour of u lies in K and is thus adjacent to u'), respectively. In the second case, we can replace the parts $p(u)$ and $p(u')$ with $p(u) \setminus \{u\} \cup \{z\}$ and $p(u') \setminus \{z\} \cup \{u\}$, respectively. In both cases, it can be checked that another realisation of π in G results, and it verifies that u and u' always belong to the same part, provided u belongs to a part of size at least 2. The same holds for v and v', and w and w'.

So, from now on, we can assume that, by $\mathcal{R}$, if $|p(u)| > 1$ then $p(u) = p(u')$, if $|p(v)| > 1$ then $p(v) = p(v')$, and if $|p(w)| > 1$ then $p(w) = p(w')$. Since $x, y \notin \{u', v', w'\}$, note that if $|p(x)| \geq 3$, then $p(x)$ contains a vertex $z \in K \setminus \{x, y\}$ (which can lie in $\{u', v', w'\}$), and the presence of z in $p(x)$ implies that $\mathcal{R}$ also holds in G' (since zy is also an edge of G). Thus, the last case to consider is when $p(x) = \{x, y\}$. Since $n \geq 9$, there must a vertex $z \in K \setminus \{u', v', w', x, y\}$. If $|p(z)| = 1$, then by replacing $p(x)$ and $p(z)$ with $\{z, x\}$ and $\{y\}$, respectively, we get a realisation of π that holds in G'. So $|p(z)| \geq 2$. Then $p(z)$ contains a vertex in $K \setminus \{x, y, z\}$ (which can lie in $\{u', v', w'\}$), and thus, by replacing the parts $p(x)$ and $p(z)$ with $\{z, x\}$ and $p(z) \setminus \{z\} \cup \{y\}$, respectively, we reach the same conclusion as in the previous case. So, in all cases, we deduce a realisation of π in G', meaning that G' is AP, and thus that G is not minimal AP. $\square$

7 Directions for further work

Our main goal in this work was to pursue the line of research initiated by Ravaux and Baudon, Przybyło, and Woźniak in [8, 28] on minimal AP graphs. We chose to investigate basic properties of minimal AP graphs, and, in particular, whether some of their basic properties can reach extreme values.

Our guiding line in this work relates mainly to Conjecture 1.1 by Ravaux, and to the work [8] of Baudon, Przybyło, and Woźniak on the density of minimal AP graphs. In particular, our results from Sections 4 (on the minimum degree) and 6 (on the clique number) relate directly to the density of these graphs. We also considered a few other aspects, such as the minimum order of non-tree minimal AP graphs (which we determined is 9, in Section 3) and the maximum degree of minimal AP graphs (improving upon a previous result of Ravaux, in Section 5).

All our results and observations in this work open the way to more questions on minimal AP graphs. In particular, we can raise the following ones:

- Regarding our observations on small minimal AP graphs in Section 3, we note that they all share a certain number of properties. In particular, they are all of maximum degree at most 3, of minimum degree 1, they all have cut vertices, at most one cycle, etc. One can thus wonder about the existence and the minimum order of a hypothetical minimal AP graph not having one of these properties. For instance, recall that, by Theorem 2.3, the 4-pode $\mathrm{P}_4(1, 1, 4, 6)$ is minimal AP, has maximum degree 4, order 13, and one can wonder whether this is the smallest minimal AP graph with maximum degree at least 4 (recall that the maximum degree of a minimal AP graph can indeed be arbitrarily large, by Observation 5.1). One could also wonder how the number of minimal AP n-graphs grows with respect to n. Recall that we observed that this number is 1, 1, 2, 1, 4, 2, 6, 16 for $n = 3, \dots, 10$, respectively. From this, one could also question whether there are values of n that are more favourable for many minimal AP n-graphs to exist. For instance, having no perfect matching being one of the main obstructions to APness, one could suspect that, perhaps, odd values of n are more favourable. Also, this restricted sample shows that there are small values of n for which all minimal AP n-graphs are trees. Again, we wonder whether this can be somewhat generalised for larger values of n. A last question could concern the smallest minimal AP graphs having cycles sharing vertices. Note that all minimal AP graphs exhibited in Section 3 have at most one cycle, but, by a remark from [8], minimal AP graphs having intersecting cycles do exist, which can be proved by refining the arguments we used to prove Observation 5.1.

- Regarding our results from Section 4, any improvement over our upper bound in Corollary 4.2 would, of course, be interesting by itself. It would definitely be more interesting, though, to determine whether there exist minimal AP graphs with minimum degree at least 2. In any case, the larger the minimum degree of an AP graph is, the larger its density is, and the more likely it is that it is not minimal AP. Thus, a good step towards Conjecture 1.1 would be to determine whether minimal AP graphs G satisfy $\delta(G) < k$ for some absolute constant k. This question could also be investigated through the lens of other graph properties influencing the minimum degree. In particular, all minimal AP graphs we know of have cut vertices. A question is then whether there exist minimal AP graphs with arbitrarily large connectivity.

- Regarding our main result, Theorem 5.2, in Section 5, we feel this could be generalised to a more general result, stating, for instance, that, perhaps, for every $k \geq 1$ there is a function $f(k)$ such that minimal AP n-graphs with order $n \geq f(k)$ have maximum degree at most $n - k$. This could be a plausible hypothesis as such a condition would imply that the graph has a vertex that is very close to universal (a negligible fraction of the vertices apart), and thus very helpful to modify partitions, just like we did in the proof of Theorem 5.2.

- Lastly, regarding our results from Section 6, at the moment we know that the maximum clique number of a minimal AP graph is at least 3 (recall Theorem 6.1), and we wonder whether minimal AP graphs can contain cliques on more vertices. Theorem 6.2 could also be subject to further improvement. More generally speaking, the existence of dense structures in minimal AP graphs relates directly to Conjecture 1.1 and to the density questions raised in [8]. One could wonder whether, besides cliques, minimal AP graphs can contain somewhat dense structures. A way to investigate this could be by wondering about the maximum average degree (mad) of minimal AP graphs.

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