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Doubly stochastic arrays with small support

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Abstract

An $n \times m$ non-negative array with row sum m and column sum n is called doubly stochastic. We answer the problem of finding doubly stochastic arrays of smallest possible support for every $1 < n \leq m$. Any array of minimum support is extremal in the sense of convexity, while examples of extremal arrays that are not of minimum support are given. But when n, m are coprime integers, extremal arrays are precisely those of minimum support.

1 Introduction

According to the definition given by Caron et al. in [5], an $n \times m$ array $A = (a_{i,j})$ with $a_{i,j} \geq 0$ is called doubly stochastic (with uniform marginals) if

$$\sum_{i=1}^{n} a_{i,j} = n \quad \text{for all } j = 1, \ldots, m;$$
$$\sum_{j=1}^{m} a_{i,j} = m \quad \text{for all } i = 1, \ldots, n.$$

The set of all $n \times m$ doubly stochastic arrays is denoted by $\mathcal{M}(n, m)$. Furthermore, two arrays in $\mathcal{M}(n, m)$ are called *equivalent* if one can be transformed into the other by permuting rows and columns.

We should mention here that the above definition differs slightly from the usual definition for square doubly stochastic arrays (matrices). The common definition for $\mathcal{M}(n, n)$ requires the matrices to have nonnegative entries and all row and column sums equal to 1. These matrices have been studied extensively; see for example [11], Chapter 2.

An array $M \in \mathcal{M}(n, m)$ is called extremal if it cannot be represented as a convex combination of other doubly stochastic arrays different from M, that is, M is an

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extremal element in the convex set $\mathcal{M}(n,m)$. For square $n \times n$ matrices, a full characterization of the extremal matrices in $\mathcal{M}(n,n)$ is known by a classical result due to Birkhoff [1], that we state here, using the notation of [5] that we have adopted.

Birkhoff's Theorem: $M \in \mathcal{M}(n,n)$ is extremal if and only if $\frac{1}{n}M$ is a permutation matrix. That is, $M \in \mathcal{M}(n,n)$ is extremal if and only if $\frac{1}{n}M$ is equivalent to I_n, the identity matrix.

Several types of characterization of the extremal doubly stochastic arrays in $\mathcal{M}(n,m)$ exist using either a matrix representation in some normal form, graph theory or faces of polyhedra, just to mention a few. (The interested reader could look at the list presented in the introduction of [5]). We point out here that if $M \in \mathcal{M}(n,m)$ is extremal, then all its entries are integers — see the first remarks in [5].

Li et al. in [12] have characterized extremal arrays using their support, that is, the set of their nonzero entries. In particular, they proved that an array $M \in \mathcal{M}(n,m)$ is extremal if and only if its support $\operatorname{supp}(M)$ is unique in the set $\{\operatorname{supp}(A) \mid A \in \mathcal{M}(n,m)\}$ (Theorem 1 in [12]).

In addition, the support of a doubly stochastic array has attracted the attention of Kolountzakis and Papageorgiou [9] in relation with some tiling problems. If one views an $n \times m$ array as a function f on the product of cyclic groups

$$G = \mathbb{Z}_n \times \mathbb{Z}_m$$

then, with the subgroups

$$G_1 = \mathbb{Z}_n \times \{0\}, \quad G_2 = \{0\} \times \mathbb{Z}_m,$$

the constant row sum and the constant column sum properties of the array are written as

$$\sum_{g \in G_2} f(x-g) = m, \quad \sum_{g \in G_1} f(x-g) = n, \tag{1.1}$$

respectively, valid for all $x \in G$. In this language one seeks a nonnegative function f on G, of as small a support as possible, which *tiles* simultaneously with the set of translates G_1 as well as G_2 (see [9] for a more precise definition).

These problems, of tiling simultaneously with various subgroups, derive [7] from a classical problem of Steinhaus who asked if there is a subset of the plane which tiles the plane simultaneously with all rotates of the lattice $\mathbb{Z}^2$. This problem is still very much open in case one asks for a measurable subset of the plane [10] but the answer is known to be affirmative without the measurability requirement [6]. Interestingly, in dimension 3 and higher the situation is the exact opposite: no measurable Steinhaus sets exist [10, 8] but we do not know if such sets exist if we drop measurability [6]. In [10] the problem was first investigated of how to find a *function* f (as opposed to indicator function for Steinhaus sets) on the plane which tiles simultaneously with a finite set of rotates of $\mathbb{Z}^2$ and whose support has small *diameter*. This problem was continued in [9] by examining the problem in a more general finite abelian group setting, the prototype of which is to ask for a function f on G satisfying (1.1) and has small support.

In [9] the quantity $S(n, m)$ was defined as follows.

Definition 1.1. $S(n, m) = \min\{|\operatorname{supp} A| : A \in \mathcal{M}(n, m)\}$.

The arrays $M \in \mathcal{M}(n, m)$ with $|\operatorname{supp} M| = S(n, m)$ are called **minimum** arrays in $\mathcal{M}(n, m)$. Furthermore, we call a column of $A \in \mathcal{M}(n, m)$ a **monocolumn** if it contains exactly one non-zero entry, which obviously should equal n.

It was shown (see Theorem 4.3 and Lemma 4.5 in [9]) that $S(n, kn) = kn$ while $S(n, kn + 1) = (k + 1)n$. In addition, a question has been raised about the value of $S(n, kn + r)$ for $1 < r < n$. Our main theorem in this note gives a complete answer to Question 7 in [9] and states the following.

Theorem I. *For all integers $n, m > 1$, we have $S(n, m) = n + m - \gcd(n, m)$.*

According to Corollary 2 in [12] an array $A \in \mathcal{M}(n, m)$ is not extremal if and only if there exists $B \in \mathcal{M}(n, m)$ with $\operatorname{supp} B \subsetneq \operatorname{supp} A$. Hence every minimum array in $\mathcal{M}(n, m)$ is also extremal. This gives an easy way to verify that an array is extremal just by looking at the size of its support, if this happens to be minimum. But there are extremal arrays that are not minimum (some examples are given at the end of this note) so the condition on the size of the support is only sufficient. Nevertheless, when n, m are coprime integers it is also necessary as the next result states.

Theorem II. *Let n, m be coprime integers. Then $M \in \mathcal{M}(n, m)$ is extremal if and only if M is minimum. That is, M is extremal if and only if*

$$|\operatorname{supp} M| = n + m - 1.$$

The rest of the paper contains a method to construct minimum arrays in $\mathcal{M}(n, m)$. In addition, a family of examples of extremal arrays whose size of support is one more than the minimum is constructed. Finally, a few more examples of arrays are given as counterexamples to possible generalizations.

2 Main Results

We start with a method to produce doubly stochastic arrays of size $n \times m$ for all integers $1 \leq n \leq m$ that, as we will see, are minimum.

It is already known (see Proposition 4 in [5]), that in the case $m = kn$ the array $E(n, kn) \in \mathcal{M}(n, kn)$ defined as

$$E(n,kn) = \underbrace{\begin{pmatrix} \overbrace{n\cdots n}^{k} & \cdots & \cdots \\ \cdots & \overbrace{n\cdots n}^{k} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \overbrace{n\cdots n}^{k} \end{pmatrix}}_{kn \text{ columns}} \Bigg\} n \text{ rows} \tag{2.1}$$

is an extremal array of size $n \times kn$. Furthermore, $E(n,kn)$ is minimum since it has exactly one element per column. So, $|\operatorname{supp}(E(n,kn))| = kn = S(n,kn)$.

Assume now that $n, m \in \mathbb{N}$ are given with $1 < n \leq m$. We use the Euclidean algorithm applied to n, m to produce as many extremal arrays of type (2.1) as the steps of the algorithm. That is, assume that the Euclidean algorithm goes as follows:

$$\begin{aligned} m &= k_1 n + r_1 \\ n &= k_2 r_1 + r_2 \\ r_1 &= k_3 r_2 + r_3 \\ &\vdots \\ r_{t-2} &= k_t r_{t-1} + r_t \\ r_{t-1} &= k_{t+1} r_t. \end{aligned} \tag{2.2}$$

Then at every step we produce the arrays $E(n,k_1n), E(r_1,k_2r_1), \cdots, E(r_t,k_{t+1}r_t)$. We put them together in a block form to make an $n \times m$ array $\mathbb{F}(n,m)$ as follows .

$$\mathbb{F}(n,m) = \left(\begin{array}{c:c:c:c} & \multicolumn{3}{c}{E(r_1,k_2r_1)^T} \\ \hdashline E(n,k_1n) & & \multicolumn{2}{c}{E(r_3,k_4r_3)^T} \\ & E(r_2,k_3r_2) & \multicolumn{2}{c}{} \\ \cdashline{3-4} & & E(r_4,k_5r_4) & \ddots \\ & & & B \end{array}\right)$$

where $B = \begin{cases} E(r_t,k_{t+1}r_t)^T & \text{if } t \text{ is odd} \\ E(r_t,k_{t+1}r_t) & \text{if } t \text{ is even.} \end{cases}$

To clarify our method we compute $\mathbb{F}(8,27)$. The Euclidean Algorithm for $(8,27)$

is

$$\begin{aligned} 27 &= 3 \cdot 8 + 3 \\ 8 &= 2 \cdot 3 + 2 \\ 3 &= 1 \cdot 2 + 1 \\ 2 &= 2 \cdot 1. \end{aligned}$$

Hence we form the arrays

$$E(8, 3 \cdot 8) = \left.\underbrace{\begin{pmatrix} \overbrace{8 \cdots 8}^{3} & \cdots & \cdots \\ \cdots & \overbrace{8 \cdots 8}^{3} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \overbrace{8 \cdots 8}^{3} \end{pmatrix}}_{24} \right\} 8$$

$$E(3, 2 \cdot 3)^T = \begin{pmatrix} 3 & & \\ 3 & & \\ & 3 & \\ & 3 & \\ & & 3 \\ & & 3 \end{pmatrix}, \qquad E(2, 1 \cdot 2) = \begin{pmatrix} 2 & \\ & 2 \end{pmatrix}, \qquad E(1, 2 \cdot 1)^T = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Putting them together we get

$$\mathbb{F}(8, 27) =$$

$$\begin{pmatrix} 8 & 8 & 8 & & & & & & & & & & & & & 3 & & \\ & & & 8 & 8 & 8 & & & & & & & & & & 3 & & \\ & & & & & & \cdots & & & & & & & & & & 3 & \\ & & & & & & & \cdots & & & & & & & & & 3 & \\ & & & & & & & & \cdots & & & & & & & & & 3 \\ & & & & & & & & & \cdots & & & & & & & & 3 \\ & & & & & & & & & & 8 & 8 & 8 & & & 2 & & 1 \\ & & & & & & & & & & & & & 8 & 8 \; 8 & & 2 & 1 \end{pmatrix}.$$

In order to prove Theorem 1 we need the proposition that follows. Its proof uses a characterization of extremal arrays according to which an array $A \in \mathcal{M}(n, m)$ is extremal if and only if there is no "cycle" in its support. (See Proposition 2 in [5] or [2] and [3] for an approach using graph theory. This approach is explained in detail later in the paper.)

Proposition 2.1. *Let $m = kn + r$ with $0 < r < n$. Then there exists a minimum array $A \in \mathcal{M}(n, m)$ with exactly kn monocolumns. In other words, there exists a minimum array A so that every row of A has exactly k entries equal to n.*

Proof. Assume the proposition does not hold for $\mathcal{M}(n, m)$. Let p be the maximum number of monocolumns a minimum array in $\mathcal{M}(n, m)$ can have; hence $p < kn$. We define

$$X := \{A \in \mathcal{M}(n, m) \mid A \text{ is minimum and has exactly } p \text{ monocolumns}\}.$$

Clearly every $A \in X$ contains at least one row that does not have k entries equal to n (or else $p = kn$). Among the entries of those rows (the rows that contain less than k entries equal to n), we write $m(A)$ for the maximum entry strictly less than n. Let

$$x := \max\{m(A) \mid A \in X\}$$

and assume $A_0 \in X$ is such that $m(A_0) = x$. i Clearly $0 < x < n$ and we assume that x is in the (i_0, j_0) entry of the matrix A_0. Then looking at the j_0 column of A_0 we deduce that there exist positive integers $t_1, \dots, t_l$ in the j_0 column of A_0 apart from x such that $0 < t_1, \dots, t_l < n$ while

$$\sum_{i=1}^{l} t_i + x = n.$$

Similarly looking at the i_0 row of A_0 we conclude that there exist positive integers $s_1, \dots s_u$ in the i_0-row of A_0 apart from x such that $0 < s_1, \dots, s_u < n$, while

$$\sum_{j=1}^{u} s_j + x \geq n + r$$

where the last inequality follows from the fact that at the i_0-row of A_0 there exist less than k entries equal to n, while the sum of all the elements of the row equals $m = kn + r$. Clearly $l, u > 0$ while $\sum t_i < \sum s_j$.

Case 1 Assume first that $t_1 \leq s_j$ for some $j = 1, \dots, u$.

If t_1 is in (a_1, j_0) position of A_0 and s_j is in the (i_0, b_j) one, observe that the entry in the (a_1, b_j) position of A_0 equals 0 because otherwise the entries

$$(a_1, j_0), (i_0, j_0), (i_0, b_j), (a_1, b_j)$$

form a non-zero "square" in A_0, contradicting the fact that A_0 is minimum and thus extremal; hence no cycle is allowed in its support.

Now we construct a matrix B from A_0 in the following way. Every entry of B is identical with the corresponding entry of A apart from the four entries lying in the positions $(a_1, j_0), (i_0, j_0), (i_0, b_j), (a_1, b_j)$. In those positions the entries of $A_)$ were $t_1, x, s_j, 0$ (in the order they appear) and we replace them with the entries $0, x + t_1, s_j - t_1, t_1$ respectively. That is,

$$A_0 = \begin{pmatrix} t_1 & 0 \\ x & s_j \end{pmatrix} \longrightarrow B = \begin{pmatrix} 0 & t_1 \\ x + t_1 & s_j - t_1 \end{pmatrix}.$$

Clearly $B \in \mathcal{M}(n,m)$ (the row and column sums have remained unchanged). Furthermore, $|\operatorname{supp} B| \leq |\operatorname{supp} A_0|$ and thus they are equal as A_0 is minimum. Hence B is also minimum. In addition, the monocolumns of A_0 have been transferred unchanged to monocolumns of B (as $s_j < n$). Hence the number of monocolumns of B cannot be less than the number of monocolumns of A_0 and thus it is exactly p (p being maximum). We conclude that $x + t_1 < n$ while $B \in X$. Now, the i_0-row of B has less than k entries equal to n (actually it is the same number as the one in the i_0-row of A_0) and in position (i_0, j_0) its entry is $x + t_1 > x$. Hence

$$m(B) \geq x + t_1 > x = \max\{m(A) \mid A \in X\}.$$

This final contradiction finishes Case 1.

Case 2 Assume now that $t_1 > s_j$ for all $j = 1, \ldots, u$.

Assume again that t_1 is in (a_1, j_0) position of A_0 while s_1 is in (i_0, b_1) and observe (as in Case 1) that the entry in the (a_1, b_1) position of A_0 equals 0.

Now we construct a matrix B from A_0 in a similar way as in Case 1. That is, every entry of B is identical with the corresponding entry of A apart from the four entries lying in the positions $(a_1, j_0), (i_0, j_0), (i_0, b_1), (a_1, b_1)$. In those positions the entries of A_0 were $t_1, x, s_1, 0$ (in the order they appear) and we replace them with the entries $t_1 - s_1, x + s_1, 0, s_1$ respectively. That is,

$$A_0 = \begin{pmatrix} & & \\ & t_1 & 0 \\ & & \\ & x & s_1 \\ & & \end{pmatrix} \longrightarrow B = \begin{pmatrix} & & \\ & t_1 - s_1 & s_1 \\ & & \\ & x + s_1 & 0 \\ & & \end{pmatrix}.$$

A similar argument as in Case 1 implies that $B \in X$ while

$$m(B) \geq x + s_1 > x = \max\{m(A) \mid A \in X\}.$$

This final contradiction finishes Case 2 and completes the proof of the proposition. □

An immediate consequence of Proposition 2.1 is the following two corollaries.

Corollary 2.2. *Let $m = kn + r$ with $0 < r < n$. Then there exists a minimum array in $\mathcal{M}(n,m)$ that is equivalent to*

$$B = \left(\begin{array}{c|c} E(n,kn) & \hat{B}^T \end{array} \right)$$

where $\hat{B} \in \mathcal{M}(r,n)$.

Corollary 2.3. *Let $m = kn + r$ with $0 < r < n$. Assume $A = \left(\begin{array}{c|c} E(n,kn) & \hat{A}^T \end{array} \right)$ where $\hat{A} \in M(r,n)$. If $\hat{A}$ is minimum in $\mathcal{M}(r,n)$ then A is minimum in $M(n,m)$.*

Proof. According to Corollary 2.2 there exists a minimum array $B \in \mathcal{M}(n,m)$ so that $B = \left(\begin{array}{c|c} E(n,kn) & \hat{B}^T \end{array} \right)$ with $\hat{B} \in \mathcal{M}(r,n)$. Hence $|\operatorname{supp} \hat{A}| \leq |\operatorname{supp} \hat{B}|$ as $\hat{A}$ is minimum in $\mathcal{M}(r,n)$. Hence

$$|\operatorname{supp} B| = kn + |\operatorname{supp} \hat{B}| \geq kn + |\operatorname{supp} \hat{A}| = |\operatorname{supp} A|$$

and thus $|\operatorname{supp} A| = |\operatorname{supp} B|$ and A is minimum. $\square$

We are now ready to prove Theorem I that we restate using the arrays $\mathbb{F}(n,m)$.

Lemma 2.4. *The arrays $\mathbb{F}(n,m)$ are minimum and thus extremal in $\mathcal{M}(n,m)$. In addition,*

$$S(n,m) = |\operatorname{supp}\, \mathbb{F}(n,m)| = n + m - \gcd(n,m).$$

Proof. As we have already observed, every minimum array is also extremal. To show that $\mathbb{F}(n,m)$ is minimum we induct on the number of steps needed to complete the Euclidean Algorithm. Note that in view of our notation above, this number is $t+1$. If $t=0$, that is, $m = kn$, then the array $E(n,m)$ is of minimum support. So our induction begins.

For the inductive step, observe that if the Euclidean algorithm starts with $m = k_1 n + r_1$ then our construction guarantees that $\mathbb{F}(n,m)$ is the sum of the following two arrays, whose blocks are associated with the same column partition:

$$A = \left(\begin{array}{c|c} E(n,k_1 n) & \mathbf{0} \end{array} \right) \text{ and } B = \left(\begin{array}{c|c} \mathbf{0} & \mathbb{F}(r_1,n)^T \end{array} \right).$$

As $\mathbb{F}(n,m) = A + B$, its first $k_1 n$ columns are monocolumns, exactly those of $E(n,k_1 n)$. Hence according to Corollary 2.3 the array

$$\mathbb{F}(n,m) = \left(\begin{array}{c|c} E(n,k_1 n) & \mathbb{F}(r_1,n)^T \end{array} \right)$$

is minimum if $\mathbb{F}(r_1,n)$ is minimum in $\mathcal{M}(r_1,n)$. The steps needed in the Euclidean algorithm for (r_1,n) are one less than those needed for the pair (n,m). Hence the inductive hypothesis implies that $\mathbb{F}(r_1,n)$ is minimum in $\mathcal{M}(r_1,n)$. Therefore, $\mathbb{F}(n,m)$ is minimum in $\mathcal{M}(n,m)$ and $S(n,m) = |\operatorname{supp}\, \mathbb{F}(n,m)|$.

To compute $|\operatorname{supp}\, \mathbb{F}(n,m)|$ we note that in view of (2.2) and the way $\mathbb{F}(n,m)$ is constructed we get

$$\begin{aligned}
|\operatorname{supp}\, \mathbb{F}(n,m)| &= |\operatorname{supp} E(n,k_1 n)| + |\operatorname{supp} E(r_1,k_2 r_1)| + \cdots + |\operatorname{supp} E(r_t,k_{t+1} r_t)| \\
&= k_1 n + k_2 r_1 + \cdots + k_t r_{t-1} + k_{t+1} r_t \\
&= m - r_1 + n - r_2 + \cdots + r_{t-2} - r_t + r_{t-1} \\
&= m + n - r_t.
\end{aligned}$$

But the last non-zero remainder in the Euclidean Algorithm (that is, r_t) is the greatest common divisor of (n,m). This completes the proof of the theorem. $\square$

According to Proposition 4 in [5], when $r = 0$, that is, $m = kn$, all the extremal arrays in $\mathcal{M}(n, kn)$ are equivalent to $E(n, kn)$ and thus are minimum. Hence minimum and extremal arrays coincide in $\mathcal{M}(n, kn)$.

Furthermore, according to Proposition 6 of [5], the same holds when $r = 1$. That is, if $m = kn + 1$, every extremal array M in $\mathcal{M}(n, m)$ satisfies

$$|\operatorname{supp} M| = (k + 1)n = n + m - 1 = S(n, m)$$

and thus M is extremal if and only if it is minimum.

This neat characterization of extremal arrays does not hold for $r > 1$ in general. A counterexample is given by the extremal 4×6 array

$$T = \begin{pmatrix} 2 & 2 & & & 2 & \\ 2 & & 4 & & & \\ & 2 & & 4 & & \\ & & & & 2 & 4 \end{pmatrix} \tag{2.3}$$

whose support contains nine non zero entries, while $S(4, 6) = 8$. One can check that the array is extremal using, for example, Proposition 2 in [5].

Nevertheless, when $\gcd(n, m) = 1$, extremal and minimum arrays in $\mathcal{M}(n, m)$ coincide. This is our Theorem II, that we are now ready to prove.

Proof of Theorem II. In view of Theorem 5 in [5], every extremal array $M \in \mathcal{M}(n, m)$ (with $m = kn + r$) is equivalent to the sum of two arrays M_B and M_R, where every row of M_B has exactly $k+1$ positive entries while M_R has at most $r-1$ positive entries. Hence every extremal array $M \in \mathcal{M}(n, m)$ satisfies

$$(k + 1)n \leq |\operatorname{supp} M| \leq (k + 1)n + (r - 1).$$

On the other hand, if $\gcd(n, m) = 1$ and $m = kn + r$, we get

$$S(n, m) = n + m - 1 = (k + 1)n + (r - 1).$$

We conclude that for every extremal M we have

$$S(n, m) \leq |\operatorname{supp} M| \leq (k + 1)n + (r - 1) = S(n, m).$$

Hence $|\operatorname{supp} M| = n + m - 1$, and the proposition follows. □

The array T in (2.3) is not the only example of an extremal array that is not minimum, but it is of smallest dimensions. Actually, we can produce arbitrarily large extremal non-minimum arrays, as the next result states.

Theorem III. *For every pair of integers n, m that satisfy*

$$m = k_1 n + d \quad \text{where} \quad n > d > 1 \quad \text{and} \quad n = k_2 d, \tag{2.4}$$

there exists an extremal array in $\mathcal{M}(n, m)$ that is not minimum.

For its proof we will use a characterization of extremal arrays using their associated graphs given by Brualdi [2]. We first define the associated graph $\mathcal{G}(A)$ of any $n \times m$ array $A = (a_{i,j})$ with $a_{i,j} \geq 0$ as follows. For every row i and every column j we get a node x_i and y_j respectively, for $1 \leq i \leq n$ and $1 \leq j \leq m$. There is an edge joining x_i and y_j if and only if $a_{i,j} > 0$. Then the following theorem holds; see [2] and [3].

Theorem: An array $M \in \mathcal{M}(n,m)$ is extremal if and only if the connected components of $\mathcal{G}(M)$ are trees. Equivalently, $\mathcal{G}(M)$ has no cycles.

We are now ready to prove Theorem III.

Proof. Assume n, m are as above; then $\gcd(n,m) = d$ while $S(n,m) = n + m - d$. The Euclidean Algorithm stops in two steps and our method produces

$$\mathbb{F}(n,m) = \big(E(n, k_1 n) \quad | \quad E(d, k_2 d)^T\big)$$

which is equivalent to the following array in block form

$$X = \underbrace{\begin{pmatrix} B & & \\ & \ddots & \\ & & B \end{pmatrix}}_{d\ B\text{-blocks}}$$

where every block B is

$$B = \left.\underbrace{\begin{pmatrix} d & \overbrace{n \cdots n}^{k_1} & & \\ d & & \overbrace{n \cdots n}^{k_1} & \\ \vdots & & \ddots & \\ d & & & \overbrace{n \cdots n}^{k_1} \end{pmatrix}}_{k_1 k_2 + 1 \text{ columns}}\right\} k_2 \text{ rows.}$$

As $d \geq 2$, there exist at least two blocks in the array X. We replace the first two B-blocks in X with the $2k_2 \times 2(k_1k_2+1)$ array $C =$

$$\left(\begin{array}{cccccccccc} d & n-d & \overbrace{n \cdots n}^{k_1-1} & & & & d & & & \\ d & 0 & & \overbrace{n \cdots n}^{k_1} & & & 0 & & & \\ \vdots & & & \ddots & & & & & & \\ d & & & & \overbrace{n \cdots n}^{k_1} & & & & & \\ & d & & & & \overbrace{n \cdots n}^{k_1} & & & & \\ & & & & & & d & \overbrace{n \cdots n}^{k_1} & & \\ & & & & & & \vdots & & \ddots & \\ & & & & & & d & & & \overbrace{n \cdots n}^{k_1} \end{array}\right) \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} k_2+1 \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} k_2-1 \end{array} .$$

So we get

$$Y = \underbrace{\begin{pmatrix} C & & & \\ & B & & \\ & & \ddots & \\ & & & B \end{pmatrix}}_{d-2\ B\text{-blocks}}.$$

Observe that the array T in (2.3) is a special case of Y when $d = 2 = k_2$ and $k_1 = 1$.

Clearly Y is not minimum as

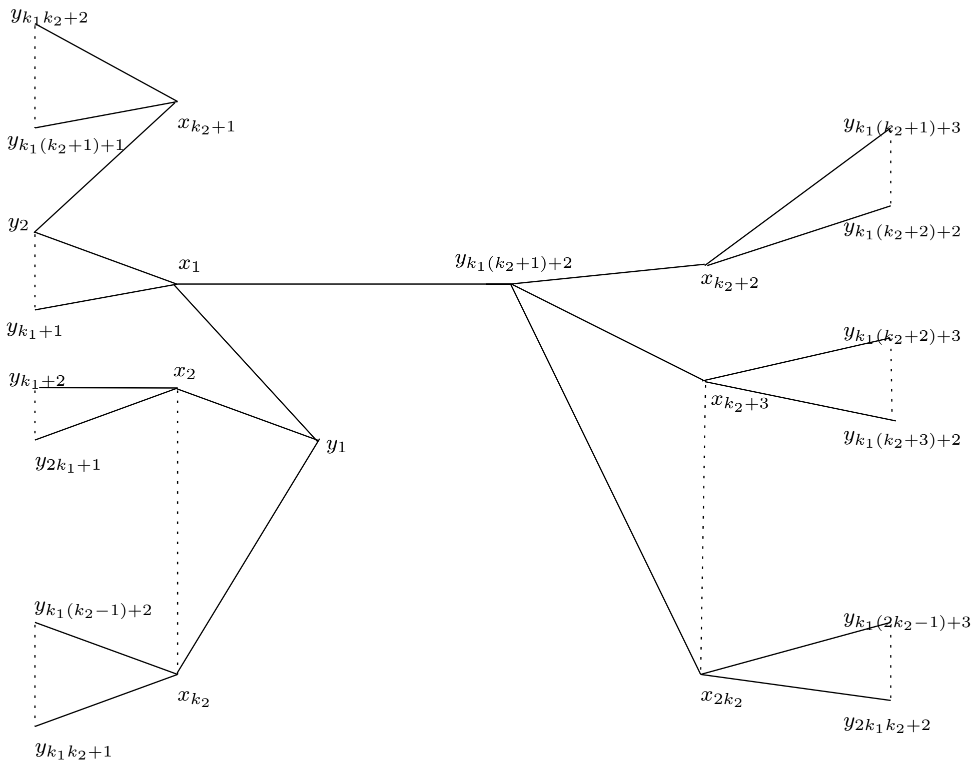
$$|\operatorname{supp} C| = k_1 + 2 + (k_1 + 1) \cdot (2k_2 - 1) = 2k_2(k_1 + 1) + 1 = 2|\operatorname{supp} B| + 1$$

and therefore

$$|\operatorname{supp} Y| = |\operatorname{supp} C| + (d-2)|\operatorname{supp} B| = d|\operatorname{supp} B| + 1 = |\operatorname{supp} X| + 1.$$

It remains to show that Y is extremal. Y is defined as a direct sum of the block arrays C and $(d-2)$ copies of B. Each one of those blocks contributes to the graph $\mathcal{G}(Y)$ one or more connected components. Clearly those components that are associated with B are trees. (This can be seen either directly from the array B or from the fact that X is extremal and X is a direct sum of d blocks, all equal to B.)

We conclude that Y is extremal if and only if the associated graph $\mathcal{G}(C)$ of C is a tree. This is indeed so, as the graph $\mathcal{G}(C)$ is



Hence Y is extremal and the theorem follows. □

We conclude this note with a few more examples of arrays that serve as counterexamples to possible generalizations of the results mentioned.

Remark 1. The array

$$F = \begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

is an element of $\mathcal{M}(3,4)$ with $|\operatorname{supp} F| = 7 = S(3,4) + 1$ but it is not extremal. Hence it is not the case that any doubly stochastic array in $\mathcal{M}(n,m)$ whose support is just one above the minimum support of $\mathcal{M}(n,m)$ must be extremal.

Remark 2. Clearly a possible generalization of Birkhoff's theorem to non-square doubly stochastic arrays, stating that any two extremal arrays in $\mathcal{M}(n,m)$ are equivalent, fails. This can easily be seen as there exist plenty of examples of extremal arrays $A, B \in \mathcal{M}(n,m)$ with $|\operatorname{supp} A| \neq |\operatorname{supp} B|$. As minimum and extremal arrays coincide in $\mathcal{M}(n,n)$, we entertained the idea that, maybe, any two minimum arrays in $\mathcal{M}(n,m)$ are equivalent. (If this were true Birkhoff's theorem would be a special case.) But this fails too, as the next two minimum arrays in $\mathcal{M}(4,5)$ prove.

$$\begin{pmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 1 \\ 3 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 3 \end{pmatrix} \text{ and } \begin{pmatrix} 4 & 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 & 1 \\ 0 & 0 & 0 & 4 & 1 \end{pmatrix}.$$

Nevertheless, we have not managed, so far, to produce two minimum arrays whose set of entries are equal (counting multiplicities) without being equivalent. We should mention here that the way $\mathbb{F}(n,m)$ are constructed ensures that the entries of $\mathbb{F}(n,m)$ are $\{n, r_1, r_2, \dots r_t\}$ (using the notation in (2.2)) appearing with multiplicities

$$\{k_1 n,\ k_2 r_1,\ k_3 r_2,\ \dots,\ k_{t+1} r_t\}$$

respectively.

Since the first version of this paper appeared, a partial answer to Remark 2 above was given in [4]. In the same paper Etkind and Lev provide an alternative way to prove Theorems 1 and 2 and a generalization of Theorem 3.

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