AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **85(3)** (2023), Pages 308–312

A note on the number of regions in a line arrangement

Dickson Y. B. Annor* Michael S. Payne†

Department of Mathematical and Physical Sciences
La Trobe University
Bendigo, Victoria 3552
Australia
d.annor@latrobe.edu.au m.payne@latrobe.edu.au

Abstract

For an arrangement of n lines in the real projective plane, we denote by f the number of regions into which the real projective plane is divided by the lines. Using Bojanowski's inequality (2003), we establish a new lower bound for f. In particular, we show that if no more than $\frac{2}{3}n$ lines intersect at any point, then $f \geq \frac{1}{6}n^2$.

1 Introduction

Let $\mathcal{L}$ be an arrangement of $n \geq 2$ lines in the real projective plane $\mathbb{RP}^2$ and let m denote the maximum number of lines from $\mathcal{L}$ intersecting at one point. The lines from $\mathcal{L}$ divide $\mathbb{RP}^2$ into polygonal regions which are the connected components of the complement of the union of the lines. Denote the number of regions by f. The question we are interested in is: how many regions can be obtained (under all possible arrangements $\mathcal{L}$ of n lines)?

Below we collect some known lower bounds for f in terms of n and m.

- $f \geq 2n - 2$, if $m < n$, Grünbaum [5];
- $f \geq 3n - 6$, if $m \leq n - 2$, Grünbaum [5];
- $f \geq m(n + 1 - m)$, Arnol'd [1];
- $f \geq \frac{n(n-1)}{2(m-1)}$, if $m > 2$, Arnol'd [1];

* Dickson Annor is supported by a La Trobe Graduate Research Scholarship.
† Michael Payne is supported by a DECRA from the Australian Research Council.

ISSN: 2202-3518

- $f \geq (m+1)(n-m)$, Arnol'd [1] and Purdy [9];
- $f \geq (r+1)(n-r)$, if $m \leq n-r$ and $n \geq \frac{r^2+r}{2}+3$ for some $r \in \mathbb{Z}$, Shnurnikov [10];
- $f \geq 2\left(\frac{n^2-n+2m}{m+3}\right)$, Shnurnikov [10];
- $f \geq \frac{(3m-10)n^2+(m^2-6m+12)n}{m^2+3m-18}+1$, if $5 \leq m < n-2$, Shnurnikov [11].

In this paper we use Bojanowski's inequality [2] to establish a new lower bound for f. This inequality, which we introduce in Section 2, was derived from the work of Langer [12] and has taken some time to become widely known by the combinatorics community [8, 13]. Our main result states that:

Theorem 1 *Let $\mathcal{L}$ be an arrangement of n lines in the real projective plane such that $m \leq \frac{2}{3}n$. Then*

$$f \geq \frac{(m+2)n^2+(3m-6)n}{6m}+1 \geq \frac{1}{6}n^2.$$

We remark that to the best of our knowledge, if $m(n)$ is a sublinear but increasing function of n, then this is the first quadratic lower bound on f. For example consider the case $m=\sqrt{n}$ in the previously known inequalities given above.

2 Bounds for Number of Regions

For an arrangement of lines $\mathcal{L}$ in the projective plane we denote by t_k, $2 \leq k \leq n-1$, the number of intersection points where exactly k lines of the arrangement are incident. The following are some known relations for values of t_k.

- $t_2 \geq 3+\sum_{k\geq 4}(k-3)t_k$, Melchior [7];
- $t_2 \geq \frac{6}{13}n$ for $n \geq 8$, Csima and Sawyer [3];
- $t_2+\frac{3}{4}t_3 \geq n+\sum_{k\geq 5}(2k-9)t_k$, if $t_{n-1}=t_{n-2}=0$, Hirzebruch [6];
- $t_2+\frac{3}{4}t_3 \geq n+\sum_{k\geq 5}\left(\frac{1}{4}k^2-k\right)t_k$, if $t_k=0$ for $k>\frac{2}{3}n$, Bojanowski [2];
- $t_2 \geq \frac{1}{2}n$ and $t_2 \geq 3\lfloor\frac{1}{4}n\rfloor$ for sufficiently large, even and odd n, respectively, Green and Tao [4].

Perhaps it is worth mentioning here that both Bojanowski [2] and Hirzebruch [6] inequalities hold for arrangements of complex lines in the complex projective plane and consequently, they also hold for arrangements of lines in the real projective plane. To the best of our knowledge, Bojanowski's inequality [2] is the strongest known inequality for line arrangements with $m \leq \frac{2}{3}n$ [8].

Proof of Theorem 1. Let $\mathcal{L}$ be an arrangement of n lines. If we add lines one by one, then the number of new regions created by each line is equal to the number of intersection points with previously added lines. In this process, a point with k lines passing through it is intersected $k-1$ times. Thus, the number of regions, including 1 for the first line, is

$$f = 1 + \sum_{k=2}^{m}(k-1)t_k. \tag{1}$$

Note that (1) can be obtained by using the fact that the Euler characteristic of the real projective plane is 1. The number of pairs of lines in $\mathcal{L}$ is equal to $\frac{n(n-1)}{2}$. In a projective plane, every pair of lines intersects at exactly one point, and if k lines meet at a point, we get $\frac{k(k-1)}{2}$ of such pairs which cross at that point. Since $t_k = 0$ for $k > m$, we obtain

$$n(n-1) = \sum_{k=2}^{m} k(k-1)t_k. \tag{2}$$

Suppose we are given an inequality

$$\sum_{k=2}^{m} \alpha_k t_k \geq \alpha_0 \tag{3}$$

where $\alpha_0, \alpha_2, \alpha_3, \ldots, \alpha_m$ are some real numbers, and suppose that for some $c_1, c_2 > 0$ the inequality

$$c_1 k(k-1) + c_2\alpha_k \leq k-1 \tag{4}$$

is satisfied for all $2 \leq k \leq m$. Multiply both sides of (4) by t_k and sum up for $k = 2, 3, \ldots, m$ to obtain

$$c_1 \sum_{k=2}^{m} k(k-1)t_k + c_2 \sum_{k=2}^{m} \alpha_k t_k \leq \sum_{k=2}^{m}(k-1)t_k$$

since $t_k \geq 0$. This is equivalent to

$$c_1 n(n-1) + c_2 \sum_{k=2}^{m} \alpha_k t_k \leq f-1. \tag{5}$$

Using (3) and (5) and the fact that $c_2 > 0$, we obtain

$$f \geq c_1 n(n-1) + c_2\alpha_0 + 1 \tag{6}$$

for positive c_1, c_2, satisfying (4). For $m \leq \frac{2}{3}n$ we use Bojanowski's inequality [2]

$$t_2 + \frac{3}{4}t_3 + \sum_{k\geq 5}\left(k - \frac{1}{4}k^2\right)t_k \geq n$$

in the form (3) to obtain the following

$$\alpha_0 = n, \quad \alpha_2 = 1, \quad \alpha_3 = \frac{3}{4}, \quad \alpha_4 = 0, \quad \alpha_k = \left(k - \frac{1}{4}k^2\right) \quad \text{for } k \geq 5.$$

From (4) we get

$$1 \geq 2c_1 + c_2, \quad 2 \geq 6c_1 + \frac{3}{4}c_2, \quad 3 \geq 12c_1, \quad \text{for } k = 2, 3, \text{and } 4, \text{respectively},$$
$$0 \geq c_1 k(k-1) + c_2\left(k - \frac{1}{4}k^2\right) - (k-1) \quad \text{for} \quad 5 \leq k \leq m.$$

For $m \geq 2$, let us take the positive numbers

$$c_1 = \frac{m+2}{6m}, \qquad c_2 = \frac{2(m-1)}{3m}.$$

Now we need to check these inequalities for $2 \leq k \leq m$ and for the given c_1, c_2. The first three are easy to check, so we verify the last one for $5 \leq k \leq m$. Thus,

$$c_1 k(k-1) + c_2\left(k - \frac{1}{4}k^2\right) - (k-1) = \frac{0.5(k-2)(k-m)}{m} \leq 0,$$

because $k \leq m$ and $k \geq 5$. So, we obtain (6) for the given c_1, c_2, and hence, the inequality of the theorem. □

Note that lower bounds on f in the form (6) were obtained by Shnurnikov in [11]. In [11] he applied Hirzebruch's inequality [6] to obtain the result mentioned in Section 1.

It is natural to ask under which assumptions the inequality of Theorem 1 is stronger than previously known inequalities. The inequality in Theorem 1 is quadratic in n. So, it suffices to compare it to those inequalities mentioned in Section 1 that are quadratic in n for some function $m(n)$. In particular, the results of Arnol'd [1] and Purdy [9] become quadratic in n if $m(n) = \frac{n}{p}$ where p is a real number greater than 1. A simple calculation shows that Theorem 1 is weaker than those inequalities when $p \in \left(3 - \sqrt{3}, 3 + \sqrt{3}\right)$ and it is also weaker than the second of the two listed inequalities from Shnurnikov [10] when $m \leq 5$. On the other hand, Theorem 1 is stronger than all the inequalities mentioned in Section 1 whenever $7 \leq m \leq \dfrac{n}{5}$ and for $m = 6$ we have equality with Shnurnikov [11].

Acknowledgements

We are grateful to Yuri Nikolayevsky for his interest in this work and for his useful comments and suggestions which improved the presentation. Also, we are thankful to the referees for their careful reading and valuable suggestions that improved the paper.

References

[1] V. I. Arnol'd, Into how many regions do n lines divide the plane?, *Matem. Pros., Ser. 3* **12** (2008), 95.

[2] R. Bojanowski, Zastosowania uogólnionej nierównosci Bogomolova-Miyaoka-Yau, Master Thesis (in Polish), `http://www.mimuw.edu.pl/~alan/postscript/bojanowski.pdf` (2003).

[3] J. Csima and E. T. Sawyer, There exist $6n/13$ ordinary points, *Discrete Comput. Geom.* **9(2)** (1993), 187–202.

[4] B. Green and T. Tao, On sets defining few ordinary lines, *Discrete Comput. Geom.* **50(2)** (2013), 409–468.

[5] B. Grünbaum, *Arrangements and spreads*, Conf. Board of the Math. Sciences Regional Conf. Series in Math., No. 10, American Mathematical Society, Providence, R.I. (1972).

[6] F. Hirzebruch, Singulaties of algebraicnsurfaces and characteristic numbers, *Contemp. Math.* **58** (1986), 141–155.

[7] E. Melchoir, Über Vielseite der projektiven Ebene, *Dtsch. Math.* **5** (1941), 4–306.

[8] P. Pokora, Hirzebruch-type inequalities viewed as tools im combinatorics, *Electron. J. Combin.* **28(1)** (2021), #P1.9 (22 pp.).

[9] G. Purdy, On the number of regions determined by n lines in the projective plane, *Geom. Dedicata.* **9(1)** (1980), 107–109.

[10] I. N. Shnurnikov, Into how many regions do n lines divide a plane if at most $n-k$ of them are collinear?, *Moskov. Univ. Math. Bull.* **65(5)** (2010), 32–36.

[11] I. N. Shnurnikov, A t_k inequality for arrangements of pseudolines, *Discrete Comput. Geom.* **55(2)** (2016), 284–295.

[12] A. Langer, Logarithmic orbifold Euler numbers of surfaces with applications, *Proc. London Math. Soc. (3)* **86(2)** (2003), 358–396.

[13] F. de Zeeuw, Spanned lines and Langer's inequality, (2018), `https://arxiv.org/abs/1802.08015`.

(Received 31 May 2022; revised 10 Nov 2022)