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# The lattice of arithmetic progressions

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**Abstract**

In this paper we investigate properties of the lattice $L_n$ of subsets of $[n] = \{1, \dots, n\}$ that are arithmetic progressions, under the inclusion order. For $n \geq 4$, this poset is not graded and thus not semimodular. We give three independent proofs of the fact that for $n \geq 2$, $\mu_n(L_n) = \mu(n-1)$, where $\mu_n$ is the Möbius function of $L_n$ and $\mu$ is the classical (number-theoretic) Möbius fnction. We also show that $L_n$ is comodernistic, which implies that $L_n$ is EL-labelable. Comodernism is then used to prove that the order complex $\Delta_n$ of the lattice is either contractible or homotopy equivalent to a sphere.

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# 1 Introduction

The additive structure of certain subsets of additive groups has long been a topic of interest in number theory and combinatorics. A class of sets with a great deal of additive structure is the set of *arithmetic progressions*. These are sets of the form

$$\{a, a+r, \ldots, a+(k-1)r\}$$

where the *base point* $a$ and *step size* (or simply *step*) $r$ are elements of an additive group and the *length* $k$ is an integer. In this paper we take our underlying additive group to be the integers $\mathbf{Z}$. The business of finding arithmetic progressions in sets of integers goes back to a classical 1927 theorem of van der Waerden [23], which states that any colouring of the integers with finitely many colours gives rise to monochromatic arithmetic progressions of arbitrary length. This was generalised by Szemerédi, who in 1975 proved the existence of arithmetic progressions of arbitrary length in any set of positive upper density [22]. More recently, Green and Tao showed that the same conclusion holds in the primes [14].

Set systems consisting of arithmetic progressions have received some attention in the realm of topology. The topology on $\mathbf{Z}$ generated by (infinite) arithmetic progressions $a + k\mathbf{Z}$ was used by Furstenberg to give an alternative proof of the infinitude of primes [10]. This topology came to be known as Golomb's topology, after Golomb, who studied its properties more systematically in a 1959 paper [12]. We will restrict ourselves to a finite subset of $\mathbf{Z}$ and study the set of arithmetic progressions itself, rather than the topology it forms a basis of. As with any set of subsets, it is partially ordered by inclusion, and in the present paper we investigate the structure induced by this ordering.

We shall also investigate topological properties of the order complex associated to this lattice. Several other simplicial complexes related to number-theoretic objects have recently appeared in the literature. The simplicial complex of squarefree positive integers less than or equal to $n$ was studied in a 2011 paper by Björner [4], and a 2017 paper [7] of Ehrenborg, Govindaiah, Park and Readdy introduces a simplicial complex called the van der Waerden complex $\mathrm{vdW}(n, k)$, whose facets correspond to arithmetic progressions of length $k$ in $\{1, \ldots, n\}$. A subsequent paper of Hooper and Van Tuyl characterised the pairs $(n, k)$ for which $\mathrm{vdW}(n, k)$ is shellable [16]. The simplicial complexes arising from our posets are different in that the vertices are themselves arithmetic progressions.

For brevity, we let $[n]$ denote the set $\{1, 2, \ldots, n\}$ and we will also sometimes write sets of integers by concatenating their elements: so, for example, 135 is shorthand for $\{1, 3, 5\}$. For $n \geq 1$ we let $L_n$ denote the partially-ordered set (poset) of all finite integer arithmetic progressions contained in $[n]$ including trivial progressions of length 1 and 2 as well as the empty set $\emptyset$. When it is convenient, we artificially define $L_0 = \{\emptyset\}$. Small examples are depicted in Figure 1. In each figure, every point represents a progression, progressions with the same cardinality are placed on the same row, and these rows are ordered lexicographically (the reader may find it instructive to spend a minute or two trying to pencil in labels for $L_4$ and $L_5$).

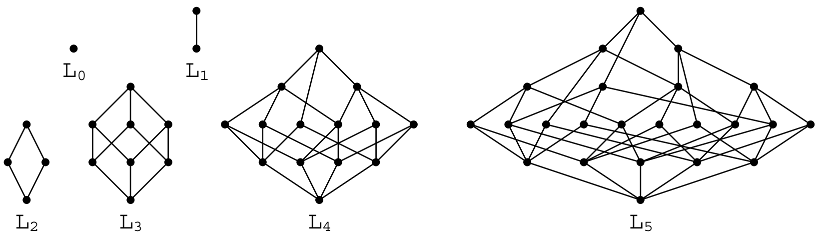


Figure 1: Hasse diagrams of $L_n$ for small values of $n$.

The notation $L_n$ is motivated by the fact that $L_n$ is a lattice. The meet of two elements is simply the set-theoretic intersection, since the intersection of two integer arithmetic progressions is a (possibly empty) arithmetic progression. By induction, one finds that the meet of any finite number of points is well-defined in $L_n$, and this as well as the existence of a maximum element $12\cdots n$ implies the existence of a join of two arbitrary elements $x_1, x_2 \in L_n$.

The poset $L_n$ is not graded for $n \geq 4$. To see this, note that

$$1 < 14 < [4] < [5] < \cdots < [n] \qquad \text{and} \qquad 1 < 12 < 123 < 1234 < \cdots < [n]$$

are both maximal chains, but the first has length $n-2$ while the second has length $n-1$. Since the posets $L_n$ for $n \geq 4$ are not graded, they are also not (upper) semimodular. Indeed, 12 and 14 both cover $12 \wedge 14 = 1$, but $12 \vee 14 = 1234$ does not cover 12.

For two elements $x \leq y$ in a poset $X$, the *interval* $[x, y]$ is the set of all $z \in X$ satisfying $x \leq z \leq y$. If $X$ has a minimum element $\widehat{0}$, then we can define the the *principal (order) ideal generated by* $x$, denoted $\downarrow x$, to be the interval $[\widehat{0}, x]$. A poset is said to be *locally finite* if every interval is finite. The *Möbius function* $\mu_X$ of a locally finite poset $X$ is the function from intervals of the poset to the complex field $\mathbf{C}$ given by the formulas $\mu_X(x, x) = 1$ for all $x \in X$ and

$$\mu_X(x, y) = -\sum_{x \leq z < y} \mu_X(x, z) \tag{1.1}$$

for all $x \leq y$ in $X$, where we have abbreviated $\mu_X\big([x, y]\big)$ by $\mu_X(x, y)$. If the poset $X$ is a lattice, with minimum element $\widehat{0}$ and maximum element $\widehat{1}$, then $X = [\widehat{0}, \widehat{1}]$ and it makes sense to write $\mu_X(X)$ for $\mu(\widehat{0}, \widehat{1})$. In the case that $X$ is the set of all positive integers, ordered by divisibility, then $\mu_X(m, n) = \mu(n/m)$, where $\mu$ is the classical Möbius function. Recall that $\mu(s) = 1$ if $s = 1$ or $s$ is a product of an even number of distinct primes, $\mu(s) = -1$ if $s$ is a product of an odd number of distinct primes, and $\mu(s) = 0$ if $s$ is divisible by a perfect square. We centre our discussion around the following main result.

**Theorem 1.1** *Let $\mu_n = \mu_{L_n}$ be the Möbius function of the lattice of arithmetic progressions $L_n$. We have $\mu_0(L_0) = 1$, $\mu_1(L_1) = -1$, and $\mu_n(L_n) = \mu(n-1)$ for $n \geq 2$, where $\mu$ is the classical Möbius function.*

We now briefly outline the paper. In Section 2, we develop some properties of the number $p_{nk}$ of arithmetic progressions of size $k$ in $[n]$ and show that these quantities arise in a recurrence that proves Theorem 1.1 directly from the definition of the Möbius function. In Section 3, we count chains in $L_n$ in order to gain information about the order complex of $L_n$ and derive the same recurrence in a slightly different manner. We then proceed in Section 4 to study the set of coatoms in $L_n$ in order to give a general formula for $\mu_n$, evaluated at an arbitrary interval of $L_n$. As a corollary, we obtain a third proof of Theorem 1.1 that is of a rather different nature than the first two proofs. In Section 5, we explicitly compute the homology groups of the order complex $\Delta_n$ of $L_n$. In Section 6, we prove that $L_n$ is comodernistic, a property recently introduced by Schweig and Woodroofe that in particular implies that $\Delta_n$ is shellable for all $n$ [21]. Lastly, in Section 7, we use lemmas proved in previous sections to show that $L_n$ is EL-labelable, that $\Delta_n$ is either contractible or has the homotopy type of a sphere, and that $L_n$ is complemented if and only if $n-1$ is squarefree.

## 2 The number of arithmetic progressions

Our starting point is the number $p_{nk}$ of arithmetic progressions of length $k$ contained in $[n]$. It was shown in [11] that for $2 \le k \le n$,

$$p_{nk} = \sum_{r=1}^{\lfloor (n-1)/(k-1) \rfloor} \big(n-(k-1)r\big) = n\left\lfloor \frac{n-1}{k-1} \right\rfloor - \frac{k-1}{2}\left( \left\lfloor \frac{n-1}{k-1} \right\rfloor^2 + \left\lfloor \frac{n-1}{k-1} \right\rfloor \right).$$

(We have halved their formula here, because we consider arithmetic progressions as sets and not as ordered sequences.) We also have $p_{n0} = 1$ to count the empty progression as well as $p_{n1} = n$ to count the $n$ singletons. Values of $p_{nk}$ for small values of $n$ and $k$ are collected in Table 1. We first derive a formula for the bivariate generating function of $p_{nk}$ (see, e.g., [8] for an exhaustive reference on generating functions).

**Lemma 2.1** *For integers $n, k \ge 0$, let $p_{nk}$ denote the number of arithmetic progressions of size $k$ in the interval $[n]$. We have the formula*

$$f(z,q) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} p_{nk} z^n q^k = \frac{1}{(1-z)^2}\left(1 - z + zq + \sum_{k=2}^{\infty} \frac{(zq)^k}{1-z^{k-1}}\right)$$

*for the bivariate generating function of $p_{nk}$.*

PROOF: The sequences $\big(p_{n0}\big)_{n\ge 0}$ and $\big(p_{n1}\big)_{n\ge 0}$ are $(1,1,1,\ldots)$ and $(0,1,2,\ldots)$ respectively, so that the coefficient of $q^0$ in $f(z,q)$ is $1/(1-z)$ and the coefficient of $q$ is $z/(1-z)^2$. For $k \ge 2$, there are $n-1$ possible base points and for each base point $a$, the number of possible step sizes is $\lfloor (n-a)/(k-1) \rfloor$. So

$$\sum_{n=2}^{\infty} p_{nk} z^n = \sum_{n=2}^{\infty}\sum_{a=1}^{n-1} \left\lfloor \frac{n-a}{k-1} \right\rfloor z^n = \sum_{n=0}^{\infty}\sum_{a=1}^{n-1} \left\lfloor \frac{a}{k-1} \right\rfloor z^n = \frac{1}{1-z}\sum_{n=0}^{\infty} \left\lfloor \frac{n}{k-1} \right\rfloor z^n \quad (2.1)$$

| $n$ | $p_{n0}$ | $p_{n1}$ | $p_{n2}$ | $p_{n3}$ | $p_{n4}$ | $p_{n5}$ | $p_{n6}$ | $p_{n7}$ | $p_{n8}$ | $p_{n9}$ | $p_{n(10)}$ | $p_{n(11)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | | |
| 1 | 1 | 1 | | | | | | | | | | |
| 2 | 1 | 2 | 1 | | | | | | | | | |
| 3 | 1 | 3 | 3 | 1 | | | | | | | | |
| 4 | 1 | 4 | 6 | 2 | 1 | | | | | | | |
| 5 | 1 | 5 | 10 | 4 | 2 | 1 | | | | | | |
| 6 | 1 | 6 | 15 | 6 | 3 | 2 | 1 | | | | | |
| 7 | 1 | 7 | 21 | 9 | 5 | 3 | 2 | 1 | | | | |
| 8 | 1 | 8 | 28 | 12 | 7 | 4 | 3 | 2 | 1 | | | |
| 9 | 1 | 9 | 36 | 16 | 9 | 6 | 4 | 3 | 2 | 1 | | |
| 10 | 1 | 10 | 45 | 20 | 12 | 8 | 5 | 4 | 3 | 2 | 1 | |
| 11 | 1 | 11 | 55 | 25 | 15 | 10 | 7 | 5 | 4 | 3 | 2 | 1 |

Table 1: The number $p_{nk}$ of arithmetic progressions of size $k$ in $\{1, 2, \ldots, n\}$

where we have added the empty terms for $n = 0$ and $n = 1$ and reversed the order of summation in the second equality. Note that

$$\begin{aligned}\sum_{n=0}^{\infty} \left\lfloor \frac{n}{k-1} \right\rfloor z^n &= \sum_{i=1}^{\infty} \sum_{n=i(k-1)}^{\infty} z^n \\ &= \sum_{i=1}^{\infty} \frac{z^{i(k-1)}}{1-z} \\ &= \frac{1}{1-z} \cdot \frac{1-(1-z^k)}{1-z^{k-1}} \\ &= \frac{z^k}{(1-z^{k-1})(1-z)}.\end{aligned} \tag{2.2}$$

Putting everything together, we find that

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} p_{nk} z^n q^k = \frac{1}{1-z} + \frac{z}{(1-z)^2} q + \sum_{k=2}^{\infty} \frac{z^k}{(1-z^{k-1})(1-z)^2} q^k, \tag{2.3}$$

which simplifies to the formula we were looking for. □

Because $p_{nk} = 0$ when $k > n$, the horizontal generating functions $f_n(q)$ are polynomials $\sum_{k=0}^{n} p_{nk} q^k$. For instance, since $L_0$ through $L_3$ are just Boolean lattices (consisting of all subsets of a finite ground set), we have $f_n(q) = (1 + q)^n$. When $n = 4$, we have $f_4(q) = 1 + 4q + 6q^2 + 2q^3 + q^4$, which is irreducible in $\mathbf{Z}[q]$ by Cohn's criterion [5], since $f_4(10) = 12641$ is prime. It can also be checked computationally that $f_n(q)$ is irreducible for $5 \le n \le 10$, and there is no reason to suspect that this polynomial has a neat factorisation for any larger values of $n$. As a corollary of the above lemma, we obtain a nice formula for $f_n(1) = |L_n|$, the number of elements in the lattice. We use the notation $j \mid k$ to indicate that $k$ is an integer multiple of $j$.

**Corollary 2.2** *For $n \in \mathbf{N}$, the poset $L_n$ has*

$$|L_n| = 1 + n + \sum_{a=1}^{n-1} \sum_{r=1}^{a} \tau(r)$$

*elements, where $\tau(r) = \sum_{d|r} 1$ is the divisor function.*

Proof: We write

$$|L_n| = f_n(1) = 1 + n + \sum_{a=1}^{n-1} \sum_{r=1}^{a} \left\lfloor \frac{a}{r} \right\rfloor$$

and then apply the elementary identity $\sum_{k=1}^{n} \tau(k) = \sum_{k=1}^{n} \lfloor n/k \rfloor$. □

The sequence $(|L_n| - 1)_{n \geq 1}$ appears in the On-line Encyclopedia of Integer Sequences under the entry A051336. We are almost ready to give the first proof of Theorem 1.1, which expresses the Möbius function of $L_n$ as a recurrence defined in terms of $p_{nk}$. We start with the following observation.

**Lemma 2.3** *Let $n \geq 1$ and let $1 \leq k \leq n$. For any progression $x \in L_n$ of cardinality $k$, we have $\mu_n(\emptyset, x) = \mu_k(\emptyset, [k])$.*

Proof: Because $x = \{a, a + r, \ldots, a + (k - 1)r\}$ is a progression, one obtains an isomorphism of posets between the ideal $\downarrow x$ and $L_k$ by relabelling the element $a + ir$ with $i + 1$ for $0 \leq i < k$. □

The rest of the proof involves setting up and solving a certain recurrence.

First proof of Theorem 1.1. Let $M_n = \mu_n(L_n)$ for short. The case $n = 0$ is trivial. For $n \geq 1$, we must subtract $\mu_k(\emptyset, x)$ for every progression $x \in L_n^* = L_n \setminus \{[n]\}$. By Lemma 2.3 and the fact that there are $p_{nk}$ progressions of size $k$ in $L_n$, we have the recurrence

$$M_n = -\sum_{x \in L_n^*} \mu_n(\emptyset, x) = -\sum_{k=0}^{n-1} M_k p_{nk}. \tag{2.4}$$

We can then compute $M_1 = -1$ and $M_2 = 1 = \mu(1)$. For $n > 2$ we now proceed by strong induction; suppose that $M_k = \mu(k - 1)$ for all $2 \leq k < n$. We expand the above recurrence to

$$
\begin{aligned}
M_n &= -\Big(M_0 p_{n0} + M_1 p_{n1} + \sum_{k=2}^{n-1} M_k p_{nk}\Big) \\
&= -\Big(1 - n + \sum_{k=2}^{n-1} \mu(k-1) \sum_{r=1}^{\lfloor (n-1)/(k-1) \rfloor} \big(n - (k-1)r\big)\Big) \qquad (2.5) \\
&= -\Big(1 - n - \mu(n-1) + \sum_{k=1}^{n-1} \mu(k) \sum_{r=1}^{\lfloor (n-1)/k \rfloor} \big(n - kr\big)\Big)
\end{aligned}
$$

and sum over all possible values of $kr$ by setting $m = kr$ and summing over divisors $d$ of $m$, for $1 \leq m \leq n-1$. This gives

$$
M_n = -\Big(1 - n - \mu(n-1) + \sum_{m=1}^{n-1} \sum_{d|m} \mu(d)(n-m)\Big). \qquad (2.6)
$$

But $\sum_{d|m} \mu(d) = 0$ when $m > 1$ and when $m = 1$, the summation equals $n-1$. After cancellation, we see that the right-hand side equals $\mu(n-1)$, which is what we wanted to show. □

## 3 Chains and the order complex

An *abstract simplicial complex* is a set system $\Delta$ on a vertex set $V$ containing every singleton subset of $V$ and with the property that for every set $F \in \Delta$, all subsets of $F$ also belong to $\Delta$. The elements of $\Delta$ are called *faces*, and the *dimension* of a face $F$ is defined to be $|F| - 1$. A face is said to be *maximal* if it is not strictly contained in another face, and the dimension of $\Delta$ is the maximum dimension of a (maximal) face in $\Delta$. For our purposes, simplicial complexes will contain the empty set, a face of dimension $-1$. We will require various notions from topology in this section. Any definitions that we do not recall here can be found in any introductory textbook, such as [19], for example.

A *chain* of length $k$ in a poset $X$ is a set $\{x_1, x_2, \ldots, x_{k+1}\} \subseteq X$ such that $x_1 < x_2 < \cdots < x_{k+1}$; so a chain of length 0 is a singleton set and we will say that an empty chain has length $-1$. One can associate a simplicial complex, called the *order complex*, to any lattice (with bottom element $\widehat{0}$ and top element $\widehat{1}$) by taking $L \setminus \{\widehat{0}, \widehat{1}\}$ as the vertex set and letting the faces be chains in this modified poset. Let $L'_n$ denote the poset $L_n$ with the minimum element $\emptyset$ as well as the maximum element $[n]$ removed. Note that chains in $L'_n$ of length $k-2$ are in bijection with chains of length $k$ in $L_n$ that contain both $\emptyset$ and $[n]$, which we shall count in the next lemma.

| $n$ | $b_{n1}$ | $b_{n2}$ | $b_{n3}$ | $b_{n4}$ | $b_{n5}$ | $b_{n6}$ | $b_{n7}$ | $b_{n8}$ | $b_{n9}$ | $b_{n(10)}$ | $b_{n(11)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | | |
| 2 | 1 | 2 | | | | | | | | | |
| 3 | 1 | 6 | 6 | | | | | | | | |
| 4 | 1 | 12 | 24 | 12 | | | | | | | |
| 5 | 1 | 21 | 68 | 72 | 24 | | | | | | |
| 6 | 1 | 32 | 144 | 244 | 180 | 48 | | | | | |
| 7 | 1 | 47 | 283 | 666 | 764 | 432 | 96 | | | | |
| 8 | 1 | 64 | 486 | 1510 | 2436 | 2164 | 1008 | 192 | | | |
| 9 | 1 | 85 | 799 | 3117 | 6534 | 8028 | 5816 | 2304 | 384 | | |
| 10 | 1 | 109 | 1232 | 5860 | 15368 | 24524 | 24516 | 15040 | 5184 | 768 | |
| 11 | 1 | 137 | 1838 | 10418 | 33049 | 65402 | 84284 | 70992 | 37760 | 11520 | 1536 |

Table 2: The number $b_{nk}$ of chains of length $k-2$ in $L'_n$

**Lemma 3.1** *The number $b_{nk}$ of chains of length $k$ in $L_n$ that contain $\emptyset$ and $[n]$ satisfies the recurrence*

$$b_{nk} = \sum_{i=1}^{n-1} p_{ni} b_{i(k-1)}, \tag{3.1}$$

*for $2 \le k \le n$, with $b_{n1} = 1$ for all $n$ and $b_{nk} = 0$ whenever $k > n$.*

Proof: The case $k = 1$ is trivial and it is clear that $b_{nk}$ should be zero for $k > n$. In the other cases, we are counting chains $\emptyset \subset x_1 \subset \cdots \subset x_k \subset [n]$ (we require strict inclusion here). We split up the cases by the second-greatest element $x_k$ of the chain. It is clear that the subchain $\{\emptyset, x_1, \ldots, x_k\}$ is a chain containing both the maximum and minimum element of the ideal $\downarrow x_k$, which, by Lemma 2.3, is isomorphic to $L_i$, where $i$ is the size of $x_k$ (as a set). Thus the number of such chains is $b_{m(k-1)}$. There were $p_{ni}$ choices for the element of size $i$, and summing over all possible $i$ gives the recurrence above. □

For small values of $n$ and $k$, the values $b_{nk}$ are displayed in Table 2. As a slight digression, note for analogy that if in the recurrence (3.1) we replace $p_{nk}$ with $\binom{n}{k}$, we obtain the array of numbers $k!\left\{{n \atop k}\right\}$, where $\left\{{n \atop k}\right\}$ is a Stirling number of the second kind (see, e.g., [13]). These numbers count the number of ways to partition $n$ numbers into $k$ nonempty subsets, and for each such partition $S_1, S_2, \ldots, S_n$, we obtain $k!$ chains in the Boolean lattice that contain both $\emptyset$ and $[n]$ (for each permutation $\sigma$ in $\mathfrak{S}_n$, we have the chain $\{\emptyset, S_{\sigma(1)}, S_{\sigma(1)} \cup S_{\sigma(2)}, \ldots, [n]\}$).

Returning to our numbers $b_{nk}$, we see that for $-1 \le k \le n-2$, the number of $k$-dimensional faces of $\Delta_n$ is $b_{n(k+2)}$. Hence $\Delta_n$ is an $(n-2)$-dimensional simplicial complex. Let $\widetilde{\chi}(\Delta_n) = \chi(\Delta_n) - 1$ be the reduced Euler characteristic of the order complex. We have

$$\widetilde{\chi}(\Delta_n) = \sum_{k=1}^{n}(-1)^k b_{nk} \tag{3.2}$$

for $n \geq 1$. Using the fact that the Möbius function of a poset with a maximum and minimum element artificially adjoined equals the reduced Euler characteristic of its order complex, we obtain an alternative proof of Theorem 1.1.

Second proof of Theorem 1.1. Let $M_n = \sum_{k=1}^{n}(-1)^k b_{nk}$. We compute $M_0 = 1$ and $M_1 = -1$ by hand. To complete the proof, it suffices to show that $\widetilde{\chi}(\Delta_n) = M_n = \mu(n-1)$ for all $n \geq 2$. The base case $M_2 = 1$ follows from a direct computation, and for $n > 2$, we have

$$\begin{aligned} M_n &= \sum_{k=1}^{n}(-1)^k b_{nk} \\ &= -1 + \sum_{k=2}^{n}(-1)^k \sum_{i=1}^{n-1} p_{ni} b_{i(k-1)} \\ &= -1 + \sum_{i=1}^{n-1} p_{ni} \sum_{k=2}^{n}(-1)^k b_{i(k-1)}, \end{aligned} \tag{3.3}$$

by Lemma 3.1. We can pull out one of the $-1$ factors and reindex to obtain

$$M_n = -\left(1 + \sum_{i=1}^{n-1} p_{ni} \sum_{k=1}^{i}(-1)^k b_{ik}\right). \tag{3.4}$$

Note that the upper index in the inner summation has been changed to $i$, since $b_{ik} = 0$ when $k > i$. By the induction hypothesis, this inner sum is $M_i$, so

$$M_n = -\left(1 + \sum_{i=1}^{n-1} p_{ni} M_i\right) = \sum_{i=0}^{n-1} p_{ni} M_i, \tag{3.5}$$

which is the recurrence (2.4) we encountered in the first proof of this theorem. The rest of the proof proceeds exactly as before. □

## 4 Coatoms

We now set out to compute $\mu_n(x_1, x_2)$ for arbitrary progressions $x_1$ and $x_2$ in $L_n$. Towards this goal, we will need to study the *coatoms* of $L_n$, the elements covered by $[n]$. It turns out that we can give an explicit description of the set of coatoms in $L_n$.

**Lemma 4.1** *Let $A_n \subseteq L_n$ be the set of coatoms. We have $A_1 = \{\emptyset\}$, $A_2 = \{1, 2\}$, and $A_3 = \{12, 13, 23\}$. For $n \geq 4$, we have $A_n = B_n \cup C_n$, where $B_n = \{12 \cdots (n-1), 23 \cdots n\}$, and*

$$C_n = \begin{cases} \{1n\}, & \text{if } n-1 \text{ is prime;} \\ \{\{1, 1+p, 1+2p, \ldots, n\} : p \text{ prime},\ p \mid (n-1)\}, & \text{otherwise.} \end{cases} \tag{4.1}$$

*In particular, the size of $A_n$ is $\omega(n-1)+2$, where $\omega(n)$ is the number of distinct prime divisors of $n$.*

PROOF: The small cases are easily computed explicitly. When $n \geq 4$ there are only two elements of size $n-1$, and the fact that they are coatoms is obvious. Now any element that does not contain both 1 and $n$ cannot be a coatom, since an element of $B_n$ would contain it. The progressions that contain $1n$ are of the form $x_d = \{1, d+1, 2d+1, \ldots, n\}$ for divisors $d$ of $n-1$, but note that if $d$ is composite, then $x_d$ is contained in $x_{d'}$ for any $d'$ dividing $d$. Hence the remaining coatoms are the progressions with prime steps, implying that $C_n$ is of one of the two forms above. □

Every non-top element in $L_n$ is contained in some coatom, but when $n \geq 4$, not all elements can be expressed as a *meet* of coatoms. The next lemma shows that in $L_n$, if an element can be expressed as a meet of coatoms, then this representation is unique.

**Lemma 4.2** *Let $L_n$ be the lattice of arithmetic progressions and let $A_n \subseteq L_n$ be the set of coatoms. If $x \in L_n$ can be expressed as $x = \bigwedge_{s \in S} s$ for some $S \subseteq A_n$, then $S$ is uniquely determined by $x$.*

PROOF: If $x = \emptyset$, then $S$ must equal $A_n$, since omitting one of $12\cdots(n-1)$ or $23\cdots n$ would cause one of the elements 1 or $n$ to appear in the meet, and omitting the progression with base point 1, step size $p$ (a prime dividing $n-1$), and end point $n$ will cause the $p-1$ elements

$$1 + \frac{n-1}{p}, 1 + \frac{2(n-1)}{p}, \ldots, n - \frac{n-1}{p}$$

to appear in the meet. This last observation also shows that if $x = \{1\}$ then we have omitted only $23\cdots n$ and if $x = \{n\}$, then we have omitted only $12\cdots(n-1)$.

Now suppose that $x$ is nonempty and we can write out the elements of $x = \{a, a+r, \ldots, a+(k-1)r\}$. We will consider the possible step sizes $r$. When $r = 1$, $x$ is either $12\cdots(n-1)$, $23\cdots n$, or $23\cdots(n-1)$ and in all three cases it is clear that there is only one representation of $x$ as the meet of coatoms. For $r > 1$, we find that $r$ must be the least common multiple of some primes dividing $n-1$, and there is only one way to express $r$ as a least common multiple of distinct primes, thus uniquely determining the coatoms with prime step size that are in $S$. Lastly, note that $12\cdots(n-1)$ is in $S$ if and only if $a = 1$ and $23\cdots n$ is in $S$ if and only if $a+(k-1)r = n$. □

These properties of the set of coatoms in $L_n$ imply a general formula for computing $\mu_n(x, [n])$.

**Theorem 4.3** *Let $x \neq [n]$ be a progression in $L_n$.*

$$\mu_n(x,[n]) = \begin{cases} (-1)^k, & \text{if } x \text{ is the meet of } k \text{ coatoms;} \\ 0, & \text{if } x \text{ is not a meet of coatoms.} \end{cases} \tag{4.2}$$

PROOF: We dispense first with the case where $x$ is a coatom, since then it is clear that $\mu_n(x,[n]) = -1$. Otherwise, note that $x$ is the minimum element of the interval $L = \big[x,[n]\big]$; the subset $S \subseteq L_n$ of coatoms whose meet equals $x$ is contained in this interval. By the cross-cut theorem [20],

$$\mu_n(x,[n]) = \sum_{k=1}^{|A_n|} (-1)^k N_k, \tag{4.3}$$

where $N_k$ is the number of sets of $k$ coatoms whose meet is $x$. By Lemma 4.2, $N_{|S|} = 1$ and $N_k = 0$ for all $k \neq |S|$, proving the theorem. □

It is easy to tell if a given progression $x$ is a meet of coatoms, since such $x$ have a very specific form. In particular, $x$ is a meet of coatoms of $L_n$ if and only if

$$x \cap \{2,\ldots,n-1\} = (1+d\mathbf{Z}) \cap \{2,\ldots,n-1\}$$

for some divisor $d$ of $n-1$. One can then work out the number of elements in the meet representation by taking the prime decomposition of $d$ and checking whether 1 or $n$ (or both or neither) are included in $x$. Let $\omega(n)$ be the number of distinct primes dividing an integer $n$ and let $S$ denote the set of progressions $x$ with $\mu_n(x,[n]) \neq 0$. Lemma 4.1 and Theorem 4.3 together imply that there are exactly $2^{\omega(n-1)+2}$ such elements $x$ in $L_n$. Since every squarefree divisor of $n-1$ contributes exactly four progressions to the set $S$, we can prove the elementary identity $\sum_{d|n} \big|\mu(d)\big| = 2^{\omega(n)}$ by counting $S$ in two ways.

Since, by Lemma 2.3, the ideal $\downarrow x \subseteq L_n$ is isomorphic to $L_m$ for any progression $x$ of size $m$, Theorem 4.3 immediately implies a general method for computing the Möbius function of an arbitrary interval.

**Corollary 4.4** *Let $x_1$ and $x_2$ be elements of $L_n$ with $x_1 \leq x_2$ and let $C$ be the set of elements covered by $x_2$. We have*

$$\mu_n(x_1,x_2) = \begin{cases} (-1)^k, & \text{if } x_1 \text{ is the meet of } k \text{ elements of } C; \\ 0, & \text{if } x_1 \text{ is not a meet of elements of } C. \end{cases} \tag{4.4}$$

This corollary tells us that the Möbius function of $L_n$ takes values in $\{0,\pm1\}$ no matter the interval at which it is evaluated. Posets with this property are sometimes called *totally unimodular* (see, e.g., [15]). Theorem 4.3 also allows us to give a third proof of Theorem 1.1.

Third proof of Theorem 1.1. We take $n \geq 4$; smaller cases can easily be worked out explicitly. First suppose that $n-1$ is squarefree, equalling the product of distinct primes $p_1, p_2, \ldots, p_k$, so that $\mu(n-1) = (-1)^k$. The claim is that for any nonempty progression $x \in L_n$, there is some coatom that does not contain $x$. If $x$ contains either 1 or $n$, then one of the two progressions in $L_n$ of size $n-1$ does not contain $x$. Otherwise, $x$ contains some integer $1+m$ for $1 \leq m \leq n-2$. Since $m < n-1 = \operatorname{lcm}(p_1, p_2 \ldots, p_k)$, there is some prime $p_i$ that does not divide $m$, hence $1+m$ is not contained in the coatom of step size $p_i$. There are $k+2$ coatoms in $L_n$, so Theorem 4.3 can be applied to give $\mu_n(L_n) = (-1)^{k+2} = (-1)^k = \mu(n-1)$.

Now assume that $n-1$ is divisible by $p^2$ for some prime $p$. Since the integer $(n-1)/p$ is divisible by every prime dividing $n-1$, the element $1+(n-1)/p$ belongs to every coatom of $L_n$. So $\emptyset$ cannot be expressed as a meet of coatoms and $\mu_n(L_n) = 0$. □

## 5 Homology groups of the order complex

Although less direct than the first two proofs we supplied, the proof of Theorem 1.1 given in the previous section reveals much of the internal structure of $L_n$. We now show that it can be reinterpreted to give a complete characterisation of the homology groups of $\Delta_n$, a strictly stronger result than Theorem 1.1. A simplicial complex $\Delta$, as we have defined it, is simply a set system, but $\Delta$ can be embedded in Euclidean space to give rise to a topological space $|\Delta|$ called its *geometric realisation*. We will sometimes abuse notation and ascribe topological properties of $|\Delta|$ to $\Delta$. The reduced Euler characteristic of an $n$-dimensional simplicial complex $\Delta$ can also be expressed as the alternating sum

$$\widetilde{\chi}(\Delta) = \widetilde{\chi}(|\Delta|) = \sum_{i=0}^{n} (-1)^i \operatorname{rank} \widetilde{H}_i(|\Delta|, \mathbf{Z}), \tag{5.1}$$

where $\widetilde{H}_i(|\Delta|, \mathbf{Z})$ is the $i$th reduced homology group of the topological space $|\Delta|$ (whenever we refer to a homology group, we shall understand reduced homology group).

To derive the homology groups of $L_n$, we will require the notion of cross-cuts. A *cross-cut* $C$ of a lattice $L$ (with maximum $\widehat{1}$ and minimum $\widehat{0}$) is a subset of $L$ not containing either of $\widehat{1}$ and $\widehat{0}$ such that no two elements of $C$ are comparable and every maximal chain in the lattice contains some element of $C$. A subset $S$ of $L$ is said to be *spanning* if the join of all its elements is $\widehat{1}$ and the meet of all its elements is $\widehat{0}$. For a cross-cut $C$ of a lattice $L$, we can define a simplicial complex $\Delta(C)$ whose vertices are the elements of $C$ and whose faces are given by subsets of $C$ that are *not* spanning. A paper of Folkman [9] showed that $\widetilde{H}_i\big(\Delta(C), \mathbf{Z}\big) \cong \widetilde{H}_i\big(\Delta, \mathbf{Z}\big)$ for all $i$, where $\Delta$ is the order complex of $L$. We use this to derive the homology groups of $\Delta_n$.

**Lemma 5.1** *For $n \geq 4$, let $L_n$ be the lattice of arithmetic progressions and let $\Delta_n$ be the order complex of $L'_n = L_n \setminus \{\emptyset, [n]\}$. Let $\widetilde{H}_i(\Delta_n, \mathbf{Z})$ be the ith reduced homology group of $\Delta_n$. If $n-1$ is squarefree and equal to the product of $k$ distinct primes, then*

$$\widetilde{H}_i(\Delta_n, \mathbf{Z}) = \begin{cases} \mathbf{Z}, & \text{if } i = k; \\ 0, & \text{otherwise.} \end{cases}$$

*If $n-1$ is not squarefree, then all the homology groups of $\Delta_n$ are trivial.*

Proof: Let $C$ be the set of coatoms of $L_n$, whose explicit construction was given by Lemma 4.1. Let $k = \omega(n-1)$, so that $|C| = k+2$. If $n-1$ is squarefree, then as we saw earlier in the third proof of Theorem 1.1, we can express $\emptyset$ as a meet of elements of $C$, so $C$ is a spanning set. However, any proper subset $C'$ of $C$ is not spanning, since if $c_i$ is the element of $C$ that is not in $C'$, then we can build a chain $\emptyset \subset \cdots \subset c_i \subset [n]$ that does not contain an element of $C'$. So every subset of $C$ with cardinality $k+1$ is an element of the abstract simplicial complex $\Delta(C)$, i.e., $\Delta(C)$ is the boundary of a $(k+1)$-dimensional simplex, whose $k$th homology group is $\mathbf{Z}$ and whose other reduced homology groups are all trivial.

When $n-1$ is not squarefree, the construction we gave in the third proof of Theorem 1.1 shows that $\emptyset$ is not the meet of the elements of $C$, which means that $C$ itself does not span. Hence $\Delta(C)$ is the $(k+1)$-dimensional simplex, including its interior, all of whose reduced homology groups are trivial. □

We will use Lemma 5.1 later on to prove the stronger fact that $\Delta_n$ has the homotopy type of a sphere when $n-1$ is squarefree.

## 6 Left-modularity and comodernism

An element $m$ in a lattice $L$ is *left-modular in $L$* if for all $x < y \in L$, $(x \vee m) \wedge y = x \vee (m \wedge y)$. A lattice $L$ is *comodernistic* if every interval $[x, y] \subseteq L$ has a coatom which is left-modular in $[x, y]$. The aim of this section is to show that $L_n$ is comodernistic. To do so, we will make use of two of the lemmas in the paper of Schweig and Woodroofe that introduced the definition of comodernism.

**Lemma A ([21], Lemma 2.12)** *Let $m$ be a coatom of the lattice $L$. Then $m$ is left-modular in $L$ if and only if for every $y \in L$ with $y \not\leq m$, $y$ covers $m \wedge y$.*

**Lemma B ([21], Lemma 4.1)** *Let $L'$ be a sublattice of a lattice $L$. If $m \in L'$ is a left-modular coatom in $L$, then $m$ is also left-modular in $L'$.*

Note that we have modified these lemmas slightly to suit our notation and usage; in particular, the original version of Lemma B requires only that $L'$ be a meet subsemilattice. We begin with a small lemma.

**Lemma 6.1** *For $n \geq 1$, the elements $12 \cdots (n-1)$ and $23 \cdots n$ are left-modular in $L_n$.*

Proof: Without loss of generality, let $m = 12 \cdots (n-1)$; the case where $m = 23 \cdots n$ is symmetric. Let $y \in L_n$ be such that $y \not\leq m$, so it must be that $n \in y$; hence $m \wedge y = y \setminus \{n\}$ which is covered by $y$. By Lemma A, this shows that $m$ is left-modular. □

We are now able to show that $L_n$ is comodernistic for all $n$. For brevity of notation, in the following proof we let $\uparrow_k x$ denote the *principal filter* of $x \in L_k$; that is, $\uparrow_k x = \{y \in L_k : x \leq y\}$.

**Theorem 6.2** *For all $n \geq 0$, the lattice $L_n$ is comodernistic.*

Proof: Let $[x, y]$ be an interval in $L_n$. We once again employ Lemma 2.3, which says that $\downarrow y$ is isomorphic to $L_k$ where $k = |y|$. This isomorphism sends $[x, y]$ to the interval $\uparrow_k x \subseteq L_k$, so it suffices to show that, for all $k \geq 1$ and $x \in L_k$, the principal filter $\uparrow_k x$ contains a coatom which is left-modular (in the filter). Let $A_k = B_k \cup C_k$ be the coatoms of $L_k$, with $B_k$ and $C_k$ defined as in Theorem 4.1. Clearly, the coatoms of $\uparrow_k x$ are a subset of $A_k$. If $\uparrow_k x \cap B_k \neq \emptyset$, then by Lemma 6.1, $\uparrow_k x$ contains a coatom which is left-modular in all of $L_k$, and by Lemma B it is also a left-modular coatom $\uparrow_k x$. If $\uparrow_k x \cap B_k$ is empty, then the progression $x$ must contain both 1 and $k$, so $\uparrow_k x \subseteq \uparrow_k 1k$ and in particular, every coatom of $\uparrow_k x$ is also a coatom of $\uparrow_k 1k$. By another application of Lemma B, we may reduce our proof to showing that every coatom of $\uparrow_k 1k$ is left-modular in this filter.

The coatoms of $\uparrow_k 1k$ are precisely the elements in $C_k$. If $k-1$ is prime, Lemma 4.1 tells us that $1k$ is a coatom, so $\uparrow_k 1k$ contains only the two elements $1k$ and $[k]$, the former of which is trivially left-modular in this interval. On the other hand, let $k-1$ be composite and let $m$ be a coatom of $\uparrow_k 1k$; by Lemma 4.1, $m$ is of the form $\{1, 1+p, ..., k\}$ for some $p$ dividing $k-1$. If $y \in \uparrow_k 1k$ satisfies $y \not\leq m$, then $y = \{1, 1+r, ..., k\}$ where $r$ divides $k-1$ and $p$ does not divide $r$. So $m \wedge y = \{1, 1+s, ..., k\}$ where $s = \operatorname{lcm}(r, p) = rp$, hence $m \wedge y$ is covered by $y$ and we conclude that $m$ is left-modular by Lemma A. □

## 7 EL-labelability, homotopy type, and complements

We now use the lemmas of the previous sections to demonstrate further properties of $L_n$. Here we show that $L_n$ is EL-labelable, that $\Delta_n$ is either homotopy equivalent to a point or a sphere, and that $L_n$ is complemented if and only if $n-1$ is squarefree.

### 7.1 EL-labelability

Given a lattice $L$, let $E(L)$ be the set of all $(x, y) \in L$ such that $y$ covers $x$; thus $E(L)$ is the edge set of the Hasse diagram of $L$. We say that a function $\lambda : E(L) \to \mathbf{Z}$ is an

*ER-labeling* (or *edge-rising labeling*) if for every interval $[x, y] \subseteq L$, there is a unique maximal chain $x = x_0 < x_1 < \cdots < x_s = y$ with increasing labels, that is, with

$$\lambda(x_0, x_1) < \lambda(x_1, x_2) < \cdots < \lambda(x_{s-1}, x_s).$$

Let $\mathbf{Z}^*$ denote the set of all finite sequences of integers. One defines a lexicographic partial order $\preceq$ on $\mathbf{Z}^*$ by declaring $(a_1, \ldots, a_m) \preceq (b_1, \ldots, b_n)$ if either $a_i = b_i$ for $1 \leq i \leq m$ and $m \leq n$ or else $a_i < b_i$ for the smallest $i$ with $a_i \neq b_i$. Note that the function $\lambda$ defines a map $\overline{\lambda}$ from chains in $L$ to tuples of positive integers; namely if $c$ is the chain formed by $x_0 < x_1 < \cdots < x_s$, then

$$\overline{\lambda}(c) = \big(\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{s-1}, x_s)\big).$$

Let $\lambda$ be an ER-labeling with the further property that for all $[x, y]$, the unique increasing maximal chain $m$ has $\overline{\lambda}(m) \preceq \overline{\lambda}(m')$ for all other maximal chains $m'$ in $[x, y]$. Such an ER-labeling is called an *EL-labeling* (or *edge-lexicographic* labeling). A lattice that admits an ER-labeling is said to be *ER-labelable* and one that admits an EL-labeling is *EL-labelable.*

A paper of Li showed that comodernistic lattices are EL-labelable [18], so in particular we find that for all $n \geq 0$, $L_n$ is EL-labelable. The looser property of ER-labelability is useful in certain enumerative problems. For example, it has been shown that the zeta and Möbius transforms for ER-labelable posets $P$ can be computed in at most $|E(P)|$ elementary arithmetic operations [17].

### 7.2 Homotopy type

A simplicial complex $\Delta$ is *nonpure shellable* if its facets can be given an order $C_1, C_2, \ldots, C_m$ such that for all $2 \leq k \leq m$, the facets in the complex $\big(\bigcup_{i=1}^{k-1} C_i\big) \cap C_k$ all have dimension $\dim C_k - 1$. The earliest treatment of nonpure shellable complexes was carried out by Björner and Wachs in [2] and [3]; Corollary 13.3 of the latter asserts that a nonpure shellable complex is homotopy equivalent to a wedge of spheres. Proposition 2.3 of an earlier paper by the same authors [1] states that EL-labelable posets are nonpure shellable, so $\Delta_n$ is homotopy equivalent to a wedge of spheres. In fact, $\Delta_n$ is either contractible or homotopy equivalent to a single sphere, as the following strengthening of Lemma 5.1 shows.

**Theorem 7.1** *Let $\Delta_n$ be the order complex of the lattice of arithmetic progressions $L_n$. If $n - 1$ is not squarefree, then $\Delta_n$ is contractible. Otherwise, $\Delta_n$ has the homotopy type of $S^k$, where $k$ is the number of distinct primes dividing $n - 1$.*

PROOF: We already know, from the above discussion, that $\Delta_n$ is homotopy equivalent to a wedge of spheres. If the wedge product consisted of more than one sphere, then the sum over the ranks of the reduced homology groups of $\Delta_n$ would be greater than 1. But by Lemma 5.1, this sum equals 0 when $n - 1$ is not squarefree, in which case $\Delta_n$ must have the homotopy type of a point, and when $n - 1$ is squarefree it equals 1, meaning that there is exactly one sphere in the wedge product. □

### 7.3 Complements

We finish with a miscellaneous result about complements in $L_n$. A lattice $L$ with maximum element $\widehat{1}$ and minimum element $\widehat{0}$ is said to be *complemented* if for all $x \in L$, there exists $y \in L$ such that $x \vee y = \widehat{1}$ and $x \wedge y = \widehat{0}$. The elements $x$ and $y$ are called *complements* of one another, and if we remove the condition that $x \wedge y = \widehat{0}$, then $x$ and $y$ are said to be *upper semicomplements*. The next theorem gives a necessary and sufficient condition for $L_n$ to be complemented.

**Theorem 7.2** *Let $n \geq 2$. The lattice $L_n$ is complemented if and only if $n-1$ is squarefree. In particular, if $n-1$ is not squarefree, there exists an element $x \notin \{\emptyset, [n]\}$ of $L_n$ whose only upper semicomplement is $[n]$.*

PROOF: For the "if" direction, we note that when $n-1$ is squarefree, we have $\mu_n(L_n) \neq 0$, which, by a result of Crapo (namely, the corollary to Theorem 3 in [6]), implies that $L_n$ is complemented. For the converse, suppose that $n-1$ is divisible by $p^2$ for some prime $p$. Consider the progression

$$x = \left\{1 + \frac{n-1}{p}, 1 + \frac{2(n-1)}{p}, \dots, n - \frac{n-1}{p}\right\},$$

which has length $p-1$ and is thus not empty. Note that any $x' \in L$ satisfying $x' \vee x = [n]$ must contain both 1 and $n$ and the step size $r$ must be coprime to $(n-1)/p$. We also know that $r$ must divide $n-1$. But the only such integer $r$ is 1, in which case we see that $x$ must be $[n]$. □

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