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On the broadcast dimension of a graph

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Abstract

A function $f : V(G) \to \mathbb{Z}^+ \cup \{0\}$ is a resolving broadcast of a graph G if, for any distinct $x, y \in V(G)$, there exists a vertex $z \in V(G)$ with $f(z) > 0$ such that $\min \{d(x, z), f(z) + 1\} \neq \min \{d(y, z), f(z) + 1\}$. The broadcast dimension of G is the minimum of $\sum_{v \in V(G)} f(v)$ over all resolving broadcasts f of G. The concept of broadcast dimension was introduced by Geneson and Yi in 2022 as a variant of metric dimension and has applications in areas such as network discovery and robot navigation.

In this paper, we derive an asymptotically tight lower bound on the broadcast dimension of an acyclic graph in the number of vertices, and we show that a lower bound by Geneson and Yi on the broadcast dimension of a general graph in the adjacency dimension is asymptotically tight. We also study the change in the broadcast dimension of a graph under a single edge deletion. We show that both the additive increase and decrease of the broadcast dimension of a graph under edge deletion is unbounded. Moreover, we show that under edge deletion, the broadcast dimension of any graph increases by a multiplicative factor of at most 3. These results fully answer three questions asked by Geneson and Yi.

1 Introduction

Let $G = (V(G), E(G))$ be a finite, simple, and undirected graph of order $|V(G)|$. The distance $d_G(u, v)$ between two vertices $u, v \in V(G)$ is the length of the shortest path in the graph G between u and v if they belong to the same connected component of G and infinity otherwise. We omit the subscript G if it is clear from the context. For a positive integer k and vertices $u, v \in V(G)$, we define

$$d_k(u, v) := \min\{d(u, v), k + 1\}.$$

A set $S \subseteq V(G)$ is a *resolving set* of G if, for any distinct $x, y \in V(G)$, there is a vertex $z \in S$ such that $d(x, z) \neq d(y, z)$. Intuitively, a resolving set of G is a set of landmark vertices, such that each vertex in $V(G)$ is uniquely characterized by its

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distances to the landmarks. The *metric dimension* $\dim(G)$ of G is the cardinality of a smallest resolving set of G.

Metric dimension was introduced by Slater [20] in 1975, in connection with the problem of uniquely determining the location of an intruder in a network. Harary and Melter independently introduced the same concept in [14]. Metric dimension has since been heavily studied [1, 3, 4, 6] and has applications in diverse areas such as chemistry [5], pattern recognition and image processing [19], and strategies for the Mastermind game [7]. Khuller et al. [18] considered robot navigation as another possible application of metric dimension. In that sense, a robot moving around in a space modeled by a graph can determine its distance to landmarks located at some of the vertices. The minimum number of landmarks required for the robot to uniquely determine its location on the graph is the metric dimension of the graph.

A set $A \subseteq V(G)$ is an *adjacency resolving set* of G if, for any distinct $x, y \in V(G)$, there is a vertex $z \in A$ such that $d_1(x, z) \neq d_1(y, z)$. The *adjacency dimension* $\operatorname{adim}(G)$ of G is the cardinality of a smallest adjacency revolving set of G. The concepts of adjacency resolving set and adjacency dimension were introduced by Jannesari and Omoomi [17] in 2012 as a tool for studying the metric dimension of lexicographic product graphs. The authors of [17] also considered robot navigation as a possible application of adjacency dimension: the minimum number of landmarks required for a robot moving from node to node on a graph to determine its location from only the landmarks adjacent to it is the adjacency dimension of the graph. Similarly, *truncated metric dimension* is a more general version of adjacency dimension [21], where d_k is used instead of d_1 as the distance metric, for some positive integer k.

A function $f : V(G) \to \mathbb{Z}^+ \cup \{0\}$ is a *resolving broadcast* of G if, for any distinct $x, y \in V(G)$, there is a vertex $z \in \operatorname{supp}_G(f) := \{v \in V(G) : f(v) > 0\}$ such that $d_{f(z)}(x, z) \neq d_{f(z)}(y, z)$. The *broadcast dimension* $\operatorname{bdim}(G)$ of G is the minimum of $c_f(G) := \sum_{v \in V(G)} f(v)$ over all resolving broadcasts f of G. The concepts of resolving broadcast and broadcast dimension were introduced in 2020 by Geneson and Yi [13], who noted that broadcast dimension also has applications in robot navigation. In that sense, transmitters with varying range are located at some of the vertices of a graph. A transmitter with range k has cost k for $k \in \mathbb{Z}^+ \cup \{0\}$. A robot moving around on the graph learns its distance to each transmitter that it is within range of and learns that it is out of range of the others. The minimum total cost of transmitters required for a robot to determine its location on the graph is the broadcast dimension.

We say that a resolving set, adjacency resolving set, or resolving broadcast of G is *efficient* if it achieves $\dim(G)$, $\operatorname{adim}(G)$, or $\operatorname{bdim}(G)$, respectively.

Example 1.1. The tree T in Figure 1 has different metric, adjacency, and broadcast dimension.

In [13], Geneson and Yi proved an asymptotic lower bound of $\Omega(\log n)$ on the adjacency and broadcast dimension of graphs of order n, and they further demonstrated

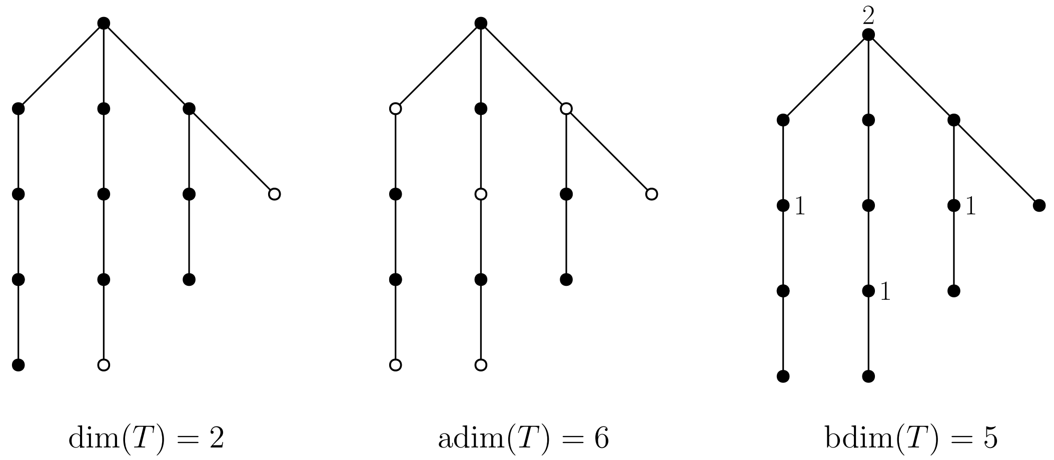

Figure 1: Three copies of tree T. An efficient resolving set is shown with open circles in the first copy; an efficient adjacency resolving set is shown with open circles in the second copy; an efficient resolving broadcast is labeled on the third copy.

that this lower bound is asymptotically tight using a family of graphs from [22].

Theorem 1.2 ([13]). *For all graphs G of order n, we have*

$$n \geq \operatorname{adim}(G) \geq \operatorname{bdim}(G) = \Omega(\log n).$$

We improve the lower bound on the broadcast dimension of acyclic graphs of order n from $\Omega(\log n)$ to $\Omega(\sqrt{n})$ and show that this improved lower bound is asymptotically tight.

Theorem 1.3. *For all acyclic graphs G of order n, we have $\operatorname{bdim}(G) = \Omega(\sqrt{n})$, and this lower bound is asymptotically optimal.*

Since the broadcast dimension is a generalization of the adjacency dimension, a natural question is how these quantities relate. Theorem 1.2 gives that $\operatorname{bdim}(G) = \Omega(\log(\operatorname{adim}(G)))$. In the following question, Geneson and Yi ask whether or not this lower bound is asymptotically optimal.

Question 1.4. ([13]). Is there a family of graphs $\{G_k\}_{k\in\mathbb{Z}^+}$ with $\operatorname{bdim}(G_k) = \Theta(k)$ and $\operatorname{adim}(G_k) = 2^{\Omega(k)}$ for every $k \in \mathbb{Z}^+$?

We resolve Question 1.4 affirmatively by constructing such a family of graphs. Thus, we complete the characterization of how the broadcast dimension of a graph G can vary in the adjacency dimension of G: $\operatorname{adim}(G) \geq \operatorname{bdim}(G) = \Omega(\log(\operatorname{adim}(G)))$, where both sides are tight. Our construction directly implies the following theorem.

Theorem 1.5. *The lower bound $\operatorname{bdim}(G) = \Omega(\log(\operatorname{adim}(G)))$ is asymptotically optimal.*

The question of the effect of vertex or edge deletion on the metric dimension of a graph was raised by Chartrand and Zhang in [6] as a fundamental question in graph theory. In [13], Geneson and Yi studied the effect of vertex deletion on the broadcast dimension of a graph, and they ask two corresponding questions for edge deletion.

Question 1.6. ([13]). Is there a family of graphs $\{G_k\}_{k\in\mathbb{Z}^+}$ such that $\mathrm{bdim}(G_k) - \mathrm{bdim}(G_k - e_k)$ can be arbitrarily large, where $e_k \in E(G_k)$?

Question 1.7. ([13]). For any graph G and any $e = uv \in E(G)$, is it true that $\mathrm{bdim}(G - e) - \mathrm{bdim}(G) \leq d_{G-e}(u, v) - 1$?

Let $e = uv$ denote an edge of a connected graph G such that $G - e$ is also a connected graph. We resolve the first question affirmatively and show that the bound proposed in the second question can fail. In fact, the value $\mathrm{bdim}(G-e)-\mathrm{bdim}(G)$ can be arbitrarily larger than $d_{G-e}(u, v)$. We also show that while $\mathrm{bdim}(G-e)-\mathrm{bdim}(G)$ can be arbitrarily large, the ratio $\frac{\mathrm{bdim}(G-e)}{\mathrm{bdim}(G)}$ is bounded from above.

Theorem 1.8. *The value* $\mathrm{bdim}(G) - \mathrm{bdim}(G - e)$ *can be arbitrarily large.*

Theorem 1.9. *The value* $\mathrm{bdim}(G - e) - \mathrm{bdim}(G)$ *can be arbitrarily larger than* $d_{G-e}(u, v)$.

Theorem 1.10. *For all graphs G and any edge $e \in E(G)$, we have* $\frac{\mathrm{bdim}(G-e)}{\mathrm{bdim}(G)} \leq 3$.

The rest of this paper is structured as follows. In Section 2, we introduce relevant terminology and notation, and we record preliminary results on the metric, adjacency, and broadcast dimension of graphs that are necessary for the rest of the paper. In Section 3, we examine the broadcast dimension of paths and cycles. In Section 4, we discuss results on the broadcast dimension of acyclic graphs and prove Theorem 1.3. In Section 5, we resolve Question 1.4 affirmatively and prove Theorem 1.5. In Section 6, we prove Theorems 1.8, 1.9, and 1.10. Finally in Section 7, we conclude with some open problems about broadcast dimension.

2 Preliminaries

In this section, we first introduce relevant terminology and notation that we will use throughout the paper. We then record some preliminary results on the metric, adjacency, and broadcast dimension of graphs. For the rest of this section, we let $f : V(G) \to \mathbb{Z}^+ \cup \{0\}$ for graph $G = (V(G), E(G))$.

We denote by P_n, C_n, and K_n the path, cycle, and complete graph on n vertices, respectively. We say $\mathrm{diam}(G) = \max\{d(u, v) \mid u, v \in V(G)\}$. We denote by $\mathbf{1}$ the vector with 1 for each entry and $\mathbf{2}$ the vector with 2 for each entry, where the length of the vector is inferred from context. For an arbitrary set S, a totally ordered set Y, and a function $g : S \to Y$, we define $\mathrm{argmax}_{x\in S}\, g(x)$ to be any $x^* \in S$ such that $g(x) \leq g(x^*)$ for all $x \in S$. We define $\mathrm{argmin}_{x\in S}\, g(x)$ analogously.

Definition 2.1. A vertex $z \in \mathrm{supp}_G(f)$ *resolves* a pair of distinct vertices $x, y \in V(G)$ if

$$d_{f(z)}(x, z) \neq d_{f(z)}(y, z).$$

In order for a vertex $z \in \mathrm{supp}_G(f)$ to resolve a pair of vertices $x, y \in V(G)$, we must have $f(z) \geq d(x,z)$ or $f(z) \geq d(y,z)$. We formally define this notion below.

Definition 2.2. A vertex $z \in \mathrm{supp}_G(f)$ *reaches* a vertex $v \in V(G)$ with respect to f if $f(z) \geq d(v,z)$, and the function f *reaches* a vertex $v \in V(G)$ if there is a vertex $z \in \mathrm{supp}_G(f)$ that reaches v.

By definition, the function f is a resolving broadcast of G if and only if every pair of distinct vertices in $V(G)$ is resolved by a vertex in $\mathrm{supp}_G(f)$. Thus, any resolving broadcast f of G must reach all but at most one vertex in $V(G)$. Equivalently, the function f is a resolving broadcast of G if and only if every vertex of G is *distinguished*; that is, every vertex of G is uniquely characterized by its distances to the vertices in $\mathrm{supp}_G(f)$ that reach it. We formally define this term below.

Definition 2.3. Let $k = |\mathrm{supp}_G(f)|$. The *broadcast representation* of a vertex $v \in V(G)$ with respect to f is the k-vector $b_f(v) = \big(d_{f(u_1)}(v,u_1), \ldots, d_{f(u_k)}(v,u_k)\big)$ for $u_i \in \mathrm{supp}_G(f)$. We say that a vertex $v \in V(G)$ is *distinguished* if it has a unique broadcast representation $b_f(v)$.

The following observations give insight into how the metric, adjacency dimension, and broadcast dimension of graphs are related and will be useful throughout the rest of the paper.

Observation 2.4. ([13])**.** The following properties hold for any graph G.

1. We have $\dim(G) \leq \mathrm{bdim}(G) \leq \mathrm{adim}(G)$.
2. If $\mathrm{diam}(G) \leq 2$, then we have $\dim(G) = \mathrm{bdim}(G) = \mathrm{adim}(G)$.

The *closed neighborhood* of a vertex $v \in V(G)$ is $N[v] = \{u \in V(G) : uv \in E(G)\} \cup \{v\}$. Two distinct vertices $u, v \in V(G)$ are called *twin vertices* if $N[u] = N[v]$.

Observation 2.5. If $u, v \in V(G)$ are twin vertices, then the following properties hold.

1. [16] For any resolving set S of G, we have that $u \in S$ or $v \in S$.
2. [17] For any adjacency resolving set A of G, we have that $u \in A$ or $v \in A$.
3. [13] For any resolving broadcast f of G, we have that $u \in \mathrm{supp}_G(f)$ or $v \in \mathrm{supp}_G(f)$.

3 Paths and Cycles

Here we restrict our attention to path and cycle graphs. It is easy to see that $\dim(P_n) = 1$ and $\dim(C_n) = 2$ for every integer $n \geq 3$. The adjacency dimension and the broadcast dimension, respectively, of paths and cycles were determined in [17] and [13].

Theorem 3.1 ([17])**.** *For every integer $n \geq 4$, we have* $\mathrm{adim}(P_n) = \mathrm{adim}(C_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor$.

Theorem 3.2 ([13]). *For every integer $n \geq 4$, we have* $\mathrm{bdim}(P_n) = \mathrm{bdim}(C_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor$.

In this section, we prove the following result on efficient resolving broadcasts of paths and cycles.

Proposition 3.3. *For every $n \in \mathbb{Z}^+$ and $G \in \{P_n, C_n\}$, if f is an efficient resolving broadcast of G, then $f(v) \leq 2$ for all $v \in V(G)$.*

We begin with two lemmas. In the proof of Theorem 3.2, Geneson and Yi proved the following useful fact, which we state here as a lemma. We include the proof for completeness.

Lemma 3.4 ([13]). *For every $n \in \mathbb{Z}^+$ and every efficient resolving broadcast f of $G \in \{P_n, C_n\}$, there is an efficient resolving broadcast f' of G with the following properties.*

1. *Every vertex reached by f is also reached by f'.*
2. *For all $v \in V(G)$, we have $f'(v) \leq 1$.*

Proof. Let G be the path $v_1, \ldots, v_n$ or the cycle $v_1, \ldots, v_n, v_1$. Let f_0 be any efficient resolving broadcast of G. If $f_0(v) \leq 1$ for all $v \in V(G)$, then we are done. Otherwise, we repeatedly modify f_i to obtain a new efficient resolving broadcast f_{i+1} that satisfies the following monovariant: for integer k, let $U_k = \{v \in V(G) : f_k(v) > 1\}$ and $S_k = \sum_{v \in U_k} f_k(v)$, then $S_{i+1} < S_i$.

Let $v_j \in V(G)$ be any vertex with $x := f_i(v_j) > 1$. If v_j is a leaf and $x = 2$, we set $f_{i+1}(v_j) = 1$ and $f_{i+1}(u) = \max\{f_i(u), 1\}$, where u is the vertex adjacent to v_j. Otherwise, we set $f_{i+1}(v_j) = x - 2$, and we let u_1 and u_2 be the vertices $v_{(j+x-1) \bmod n}$ and $v_{(j-x+1) \bmod n}$, respectively. We set $f_{i+1}(u_1) = \max\{f_i(u_1), 1\}$ and $f_{i+1}(u_2) = \max\{f_i(u_2), 1\}$. The maximum value is used for vertices assigned multiple values for f_{i+1}, and $f_{i+1}(v) = f_i(v)$ for any vertex v not assigned any value for f_{i+1}. This process will terminate after finitely many steps because of the monovariant on S_i, yielding a resolving broadcast that satisfies the description of f'. □

The proof of Lemma 3.7 uses some ideas from observations made in [2] about the metric dimension of a wheel $W_n = C_n + K_1$ for integers $n \geq 3$. To state the lemma, we need the following definition.

Definition 3.5. For a graph G, the value $\widehat{\mathrm{bdim}}(G)$ is the minimum of $\sum_{v \in V(G)} f(v)$ over all resolving broadcasts f of G such that every vertex $v \in V(G)$ is reached by at least one vertex $z \in \mathrm{supp}_G(f)$. This differs from $\mathrm{bdim}(G)$ because one vertex may be unreached by a resolving broadcast.

Observation 3.6. For all graphs G, we have

$$\widehat{\mathrm{bdim}}(G) = \mathrm{bdim}(G \cup K_1) \quad \text{and} \quad \mathrm{bdim}(G) \leq \widehat{\mathrm{bdim}}(G) \leq \mathrm{bdim}(G) + 1.$$

Lemma 3.7. *For every integer $n \geq 4$, we have* $\widehat{\mathrm{bdim}}(P_n) = \widehat{\mathrm{bdim}}(C_n) = \left\lfloor \frac{2n+3}{5} \right\rfloor$.

Proof. Let G be the path $v_1, \ldots, v_n$ or the cycle $v_1, \ldots, v_n, v_1$. First, we will show that $\widehat{\mathrm{bdim}}(G) = \mathrm{bdim}(G)$ for $n \not\equiv 1 \pmod 5$. Define $g : V(G) \to \mathbb{Z}^+ \cup \{0\}$ as follows: $g(v_i)$ is 1 if $i \equiv 2 \pmod 5$ or $i \equiv 4 \pmod 5$ and 0 otherwise. Note that g is a resolving broadcast of G that achieves $\mathrm{bdim}(G)$ given in Theorem 3.2 and that g reaches all of the vertices of G when $n \not\equiv 1 \pmod 5$.

Now, we will show that $\widehat{\mathrm{bdim}}(G) = \mathrm{bdim}(G)+1$ for $n \equiv 1 \pmod 5$. Let $n = 5x+1$ for some positive integer x; then, we have $\mathrm{bdim}(G) = \left\lfloor \frac{10x+4}{5} \right\rfloor = 2x$. It suffices to show that for any efficient resolving broadcast f of G, there is a vertex not reached by f. By Lemma 3.4, there is an efficient resolving broadcast f' of G with $f'(v) \leq 1$ for all $v \in V(G)$ that reaches all of the vertices reached by f. For the sake of contradiction, we assume that f' reaches all of the vertices, and so there is no vertex $v \in V(G)$ with $b_{f'}(v) = \mathbf{2}$.

A *gap* of graph G is a maximal connected subgraph of G that only consists of vertices that are not in $\mathrm{supp}_G(f')$. If two gaps are adjacent to the same vertex in $\mathrm{supp}_G(f')$, then we call them *neighboring gaps.* No gap can contain three vertices, since the vertex in the middle of the gap would have broadcast representation $\mathbf{2}$. Additionally, any neighboring gap of a gap that contains two vertices must contain only one vertex, since otherwise there exists five consecutive vertices of G where the vertex m in the middle is the only one in $\mathrm{supp}_G(f')$, and the two vertices adjacent to m would have the same broadcast representation.

If G is C_n, then of the $\mathrm{bdim}(G)$ gaps, at most $\left\lfloor \frac{\mathrm{bdim}(G)}{2} \right\rfloor$ gaps contain two vertices, and none contain three vertices. Thus, $n \leq 2\,\mathrm{bdim}(C_n) + \left\lfloor \frac{\mathrm{bdim}(C_n)}{2} \right\rfloor = 5x$. Similar reasoning yields $n \leq 5x$ if G were instead P_n. Since G is a graph of order $5x+1$, we have reached a contradiction. □

With the above lemma, we are now able to prove Proposition 3.3.

Proof of Proposition 3.3. Let G be the path $v_1, \ldots, v_n$ or the cycle $v_1, \ldots, v_n, v_1$, and let f be an efficient resolving broadcast of G. If $n \leq 6$, then $\mathrm{bdim}(G) \leq 2$ by Theorem 3.2, so $f(v) \leq 2$ for all $v \in V(G)$. Thus, we consider $n \geq 7$.

Let $v_i = \mathrm{argmax}_{v \in V(G)}(f(v))$. For the sake of contradiction, we assume that $f(v_i) \geq 3$. If vertex v_i were a leaf (say $i = 1$), then a function g that is identical to f, except with $g(v_3) = f(v_1) - 2$ and $g(v_1) = 1$, is a resolving broadcast of G with $c_g(G) < c_f(G)$, contradicting the efficiency of f. Thus, v_i has two neighbors. At least one of the neighbors of v_i must be reached by some other vertex $v_j \neq v_i$ or else the two neighbors of v_i would not be distinguished.

First, we will show that f is inefficient if $f(v_j) \geq 2$. Let T be the set of vertices that are reached by v_i or v_j. Note that $|T| \leq 2f(v_i) + f(v_j) + 2$. By Lemma 3.7, the vertices in T can be reached and distinguished with a total cost of $\left\lfloor \frac{2|T|+3}{5} \right\rfloor$, which is less than $f(v_i) + f(v_j)$ when $f(v_i) \geq 3$ and $f(v_j) \geq 2$.

Thus, we must have $f(v_j) = 1$, so $|T| \leq 2f(v_i) + 1$ since v_j cannot reach any vertex that v_i does not reach. By Lemma 3.7, the vertices in T can be reached and distinguished with a total cost of

$$\left\lfloor \frac{2|T| + 3}{5} \right\rfloor \leq \frac{4f(v_i) + 5}{5} < f(v_i) + 1 = f(v_i) + f(v_j).$$

This contradicts the efficiency of resolving broadcast f. □

4 Results on Acyclic Graphs

In this section, we discuss some results on the broadcast dimension of acyclic graphs, and we prove Theorem 1.3. We make use of standard terminology for trees: a *major vertex* in a tree T is a vertex of degree at least three, and a *leaf* of T is a vertex of degree one.

For any graph G, showing that a function $g : V(G) \to \mathbb{Z}^+ \cup \{0\}$ is a resolving broadcast of G gives an upper bound of $c_g(G)$ on $\operatorname{bdim}(G)$. On the other hand, obtaining a nice lower bound on $\operatorname{bdim}(G)$ is oftentimes less straightforward. The result on twin vertices from Observation 2.5 is a useful tool for lower bounding $\operatorname{bdim}(G)$. In this section, we use a different approach to derive a lower bound on the broadcast dimension of trees: we consider the number of unique broadcast representations of the vertices of a tree T with respect to various functions $f : V(T) \to \mathbb{Z}^+ \cup \{0\}$. This motivates the following definition.

Definition 4.1. For a graph G of order n and a function $f : V(G) \to \mathbb{Z}^+ \cup \{0\}$, we say that $B_G(f)$ is the number of unique broadcast representations of the vertices of G with respect to f. That is,

$$B_G(f) = |\{b_f(v) \mid v \in V(G)\}|.$$

Note that $B_G(f) = n$ if and only if f is a resolving broadcast of G.

The following lemma will be useful in the proof of Theorem 4.3.

Lemma 4.2. *Let T be a tree with resolving broadcast f, and let $a, b, v, x \in V(T)$ such that the following inequalities hold:*

$$\begin{aligned} f(a) - d(a, x) &\geq f(v) - d(v, x), \\ f(b) - d(b, x) &\geq f(v) - d(v, x), \\ f(a) - d(a, v) &\geq f(b) - d(b, v). \end{aligned}$$

Then every vertex of T that is reached by both b and v is also reached by a.

Proof. We consider four possible orientations of the vertices a, b, and v (see Figure 2).

Case 1. There is not a path in T through vertices a, b, and v.
Let c be the major vertex of T such that the path from c to a, the path from c to b, and the path from c to v do not share any edges.

In this case, $f(a) - d(a, v) \geq f(b) - d(b, v)$ implies that

$$f(a) - d(a, c) \geq f(b) - d(b, c). \tag{1}$$

If the path from x to a does not go through c, then both the path from x to b and the path from x to v must pass through c, so $f(b) - d(b, x) \geq f(v) - d(v, x)$ implies that $f(b) - d(b, c) \geq f(v) - d(v, c)$. Combining this inequality with (1), we have

$$f(a) - d(a, c) \geq f(v) - d(v, c). \tag{2}$$

Alternatively, if the path from x to a does go through c, then $f(a) - d(a, x) \geq f(v) - d(v, x)$ directly implies (2). Thus, the inequality in (2) holds no matter where vertex x is.

The inequality in (1) shows that any vertex reached by b with a path to b that goes through c is reached by a. Similarly, the inequality in (2) shows that any vertex reached by v with a path to v that goes through c is reached by a. Thus, any vertex that is reached by both b and v is also reached by a.

Case 2. $d(a, v) + d(v, b) = d(a, b)$.
If the path from x to b does not go through v, then

$$\begin{aligned} f(a) - d(a, x) \geq f(v) - d(v, x) &\implies f(a) \geq d(a, v) + d(v, x) + f(v) - d(v, x) \\ &\implies f(a) - d(a, v) \geq f(v). \end{aligned}$$

Alternatively, if the path from x to b does go through v, then replacing a with b in the above inequalities, we get $f(b) - d(b, v) \geq f(v)$, which implies that $f(a) - d(a, v) \geq f(v)$.

Thus, no matter where vertex x is, we have $f(a) - d(a, v) \geq f(v)$, which shows that a reaches all of the vertices reached by v.

Case 3. $d(b, a) + d(a, v) = d(b, v)$.

Case 4. $d(a, b) + d(b, v) = d(a, v)$.

It is easy to see that the lemma is true for Cases 3 and 4 by direct observation or by performing analysis similar to the analysis shown for Cases 1 and 2. □

Theorem 4.3. *For all trees T of order n, we have* $\operatorname{bdim}(T) \geq \sqrt{\frac{n}{6}}$.

Proof. Let T be a tree of order n, and let f be any resolving broadcast of T. We define $f' : V(T) \to \mathbb{Z}^+ \cup \{0\}$ such that $f'(v) = 0$ for all $v \in V(T)$. Note that $B_T(f') = 1$. Let $x \in V(T)$ be any vertex. We order the vertices in $\operatorname{supp}_T(f)$ so that vertex $v \in \operatorname{supp}_T(f)$ comes before vertex $u \in \operatorname{supp}_T(f)$ in the ordering only if $f(v) - d(v, x) \geq f(u) - d(u, x)$. We update the value of $f'(v)$ from 0 to $f(v)$ (notationally, $f'(v) \leftarrow f(v)$) one vertex $v \in \operatorname{supp}_T(f)$ at a time in the defined order until $f' = f$, and we consider the increase in $B_T(f')$ on each update.

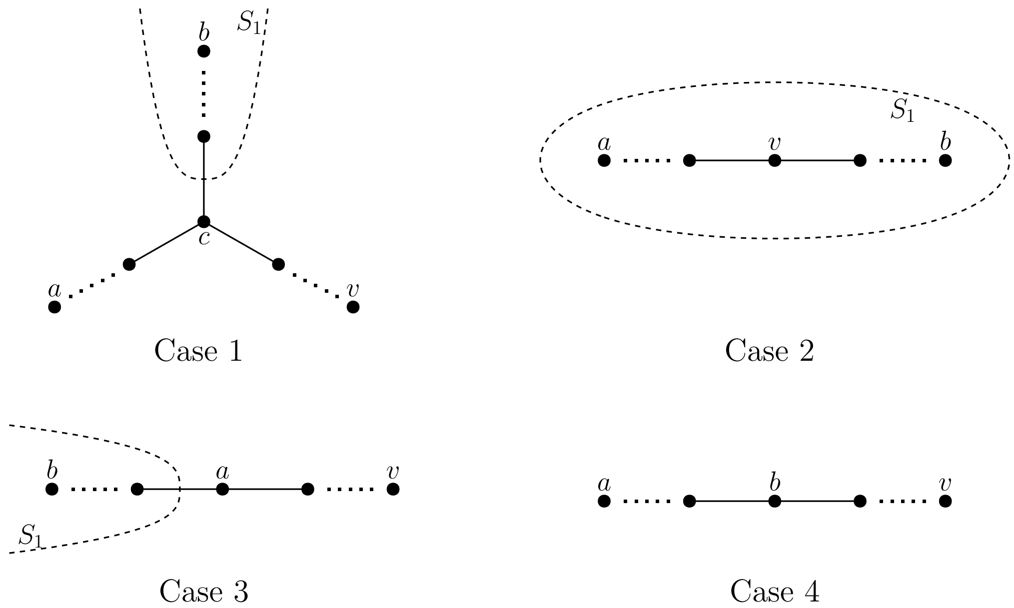

Figure 2: The four cases from the proof of Lemma 4.2 and the proof of Theorem 4.3. Note that all vertices may have larger degree than what is shown. Any non-pictured vertex of the tree that is in S (defined in the proof of Theorem 4.3) and adjacent to a vertex in S_1 is also in S_1.

For a vertex $v \in \text{supp}_T(f)$, let $W(v)$ be the set of vertices that can reach (with respect to f') at least one vertex $u \in V(T)$ that is reached by v (with respect to f). That is,

$$W(v) = \{w \mid w \neq v, w \in \text{supp}_T(f'), u \in V(T), f'(w) \geq d(u, w), f(v) \geq d(u, v)\}.$$

If $W(v) = \emptyset$, then updating $f'(v) \leftarrow f(v)$ increases $B_T(f')$ by at most $f(v) + 1$, which is upper bounded by $2\,(f\,(v))^2$ since $f(v) \geq 1$.

If $|W(v)| = 1$, then we can make the following observations about the broadcast representation $b_{f'}(u)$ of any vertex u reached by v after the update $f'(v) \leftarrow f(v)$. There are $f(v) + 1$ possible values for the entry of $b_{f'}(u)$ corresponding to vertex v and $2f(v) + 1$ possible values for the entry of $b_{f'}(u)$ corresponding to the vertex in $W(v)$. The rest of the entries of $b_{f'}(u)$ must be the maximal possible value for that entry. Thus, $B_T(f')$ increases by at most $(f(v) + 1)\,(2f(v) + 1) \leq 6\,(f\,(v))^2$ in this case.

Now we consider $|W(v)| > 1$. Let $a = \text{argmax}_{u \in W(v)}\,(f'(u) - d(u, v))$ and $b \in W(v) - \{a\}$. Let $\delta \geq 0$ such that the update $f'(v) \leftarrow f(v)$ increases $B_T(f')$ by δ.

Claim. If $f'(b)$ were instead zero, then the update $f'(v) \leftarrow f(v)$ would still increase $B_T(f')$ by at least δ.

Proof of claim. Let S be the set of vertices reached by both b and v, and let $S_0 = V(T) - S$. We consider four possible orientations of the vertices a, b, and v (see

Figure 2), and we show that, in each case, the vertices in S can be split into two (possibly empty) sets S_1 and S_2 such the three properties listed below are satisfied. Note that showing this proves the claim.

Property 1. Before updating $f'(v) \leftarrow f(v)$, every vertex in S_1 has a different broadcast representation from every vertex in $V(T) - S_1$.

Property 2. Updating $f'(v) \leftarrow f(v)$ does not increase $|\{b_{f'}(v) \mid v \in S_1\}|$.

Property 3. If $f'(b)$ were instead zero, updating $f'(v) \leftarrow f(v)$ would increase $|\{b_{f'}(v) \mid v \in S_2 \cup S_0\}|$ by at least δ.

Since we made the updates $f'(a) \leftarrow f(a)$ and $f'(b) \leftarrow f(b)$ before the update $f'(v) \leftarrow f(v)$, we have $f(a) - d(a,x) \geq f(v) - d(v,x)$ and $f(b) - d(b,x) \geq f(v) - d(v,x)$. Because of the way we chose vertex a, we have $f(a)-d(a,v) \geq f(b)-d(b,v)$. Thus, by Lemma 4.2, vertex a also reaches all of the vertices in S.

Because every vertex in S_0 is not reached by b or not reached by v, the increase in $|\{b_{f'}(v) \mid v \in S_0\}|$ after updating $f'(v) \leftarrow f(v)$ would be at least the same if $f'(b)$ were instead zero. In all four cases, if $S_1 = \emptyset$, Properties 1 and 2 are trivially satisfied, and if $S_2 = \emptyset$, Property 3 is trivially satisfied.

Case 1. There is not a path in T through vertices a, b, and v.
Let c be the major vertex of T such that the path from c to a, the path from c to b, and the path from c to v do not share any edges. Let S_1 be the set of vertices in S with a path to b that does not go through c, and $S_2 = S - S_1$. Let $u_1 \in S_1$. All other vertices with distance $d(u_1,a)$ to a and $d(u_1,b)$ to b are also in S_1 (Property 1) and have the same distance $d(u_1,a) - d(a,c) + d(c,v)$ to vertex v (Property 2). Let $u_2 \in S_2$. All of the vertices in S_2 that are distance $d(u_2,a)$ to vertex a and distance $d(u_2,v)$ to vertex v have the same distance to vertex b (Property 3).

Case 2. $d(a,v) + d(v,b) = d(a,b)$.
Let $S_1 = S$ and $S_2 = \emptyset$. Let $u_1 \in S_1$. All other vertices that are distance $d(u_1,a)$ from vertex a and distance $d(u_1,b)$ from vertex b are also in S_1 (Property 1) and are the same distance from vertex v (Property 2). Property 3 is trivially satisfied.

Case 3. $d(b,a) + d(a,v) = d(b,v)$.
Let S_1 be the set of vertices in S with a path to b that does not go through a, and $S_2 = S - S_1$. Let $u_1 \in S_1$. All other vertices with distance $d(u_1,a)$ to a and $d(u_1,b)$ to b are also in S_1 (Property 1), and they all have the same distance $d(u_1,a)+d(a,v)$ to vertex v (Property 2). Let $u_2 \in S_2$. All of the vertices in S_2 that are distance $d(u_2,a)$ to vertex a have the same distance $d(u_2,a)+d(a,b)$ to vertex b (Property 3).

Case 4. $d(a,b) + d(b,v) = d(a,v)$.
Let $S_1 = \emptyset$ and $S_2 = S$. Properties 1 and 2 are trivially satisfied. Let $u_2 \in S_2$. All of the vertices with distance $d(u_2,a)$ to vertex a and distance $d(u_2,v)$ to vertex v have the same distance to vertex b (Property 3). □

The claim implies that the change in $B_T(f')$ after updating $f'(v) \leftarrow f(v)$ when $|W(v)| > 1$ is upper bounded by the change in $B_T(f')$ after updating $f'(v) \leftarrow f(v)$

if we instead had $W(v) = \{a\}$. Thus, every update increases $B_T(f')$ by at most $6\,(f(v))^2$, and the very first update increases $B_T(f')$ by at most $2\,(f(v))^2$. Since we started out with $B_T(f') = 1$, and we must have $B_T(f') = n$ after finishing all of the updates, we have that $c_f(T) \geq \sqrt{\frac{n}{6}}$ for any resolving broadcast f of T. □

Because the broadcast dimension of a disconnected graph is at least the sum of the broadcast dimensions of all of its connected components, Theorem 4.3 directly implies the following corollaries.

Corollary 4.4. *For all acyclic graphs G of order n, we have* $\mathrm{bdim}(G) = \Omega(\sqrt{n})$.

Corollary 4.5. *For all acyclic graphs G of order n, we have* $\mathrm{adim}(G) = \Omega(\sqrt{n})$.

Corollary 4.6. *For all acyclic graphs G of order n, we have* $\mathrm{adim}(G) = O\left((\mathrm{bdim}(G))^2\right)$.

Now we will show that the bound from Theorem 4.3 is sharp up to a constant factor and that the asymptotic bounds from Corollary 4.4 and Corollary 4.6 are asymptotically optimal. We do so by finding a family of trees that achieves these bounds up to a constant factor. This family of graphs will also be used to study edge deletion in Section 6.

Definition 4.7. For every $k \in \mathbb{Z}^+ \cup \{0\}$, graph L_k is the path $v_0, \ldots, v_k$. The graph F_k is L_k with a path P_i connected to v_i for each $1 \leq i \leq k$. (See Figure 3 for the graph F_3.)

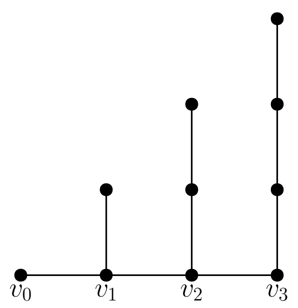

Figure 3: The graph F_3.

Theorem 4.8. *For every $k \in \mathbb{Z}^+ \cup \{0\}$, tree F_k of order $\Theta(k^2)$ has* $\mathrm{bdim}(F_k) = O(k)$ *and* $\mathrm{adim}(F_k) = \Theta(k^2)$.

Proof. The function $f_k : V(F_k) \to \mathbb{Z}^+ \cup \{0\}$ with $f_k(v_0) = f_k(v_k) = 2k$ and $f_k(v) = 0$ for all other vertices $v \in V(F_k)$ is a resolving broadcast of F_k with $c_{f_k}(F_k) = 4k$, so $\mathrm{bdim}(F_k) \leq 4k = O(k)$.

The size of any adjacency resolving set of F_k must be linear in the number of vertices in order for all of the vertices on the paths attached to L_k to be distinguished. Since tree F_k has order $\Theta(k^2)$, we have $\mathrm{adim}(F_k) = \Theta(k^2)$. □

Combining Corollary 4.4 and Theorem 4.8, we have proven Theorem 1.3.

In [13], Geneson and Yi showed that, for two connected graphs G and H such that $H \subset G$, the ratios $\frac{\dim(H)}{\dim(G)}$, $\frac{\mathrm{adim}(H)}{\mathrm{adim}(G)}$, and $\frac{\mathrm{bdim}(H)}{\mathrm{bdim}(G)}$ can be arbitrarily large. In the next result, we show that this can only be true when the graph G is not acyclic.

Proposition 4.9. *For two trees T_1 and T_2 such that $T_1 \subseteq T_2$, we have that $\dim(T_1) \leq \dim(T_2)$, $\mathrm{adim}(T_1) \leq \mathrm{adim}(T_2)$, and $\mathrm{bdim}(T_1) \leq \mathrm{bdim}(T_2)$.*

Proof. Let T be a tree with efficient resolving broadcast $f : V(T) \to \mathbb{Z}^+ \cup \{0\}$. Let $v \in V(T)$ be a leaf of T, and let $uv \in E(T)$. If $v \notin \mathrm{supp}_T(f)$, then $g : V(T - v) \to \mathbb{Z}^+ \cup \{0\}$ with $g(w) = f(w)$ for every $w \in V(T - v)$ is a resolving broadcast of graph $T - v$. If $v \in \mathrm{supp}_T(f)$, then $g : V(T - v) \to \mathbb{Z}^+ \cup \{0\}$ with $g(u) = \max\{f(v) - 1, f(u)\}$ and $g(w) = f(w)$ for every $w \in V(T - v) - \{u\}$ is a resolving broadcast of $T - v$. Thus, $\mathrm{bdim}(T - v) \leq \mathrm{bdim}(T)$ for any leaf v of T. Tree T_2 can be pruned into tree T_1 by repeatedly deleting leaves that are not in T_1. Thus, $\mathrm{bdim}(T_1) \leq \mathrm{bdim}(T_2)$. The results $\dim(T_1) \leq \dim(T_2)$ and $\mathrm{adim}(T_1) \leq \mathrm{adim}(T_2)$ follow with similar reasoning. □

5 Comparing $\mathrm{adim}(G)$ and $\mathrm{bdim}(G)$

Geneson and Yi [13] showed that, for the the d-dimensional grid graph $G_k = \Pi_{i=1}^d P_k$, we have $\mathrm{bdim}(G_k) = \Theta(k)$ and $\mathrm{adim}(G_k) = \Theta(k^d)$ for every $k \in \mathbb{Z}^+$ and any $d \geq 1$, where the constants in the bounds depend on d. In this section, we prove the following theorem.

Theorem 5.1. *There exists a family of graphs $\{G_k\}_{k\in\mathbb{Z}^+}$ with $\mathrm{bdim}(G_k) = \Theta(k)$ and $\mathrm{adim}(G_k) = 2^{\Omega(k)}$ for every $k \in \mathbb{Z}^+$.*

First, we recall the following graph notation. We denote by $G[S]$ the subgraph of G induced by $S \subseteq V(G)$. The *Cartesian product* of graphs G and H, denoted by $G\square H$, is the graph with vertex set $V(G) \times V(H) := \{(u_1, u_2) \mid u_1 \in V(G), u_2 \in V(H)\}$, where (u_1, u_2) is adjacent to (v_1, v_2) whenever $u_1 = v_1$ and $u_2v_2 \in E(H)$, or $u_2 = v_2$ and $u_1v_1 \in E(G)$.

We prove Theorem 5.1 by finding a family of graphs with the desired properties. This family of graphs is defined as follows:

Definition 5.2. Graph $\widehat{X_0}$ is the path a, b, c, and graph $\widehat{X}$ is the graph with vertex set $\{a, b, c\}$ and edge set $\{ab\}$. For $i \in \mathbb{Z}^+$, we let

$$\widehat{X_i} = \widehat{X_0}\square \underbrace{\widehat{X}\square\widehat{X} \dots \square\widehat{X}}_{i \text{ times}}.$$

For $i \in \mathbb{Z}^+ \cup \{0\}$, graph X_i is $\widehat{X_i}$ with one modification: for every $0 \leq j \leq i$, graph X_i has an additional vertex s_j that is adjacent to every vertex with a as the $(j+1)$st coordinate. (See Figure 4 for the graph X_1.)

Lemma 5.3. *We have $\mathrm{bdim}(X_k) = \Theta(k)$ for all $k \in \mathbb{Z}^+$.*

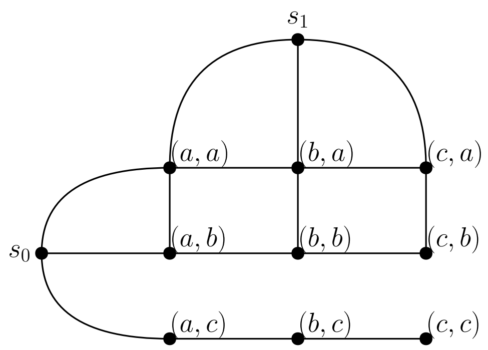

Figure 4: The graph X_1.

Proof. Let $k \in \mathbb{Z}^+$ be given. For $i \in \mathbb{Z}^+ \cup \{0\}$, we define $S_i = \{s_j \mid 0 \leq j \leq i\}$.

For $i \in \mathbb{Z}^+ \cup \{0\}$, we define the function $f_i : V(X_i) \to \mathbb{Z}^+ \cup \{0\}$ as follows: $f_i(s_0) = 3$, $f_i(s_j) = 2$ for every $1 \leq j \leq i$, and $f_i(v) = 0$ for all other vertices v. We claim that f_i is a resolving broadcast of X_i for all $i \in \mathbb{Z}^+ \cup \{0\}$. We proceed to prove this claim by induction.

In the base case $i = 0$, we have that X_0 is the path s_0, a, b, c. It is easy to see that that function f_0 with $f_0(s_0) = 3$ is a resolving broadcast of X_0. Assuming that f_{k-1} is a resolving broadcast of graph X_{k-1}, we will show that f_k is a resolving broadcast of graph X_k.

Let $u_1, u_2 \in V(X_{k-1})$ and $v_1, v_2 \in \{a, b, c\}$ such that (u_1, v_1) and (u_2, v_2) are two distinct vertices in $V(X_k)$. If $u_1 \neq u_2$, then (u_1, v_1) and (u_2, v_2) are resolved by the vertex in S_{k-1} that resolved u_1 and u_2 in X_{k-1}. Alternatively, if $u_1 = u_2$, then $v_1 \neq v_2$, and (u_1, v_1) and (u_2, v_2) are resolved by s_k. Thus, function f_k is a resolving broadcast of X_k.

Now we can upper bound the broadcast dimension of X_k:

$$\mathrm{bdim}(X_k) \leq c_{f_k}(X_k) = 3 + 2k \implies \mathrm{bdim}(X_k) = O(k).$$

By Theorem 1.2, we have $\mathrm{bdim}(X_k) = \Omega(k)$. Thus, we have $\mathrm{bdim}(X_k) = \Theta(k)$. □

Lemma 5.4. *We have* $\mathrm{adim}(X_k) = 2^{\Omega(k)}$ *for all* $k \in \mathbb{Z}^+$.

Proof. Let $k \in \mathbb{Z}^+$ be given. For $i \in \mathbb{Z}^+ \cup \{0\}$, we define $S_i = \{s_j \mid 0 \leq j \leq i\}$.

We claim that the following statement is true for all $i \in \mathbb{Z}^+ \cup \{0\}$: for any adjacency resolving set A_i of X_i, we have that $|(V(X_i) - S_i) \cap A_i| \geq 2^i$. We proceed to prove this claim by induction.

In the base case $i = 0$, for any adjacency resolving set A_0 of $X_0 = P_4$, we have by Theorem 3.1

$$|(V(X_0) - \{s_0\}) \cap A_0| \geq \left\lfloor \frac{2(4) + 2}{5} \right\rfloor - 1 = 1.$$

Now we assume that $|(V(X_{k-1}) - S_{k-1}) \cap A_{k-1}| \geq 2^{k-1}$ for any adjacency resolving set A_{k-1} of X_{k-1}.

Let H be $X_{k-1}[V(X_{k-1})-S_{k-1}]$, the subgraph induced in X_{k-1} by $V(X_{k-1})-S_{k-1}$. The induced subgraph $X_k[V(X_k) - S_k]$ contains three copies of H as subgraphs. Let H_1, H_2, and H_3 be the copies of H in $X_k[V(X_k) - S_k]$ that are induced by the sets of vertices $\{(v, a) \mid v \in V(H)\}$, $\{(v, b) \mid v \in V(H)\}$, and $\{(v, c) \mid v \in V(H)\}$, respectively.

Let $v \in V(H)$. Vertex $(v, c) \in V(H_3)$ is only adjacent to vertices in $V(X_k)-V(H_3)$ that are in S_{k-1}. In particular, the vertices $(v, c) \in V(H_3)$ and $u \in S_{k-1}$ are adjacent in X_k if and only if v and u are adjacent in X_{k-1}. Thus, we have $|V(H_3) \cap A_k| \geq 2^{k-1}$ for any adjacency resolving set A_k of X_k by the inductive hypothesis.

If $|V(H_1) \cap A_k| = 0$, then we must have $|V(H_2) \cap A_k| \geq 2^{k-1}$, in order to distinguish all of the vertices in H_2. If instead $|V(H_1) \cap A_k| = x$ for some positive integer x, then we must have $|V(H_2) \cap A_k| \geq 2^{k-1} - x$, since every vertex in $V(H_1) \cap A_k$ reaches at most one vertex in H_2. Thus, any adjacency resolving set A_k of X_k must have at least 2^{k-1} vertices in $V(H_1) \cup V(H_2)$. We have

$$|(V(X_k) - S_k) \cap A_k| = |V(H_3) \cap A_k| + |(V(H_1) \cup V(H_2)) \cap A_k| \geq 2^k$$

for any adjacency resolving set A_k of X_k, which completes the induction.

Thus, we have $|A_k| \geq 2^k$ for any adjacency resolving set A_k of X_k, so $\mathrm{adim}(X_k) = 2^{\Omega(k)}$. □

Combining Lemma 5.3 and Lemma 5.4, we have proven Theorem 5.1. We note that our construction of graph X_k has broadcast dimension that is asymptotically optimal in both its order and its adjacency dimension:

Remark 5.5. There does not exist a family of graphs $\{G_k\}_{k\in\mathbb{Z}^+}$ with $\mathrm{bdim}(G_k) = \Theta(k)$ and $\mathrm{adim}(G_k) = 2^{\omega(k)}$ for every $k \in \mathbb{Z}^+$ because $\mathrm{bdim}(G) = \Omega(\log n)$ for all graphs G of order n by Theorem 1.2.

Our result in Theorem 5.1 directly implies Theorem 1.5 and resolves Question 1.4 affirmatively. Furthermore, we can also answer Question 1.4 for acyclic graphs:

Remark 5.6. By Corollary 4.6, there does not exist a family of acyclic graphs $\{G_k\}_{k\in\mathbb{Z}^+}$ with $\mathrm{bdim}(G_k) = \Theta(k)$ and $\mathrm{adim}(G_k) = 2^{\Omega(k)}$ for every $k \in \mathbb{Z}^+$.

6 Edge Deletion

Throughout this section, we let v and e, respectively, denote a vertex and an edge of a connected graph G such that $G - v$ and $G - e$ are also connected graphs. Geneson and Yi [13] constructed families of graphs that demonstrated that both $\frac{\mathrm{bdim}(G)}{\mathrm{bdim}(G-v)}$ and $\mathrm{bdim}(G-v)-\mathrm{bdim}(G)$ can be arbitrarily large. In this section, we prove analogues of their results for the effect of edge deletion on the broadcast dimension of a graph. We prove Theorem 1.8 and Theorem 1.9, which state that both $\mathrm{bdim}(G) - \mathrm{bdim}(G - e)$

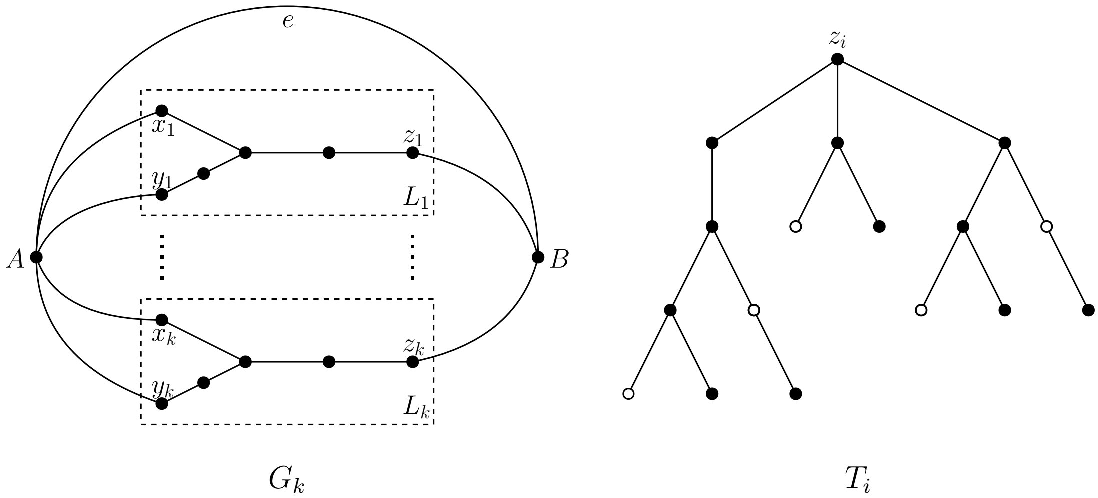

Figure 5: A graph G_k such that $\mathrm{bdim}(G_k) - \mathrm{bdim}(G_k - e) = \Omega(k)$. For every $1 \le i \le k$, vertex z_i is the root of a copy of tree T_i, shown on the right, so $|L_i| = 22$ and the degree of z_i is 5 for each $1 \le i \le k$.

and $\mathrm{bdim}(G-e) - \mathrm{bdim}(G) - d_{G-e}(u,v)$ can be arbitrarily large for $e = uv \in E(G)$. We do so by finding families of graphs that demonstrate these results. We also show that the ratio $\frac{\mathrm{bdim}(G-e)}{\mathrm{bdim}(G)}$ is bounded from above by 3, proving Theorem 1.10.

In the following theorem, we resolve Question 1.6 affirmatively by constructing a family of graphs that uses ideas from a graph constructed by Eroh et al. in [10], which they used to show that $\dim(G) - \dim(G-e)$ can be arbitrarily large.

Theorem 1.8. *The value* $\mathrm{bdim}(G) - \mathrm{bdim}(G-e)$ *can be arbitrarily large.*

Proof. Let $k \ge 2$ be an integer, and let G_k be the graph in Figure 5 with $e = AB$. For each $1 \le i \le k$, let *layer* L_i be the set of vertices indicated in Figure 5.

We define function $g : V(G_k - e) \to \mathbb{Z}^+ \cup \{0\}$ as follows:

$$g(v) = \begin{cases} 3 & \text{if } v = A, \\ 4 & \text{if } v = z_i \text{ for } 1 \le i \le k, \\ 1 & \text{if } v \text{ is a vertex on tree } T_i \text{ shown with an open circle in Figure 5,} \\ 0 & \text{otherwise.} \end{cases}$$

Because g is a resolving broadcast of graph $G_k - e$, we can upper bound the broadcast dimension of graph $G_k - e$: we have $\mathrm{bdim}(G_k - e) \le c_g(G_k - e) = 3 + 9k$.

Let $f : V(G_k) \to \mathbb{Z}^+ \cup \{0\}$ be a resolving broadcast of the graph G_k. For every pair of distinct vertices $u_1, u_2 \in V(G_k)$ with $d(u_1, A) = d(u_2, A)$ and $d(u_1, B) = d(u_2, B)$, at least one of u_1 or u_2 must be reached by a vertex in $\mathrm{supp}_{G_k}(f)$ that is on the same layer since otherwise we would have $b_f(u_1) = b_f(u_2)$. Thus, at most

$$\max_{u,v \in V(G_k)} (d(u,A)+1)(d(v,B)+1) + 1 = O(1)$$

layers of graph G_k can have a vertex that is not reached by any vertex on the same layer. The following properties must hold for the remaining $k - O(1)$ layers L_i.

1. Every vertex $v \in L_i$ is reached by a vertex in $L_i \cap \mathrm{supp}_{G_k}(f)$.
2. We have $\mathrm{supp}_{G_k}(f) \cap (L_i - V(T_i)) \neq \emptyset$, since otherwise $b_f(x_i) = b_f(y_i)$.
3. Any distinct $u, v \in V(T_i)$ with $d(u, z_i) = d(v, z_i)$ must be resolved by a vertex in L_i since $d(u, A) = d(v, A)$ and $d(u, B) = d(v, B)$.

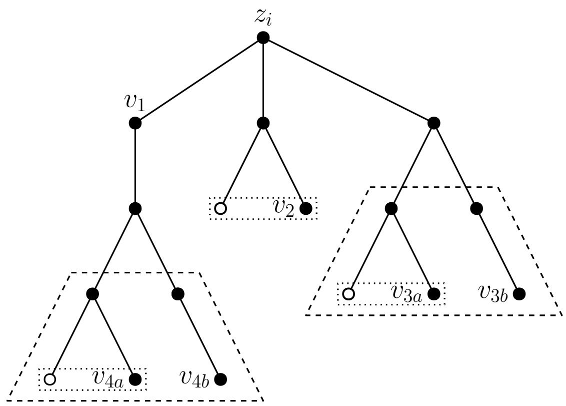

Figure 6: Tree T_i labeled for casework reference.

Refer to Figure 6 for the remainder of the proof. There are three pairs of twin vertices on tree T_i (see dotted rectangular boxes). By Observation 2.5, at least one of the vertices in each of these pairs must be in $\mathrm{supp}_{G_k}(f)$. Without loss of generality, let the three vertices that are denoted with an open circle be in $\mathrm{supp}_{G_k}(f)$. The total value assigned to each of the two groups of five vertices identified by dashed trapezoidal boxes must be at least 2 in order for the three vertices that are the same distance away from z_i in each of those groups to be distinguished.

Any assignment of a total value of 5 to T_i subject to the above constraints leaves at least four unreached vertices: v_1, v_2, v_{3a} or v_{3b}, and v_{4a} or v_{4b}. These four vertices must be reached by assigning an additional total value of at least 4 to the vertices on tree T_i (in addition to the positive value assigned to some vertex in $L_i - V(T_i)$), or by assigning an additional total value of $c < 4$ to the vertices on tree T_i and at least $5 - c$ to a vertex $v \in L_i - V(T_i)$. In either case, a total value of at least 10 must be assigned to the vertices on such a layer L_i.

Because a total value of at least 10 is assigned to at least $k - O(1)$ layers of G_k by any resolving broadcast f of G_k, we have $\mathrm{bdim}(G_k) - \mathrm{bdim}(G_k - e) \geq 10k - 9k - O(1) = \Omega(k)$. $\square$

We will prove Theorem 1.9 by constructing a family of graphs that shows that $\mathrm{bdim}(G - e) - \mathrm{bdim}(G)$ can be arbitrarily larger than $d_{G-e}(u, v)$, thus showing that the bound proposed in Question 1.7 can fail. Since we use both spider graphs and the graph F_k (see Definition 4.7) in the graph construction, we begin with two lemmas: a lemma about graphs containing spiders as subgraphs and a lemma about the graph F_k.

As we will be working with a specific family of spider graphs in the proof of Theorem 1.9, we introduce our notation for spider graphs:

Definition 6.1. A tree is called a *spider* if every vertex, except for one vertex known as the *center vertex*, has degree at most two. A *leg* of a spider graph is a path connected to the center vertex. We denote by $SP\left(\ell_1^{(x_1)}, \ldots, \ell_m^{(x_m)}\right)$ a spider of order n with x_i legs of length ℓ_i for every $1 \leq i \leq m$, where $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_m$ and $1 + \sum_{i=1}^{m} \ell_i x_i = n$.

Lemma 6.2. *For any integer $k > 1$, let G be a graph that contains spider $SP\left(3k^{(6k)}\right)$ with center c as a subgraph such that c is the only vertex on the spider that is adjacent to any vertex of the graph that is not on the spider. If a resolving broadcast f of G is efficient, then there exists a vertex $z \in \operatorname{supp}_G(f)$ with $f(z) - d(c, z) \geq 3k - 2$.*

Proof. For the sake of contradiction, consider an efficient resolving broadcast f where there is not a $z \in \operatorname{supp}_G(f)$ with $f(z) - d(c, z) \geq 3k - 2$. On each leg of the spider $SP\left(3k^{(6k)}\right)$, the three vertices farthest from c are only reached by vertices on the same leg.

Let $u, v \in V(G)$ be two distinct vertices on the legs of the spider with $d(u, c) = d(v, c)$. If neither u nor v are reached by a vertex that is on the same leg as u or v, then $b_f(u) = b_f(v)$. Thus, every vertex on the legs of the spider, except at most $3k$ of them (one vertex of each distinct distance from c) must be reached by another vertex on the same leg. On at least $6k - 3k = 3k$ legs of the spider, all vertices need to be reached by a vertex on the same leg; let L be the set of these legs.

A vertex $v \in \operatorname{supp}_G(f)$ on a leg of the spider can reach at most $2f(v) + 1$ vertices on the same leg, and we have that $2f(v) + 1 \leq 3f(v)$ with equality if and only if $f(v) = 1$. Because all of the vertices on a leg $\ell \in L$ need to be reached by a vertex on ℓ, the total value v_ℓ assigned to the $3k$ vertices on ℓ must be at least k. If $v_\ell = k$, then we must have the following assignment: the vertices on ℓ that are distance $3i - 1$ from c for $1 \leq i \leq k$ are assigned a value of 1, and the rest of the vertices on ℓ are assigned 0. However, with this assignment, there are two vertices on ℓ that have the same broadcast representation: the vertex that is distance $3k - 2$ from c and the vertex that is distance $3k$ from c are only reached by the vertex between them. Thus, $v_\ell \geq k + 1$ for every $\ell \in L$.

Consider function $f' : V(G) \to \mathbb{Z}^+ \cup \{0\}$ defined as follows. Let $f'(v) = f(v)$ for all vertices that are not on a leg in L, and let $f'(c) = 3k - 2$. The vertices on a leg in L that are distance $3i - 1$ from c for $1 \leq i \leq k$ are assigned a value of 1, and the rest of the vertices on a leg in L are assigned 0. We note that f' is a resolving broadcast of G. Moreover, we have $c_{f'}(G) < c_f(G)$ because $f'(c) - f(c) \leq 3k - 2$, and for each of the $3k$ legs $\ell \in L$, we have $\sum_{v \in \ell} f(v) - \sum_{v \in \ell} f'(v) \geq 1$. This contradicts the efficiency of f. $\square$

Lemma 6.3. *For any resolving broadcast f of F_k (see Definition 4.7) with $f(v_k) \geq k$, we have*

$$\sum_{v \in V(F_k)} f(v) \geq f(v_k) + 2k - O(1).$$

Proof. Let f be a resolving broadcast of F_k that minimizes $\sum_{v\in V(F_k)} f(v)$, under the constraint that $f(v_k) \geq k$. For $u \in V(F_k)$, we define $p(u) := \operatorname{argmin}_{i\in[0,k]} d\left(u, v_i\right)$.

Case 1. $f(w) - d\left(w, v_{p(w)}\right) \leq \left\lceil \frac{k}{2} \right\rceil$ for all $w \in V(F_k) - \{v_k\}$.
Define $f' : V(F_k) \to \mathbb{Z}^+ \cup \{0\}$ such that $f'(v_k) = f(v_k)$ and $f'(v) = 0$ for all other $v \in V(F_k)$. The number of unique broadcast representations of the vertices of F_k with respect to function f', denoted $B_{F_k}(f')$, is $k+1$. Updating $f'(w) \leftarrow f(w)$ for any $w \in \operatorname{supp}_{F_k}(f) - \{v_k\}$ introduces at most $f(w)$ new unique broadcast representations to the vertices u with $p(u) < p(w)$. Thus, every update $f'(w) \leftarrow f(w)$ for some $w \in \operatorname{supp}_{F_k}(f) - \{v_k\}$ increases $B_{F_k}(f')$ by at most

$$O\left(f(w)\right) + \sum_{i=1}^{f(w)} i \leq f(w)\left(\frac{k}{4} + O(1)\right).$$

Since we must have $\frac{k^2}{2} + O(k)$ unique broadcast representations, the lemma holds in this case.

Case 2. There is a vertex $w \neq v_k$ with $f(w) - d\left(w, v_{p(w)}\right) > \left\lceil \frac{k}{2} \right\rceil$.
Let t be the vertex on F_k farthest from vertex v_0. We must have $d(w,t) - f(w) = O(1)$, since otherwise there would be $\omega(1)$ vertices u with $p(u) \geq p(w)$ not reached by w. These vertices would be most efficiently distinguished by increasing $f(w)$, contradicting the efficiency of f. The vertices u with $p(u) < p(w)$ must be distinguished with an additional total cost of at least $p(w) - O(1)$. Thus,

$$\sum_{v \in V(F_k)} f(v) \geq f(v_k) + d(w,t) + p(w) - O(1) \geq f(v_k) + 2k - O(1),$$

as desired. □

With Lemma 6.2 and Lemma 6.3, we can prove Theorem 1.9.

Theorem 1.9. *The value* $\operatorname{bdim}(G-e) - \operatorname{bdim}(G)$ *can be arbitrarily larger than* $d_{G-e}(u,v)$*, where* $e = uv \in E(G)$.

Proof. For integer $k \geq 2$, let H_k be the graph in Figure 7, and let $e = v_i v_{i+1}$, where $i = \left\lfloor \frac{3k-2}{2} \right\rfloor$. Let S_1 and S_2 be the spider $SP\left(3k^{(6k)}\right)$ centered at v_0 and v_{3k-2}, respectively. For $u \in V(H_k)$, we define $p(u) := \operatorname{argmin}_{i\in[0,k]} d_{H_k}(u, v_i)$. We will show that for sufficiently large k, we have

$$\operatorname{bdim}\left(H_k - e\right) - \operatorname{bdim}(H_k) = d_{H_k - e}(v_i, v_{i+1}) + \Omega(k) = \frac{k}{2} + \Omega(k).$$

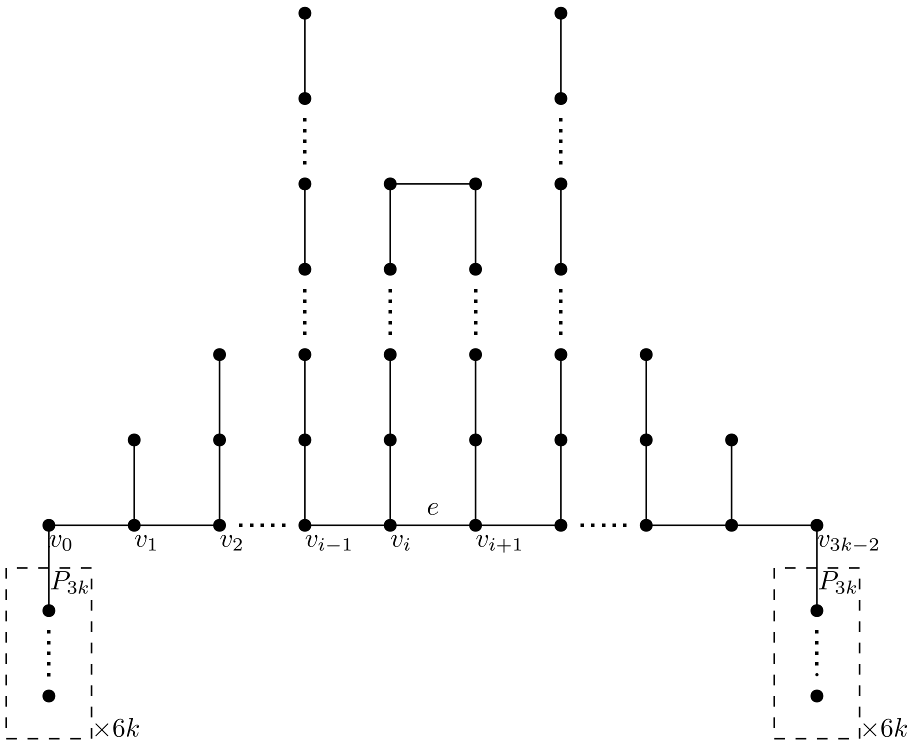

Figure 7: A graph H_k such that $\mathrm{bdim}(H_k - e) - \mathrm{bdim}(H_k)$ can be arbitrarily larger than $d_{H_k-e}(v_i, v_{i+1})$, where $e = v_iv_{i+1}$ and $i = \left\lfloor \frac{3k-2}{2} \right\rfloor$. The vertices $v_0, \dots, v_{3k-2}$ are on a path. Additionally, each v_j with $1 \le j \le i-1$ is connected to a path P_j; each v_j with $i+2 \le j \le 3k-3$ is connected to a path P_{3k-2-j}, and vertices v_i and v_{i+1} are on a cycle of length $\left\lfloor \frac{k}{2} \right\rfloor$. Finally, v_0 and v_{3k-2} are both centers of a copy of spider $SP\left(3k^{(6k)}\right)$.

Let $B = \mathrm{bdim}\left(SP\left(3k^{(6k)}\right)\right)$. Let $g : V(H_k) \to \mathbb{Z}^+ \cup \{0\}$ be the function that applies an efficient resolving broadcast of $SP\left(3k^{(6k)}\right)$ to S_1 and S_2 on graph H_k. By Lemma 6.2, there are vertices z_1' on S_1 and z_2' on S_2 with $g(z_1') - d_{H_k}(v_0, z_1') \ge 3k-2$ and $g(z_2') - d_{H_k}(v_{3k-2}, z_2') \ge 3k-2$. Function g is a resolving broadcast of H_k since every pair of distinct vertices in $V(H_k)$ that are on the same spider is clearly resolved, and every other pair of vertices is resolved by either z_1' or z_2'. Thus, $\mathrm{bdim}(H_k) \le 2B$.

Let f be an efficient resolving broadcast of the graph $H_k - e$. By Lemma 6.2, we must have vertices $z_1, z_2 \in \mathrm{supp}_G(f)$ with $f(z_1) - d_{H_k-e}(v_0, z_1) \ge 3k-2$ and $f(z_2) - d_{H_k-e}(v_{3k-2}, z_2) \ge 3k-2$.

Case 1. There does not exist a vertex z with

$$f(z) - d_{H_k-e}(v_0, z) \ge 3k-2 \quad \text{and} \quad f(z) - d_{H_k-e}(v_{3k-2}, z) \ge 3k-2.$$

In this case, z_1 and z_2 are distinct vertices. We define $c_1 := f(z_1) - 3k + 2$ and $d_1 := p(z_1)$, and we similarly define $c_2 := f(z_2) - 3k + 2$ and $d_2 := 3k - 2 - p(z_2)$. Note that $c_1 \ge d_1$ and $c_2 \ge d_2$ by Lemma 6.2.

If $d_1 > 0$, let T_1 be the F_{d_1} subgraph induced by the vertices u with $p(u) \le d_1$

that are not on a leg of spider S_1. Otherwise, let T_1 be the graph that consists of the singular vertex z_1. Let $y_1 = \text{argmax}_{V(S_1)-\{v_0\}}(f(y) - d(y, v_0))$. By Lemma 6.3, we must have

$$f(y_1) - d(y_1, v_0) + \sum_{v \in V(T_1)} f(v) \geq f(z_1) + 2d_1 + \max\{f(y_1) - d(y_1, v_0) - 2d_1, 0\} - O(1) \tag{3}$$

in order to distinguish the vertices of T_1.

On spider S_1, assigning vertex y_1 the value $f(y_1)$ only distinguishes at most $d(y_1, v_0) + f(y_1)$ vertices on the same leg of S_1. In an efficient resolving broadcast of S_1, those $d(y_1, v_0) + f(y_1)$ vertices would have instead been distinguished with a total cost of at most $\left\lceil \frac{d(y_1,v_0)+f(y_1)}{3} \right\rceil$ by assigning a value of 1 to every third vertex (see proof of Lemma 6.2). Thus, we can obtain the following bound on the total value assigned to the vertices in set $U_1 = V(S_1) - \{v_0, y_1, z_1\}$:

$$\sum_{v \in U_1} f(v) \geq B - g(z_1') - \frac{d(y_1, v_0) + f(y_1)}{3} - O(1). \tag{4}$$

Using (3) and (4), we lower bound the total value assigned to all of the vertices u with $p(u) \leq d_1$:

$$\begin{aligned}
& f(y_1) + \sum_{v \in V(T_1)} f(v) + \sum_{v \in U_1} f(v) \\
\geq\ & f(z_1) + 2d_1 + \max\{f(y_1) - d(y_1, v_0) - 2d_1, 0\} + d(y_1, v_0) + B - 3k \\
& \qquad - \frac{d(y_1, v_0) + f(y_1)}{3} - O(1) \\
\geq\ & c_1 + 2d_1 + d(y_1, v_0) + B - \frac{d(y_1, v_0) + (d(y_1, v_0) + 2d_1)}{3} - O(1) \\
\geq\ & c_1 + \frac{4d_1}{3} + B - O(1).
\end{aligned}$$

In this case, the sum of the values assigned to vertices u with $p(u) > d_1$ must be at least B in order to distinguish the vertices of spider S_2. Thus, we have

$$\text{bdim}(H_k - e) - \text{bdim}(H_k) \geq \left(\sum_{v \in V(H_k - e)} f(v) \right) - 2B \geq c_1 + \frac{4}{3} d_1 - O(1).$$

If $d_1 \geq \frac{k}{4}$, then we have

$$\text{bdim}(H_k - e) - \text{bdim}(H_k) \geq c_1 + \frac{4d_1}{3} - O(1) \geq \frac{7d_1}{3} - O(1) \geq \frac{7k}{12} - O(1) = \frac{k}{2} + \Omega(k),$$

as desired. By symmetry, if $d_2 \geq \frac{k}{4}$, then we are also done.

Now, we consider $d_1, d_2 < \frac{k}{4}$. The $\frac{k^2}{2} \pm O(k)$ vertices in region A_2 (see Figure 8) are all reached by z_2, and all but $O(k)$ of them must be reached by another vertex that is not in B_2 in order to be distinguished. Additionally, at least $\frac{k^2}{2} - O(k)$ of the vertices in A_1 must be reached by vertices not in B_1 in order to be distinguished. Vertex z_1 reaches at most $(c_1 + d_1 + O(1))k$ of the vertices in A_2, and the total value assigned to the vertices in B_1 is at least $B + c_1 + \frac{4}{3}d_1 - O(1)$. Similarly, vertex z_2 reaches at most $(c_2 + d_2 + O(1))k$ of the vertices in A_1, and the total value assigned to the vertices in B_2 is at least $B + c_2 + \frac{4}{3}d_2 - O(1)$. Any vertex v that is not in B_1 or B_2 has $g(v) = 0$ and reaches at most $k \cdot f(v) + O(1)$ of the vertices in $A_1 \cup A_2$. Thus, in this case we have

$$\mathrm{bdim}(H_k - e) - \mathrm{bdim}(H_k) \geq \frac{1}{k}\left(|A_1 \cup A_2| - O(k)\right) = k - O(1) = \frac{k}{2} + \Omega(k).$$

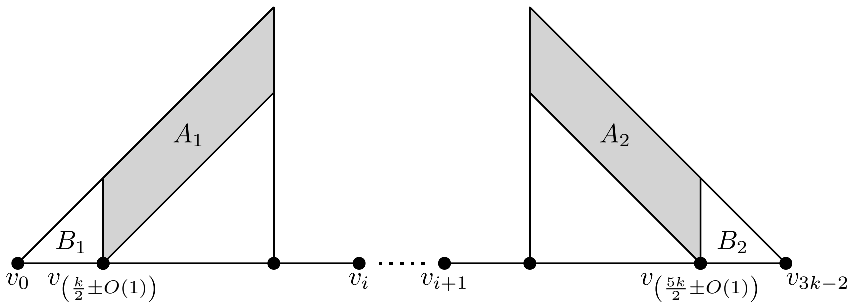

Figure 8: A geometric interpretation of graph $H_k - e$. The spiders centered at v_0 and v_{3k-2} (not pictured) are also in B_1 and B_2, respectively.

Case 2. There exists a vertex z with

$$f(z) - d_{H_k - e}(v_0, z) \geq 3k - 2 \quad \text{and} \quad f(z) - d_{H_k - e}(v_{3k-2}, z) \geq 3k - 2.$$

The assumption in this case directly implies that

$$f(z) \geq \frac{d_{H_k - e}(v_0, v_{3k-2})}{2} + 3k - O(1) = 4.75k - O(1).$$

Without loss of generality, we assume $p(z) \geq i$. Let T_1 be the F_{i-1} subgraph induced by the vertices u with $p(u) \leq i - 1$ that are not on a leg of spider S_1. By the same reasoning as in the first case, in order for the vertices on T_1 to be distinguished, an additional total value of $\frac{4}{3} \cdot d(v_0, v_{i-1}) - O(1)$ must be assigned to the vertices u with $p(u) < i$. Thus, we have

$$\begin{aligned} \mathrm{bdim}(H_k - e) - \mathrm{bdim}(H_k) &\geq f(z) - g(z_1') - g(z_2') + \frac{4}{3} \cdot \frac{3k}{2} - O(1) \\ &\geq 4.75k - 6k + 2k - O(1) \\ &= \frac{k}{2} + \Omega(k), \end{aligned}$$

as desired. □

While the value $\mathrm{bdim}(G-e)-\mathrm{bdim}(G)$ can be arbitrarily large, the ratio $\frac{\mathrm{bdim}(G-e)}{\mathrm{bdim}(G)}$ is bounded. We prove this below, using some ideas from the proof that $\dim(G-e) \leq \dim(G)+2$ in [10]. Recall that a *geodesic* is a shortest path between two points.

Theorem 1.10. *For all graphs G and any edge $e \in E(G)$, we have $\frac{\mathrm{bdim}(G-e)}{\mathrm{bdim}(G)} \leq 3$.*

Proof. Let f be an efficient resolving broadcast of G, and let vertices u and v be the endpoints of edge e. Let $b = \max_{v\in V(G)} f(v)$. We will show that function f', which is identical to f, except with $f'(u) = f'(v) = b$, is a resolving broadcast of $G - e$. Then, we will be done since

$$3\,\mathrm{bdim}(G) = 3 \sum_{w\in V(G)} f(w) \geq \sum_{w\in V(G-e)} f'(w) \geq \mathrm{bdim}(G-e).$$

Let $z \in \mathrm{supp}_G(f)$, and let x and y be two vertices with $d_{f(z)}(x, z) \neq d_{f(z)}(y, z)$ in graph G. Suppose that x and y are no longer resolved by z after the edge e is deleted; that is, $d_{f(z)}(x, z) = d_{f(z)}(y, z)$ in graph $G-e$. Then, we must have $d_G(u, z) \neq d_G(v, z)$ since removing edge $e = uv$ increases the distance from z to at least one vertex in the graph. Without loss of generality, we assume that $d_G(v, z) < d_G(u, z)$.

We consider two cases and show that u resolves x and y in graph $G - e$ in both cases; that is, we show that we have $d_{f'(u)}(x, u) \neq d_{f'(u)}(y, u)$ in graph $G - e$ in both cases.

Case 1. Removing edge e only increases the distance from z to one of x and y (say x).
Edge e must lie on every $x - z$ geodesic in G. Since $d_G(v, z) < d_G(u, z)$, we have an $x - u$ geodesic in G that does not go through edge e. Moreover, we have

$$f'(u) \geq f(z) \geq \min\{d_G(x, z), d_G(y, z)\} = d_G(x, z) \geq d_G(x, u) = d_{G-e}(x, u).$$

The above inequality shows that u reaches x with respect to f' in graph $G-e$. Thus, it remains to be shown that $d_{G-e}(x, u) \neq d_{G-e}(y, u)$ in this case.

Subcase 1. $f(z) \geq d_G(y, z) = d_{G-e}(y, z)$.
In this subcase, $d_{f(z)}(x, z) = d_{f(z)}(y, z)$ in graph $G - e$ implies that $d_{G-e}(x, z) = d_{G-e}(y, z)$, and so

$$\begin{aligned} d_{G-e}(x, u) &= d_G(x, u) = d_G(x, z) - d_G(z, u) < d_{G-e}(x, z) - d_G(z, u) \\ &= d_{G-e}(y, z) - d_G(z, u) = d_G(y, z) - d_G(z, u) \leq d_G(y, u) \\ &\leq d_{G-e}(y, u). \end{aligned}$$

Subcase 2. $f(z) < d_G(y, z) = d_{G-e}(y, z)$.
If $d_{G-e}(x, u) = d_{G-e}(y, u)$, then we have

$$\begin{aligned} d_G(y, z) &\leq d_G(y, u) + d_G(u, z) \leq d_{G-e}(y, u) + d_G(u, z) \\ &= d_{G-e}(x, u) + d_G(u, z) = d_G(x, u) + d_G(u, z) \\ &= d_G(x, z) \leq f(z), \end{aligned}$$

a contradiction.

Case 2. Removing edge e increases the distance from z to both x and y.
Edge e must lie on every $x-z$ geodesic and every $y-z$ geodesic in graph G. Since $d_G(v,z) < d_G(u,z)$, we have $d_G(u,x) < d_G(v,x)$ and $d_G(u,y) < d_G(v,y)$. Because z resolves x and y in G, at least one of x and y (say x) is reached by z in G. Then,

$$f'(u) \geq f(z) \geq d_G(x,z) \geq d_G(x,u) = d_{G-e}(x,u)$$

and

$$d_{G-e}(x,u) = d_G(x,u) \neq d_G(y,u) = d_{G-e}(y,u),$$

so vertex u resolves vertices x and y in graph $G-e$. □

7 Future Work

In Corollary 4.5, we showed that $\operatorname{adim}(G) = \Omega(\sqrt{n})$ for all acyclic graphs G of order n. To our knowledge, the best such lower bound before our work is the $\Omega(\log n)$ bound on the adjacency dimension of general graphs of order n given by Geneson and Yi in Theorem 1.2, which they showed to be asymptotically optimal using a family of graphs constructed by Zubrilina in [22]. We ask if our lower bound on the adjacency dimension of acyclic graphs is asymptotically optimal.

Question 7.1. Is there a family of acyclic graphs $\{G_k\}_{k\in\mathbb{Z}^+}$ with $\operatorname{adim}(G_k) = \Theta\left(\sqrt{|V(G_k)|}\right)$ for every $k \in \mathbb{Z}^+$?

The bounds that we derived in Theorem 4.3 and Theorem 1.10 are sharp up to a constant factor. Sharper bounds may be obtained by examining the steps of the proofs more carefully. Additionally, it would be interesting to determine the exact broadcast dimension of some special graphs for which the broadcast dimension is currently only known up to a constant factor.

Question 7.2. What is the broadcast dimension of the grid graph $P_m \square P_n$?

Question 7.3. What is the broadcast dimension of the graph F_k from Definition 4.7?

We note that the broadcast dimension of the grid graph $P_m \square P_n$ is at most $2m+2n$: for paths $P_m : x_1, x_2, \ldots x_m$ and $P_n : y_1, y_2, \ldots y_n$, the function f that assigns $m+n$ to (x_1, y_1) and (x_1, y_n) and assigns 0 to the rest of the vertices is a resolving broadcast of $P_m \square P_n$. Additionally, the broadcast dimension of F_k is at most $3k$: function f with $f(v_0) = 2k$, $f(v_k) = k$, and $f(w) = 0$ for all $w \in V(F_k) - \{v_0, v_k\}$ is a resolving broadcast of F_k. Lemma 6.3 makes partial progress towards finding the broadcast dimension of F_k.

In Section 6, we show that both $\operatorname{bdim}(G-e) - \operatorname{bdim}(G)$ and $\operatorname{bdim}(G) - \operatorname{bdim}(G-e)$ can be arbitrarily large and that $\frac{\operatorname{bdim}(G-e)}{\operatorname{bdim}(G)} \leq 3$ for all graphs G and any edge $e \in E(G)$. These results naturally lead us to ask the following question:

Question 7.4. Is $\frac{\mathrm{bdim}(G)}{\mathrm{bdim}(G-e)}$ bounded from above for all graphs G and any edge $e \in E(G)$?

On a similar note, Geneson and Yi showed in [13] that both $\frac{\mathrm{bdim}(G)}{\mathrm{bdim}(G-v)}$ and $\mathrm{bdim}(G-v) - \mathrm{bdim}(G)$ can be arbitrarily large. The corresponding problem for $\frac{\mathrm{bdim}(G-v)}{\mathrm{bdim}(G)}$ remains open.

Question 7.5. Is $\frac{\mathrm{bdim}(G-v)}{\mathrm{bdim}(G)}$ bounded from above for all graphs G and any vertex $v \in V(G)$?

To better understand how metric dimension and broadcast dimension compare to each other, it would be interesting to derive more properties of broadcast dimension that are analogues to known properties of metric dimension. For example:

Question 7.6. For a graph G and $n \in \mathbb{Z}^+$, bound $\mathrm{bdim}(G \square P_n)$ and $\mathrm{bdim}(G \square C_n)$ in terms of some function of G and n.

Question 7.7. For graphs G and H, bound $\mathrm{bdim}(G \square H)$ in terms of some function of G and H.

Question 7.8. Is determining the broadcast dimension of a graph an NP-hard problem?

It is NP-hard to determine the metric dimension and adjacency dimension of a general graph (see [12], [11], respectively). Determining the domination number of a general graph is also an NP-hard problem [12]. Heggernes and Lokshtanov [15] found a polynomial-time algorithm for computing the broadcast domination number of arbitrary graphs, and both the domination number and broadcast domination number of a tree can be determined in linear time (see [8],[9], respectively). We ask the corresponding question for the broadcast dimension of trees.

Question 7.9. Is there a polynomial-time algorithm for determining the value of $\mathrm{bdim}(T)$ for every tree T?

We refer to [13] for more open questions about broadcast dimension. Finally, we note that it would also be interesting to study the broadcast dimension of directed graphs and graphs with weighted edges.

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