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Almost resolvable odd cycle decompositions of $(K_u \times K_g)(\lambda)$

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Abstract

In this paper, we show that almost resolvable k-cycle decompositions of $(K_u \times K_g)(\lambda)$ (where $\times$ represents the tensor product of graphs) exist for all odd $k \geq 15$ with only a few possible exceptions.

1 Introduction

Throughout this paper all the graphs considered are finite. Specifically, if the graph G is simple then, for any $\lambda \geq 1$, we use $G(\lambda)$ (respectively, λG), to represent the multigraphs obtained from G by replacing each edge of G with uniform edge-multiplicity λ (respectively, λ edge-disjoint copies of G). Let C_s, K_s and $\bar{K}_s$ denote the *cycle, complete graph* and *complement* of the complete graph on s vertices, respectively. A *complete bipartite graph* with bipartition (U, V) is denoted by $K_{s,s}$, where $U = \{u_0, u_1, \ldots, u_{s-1}\}$ and $V = \{v_0, v_1, \ldots, v_{s-1}\}$. The edge set of $F_i(U, V) \subset K_{s,s}$ is defined as $\{u_j v_{j+i} : 0 \leq j \leq s-1\}$, for $0 \leq i \leq s-1$, where addition in the subscripts is taken modulo s. Clearly $F_i(U, V)$ is a 1-*factor* of $K_{s,s}$ with *distance* i from U to V. Also $\oplus_{i=0}^{s-1} F_i(U, V) = K_{s,s}$, where $\oplus$ denotes the edge-disjoint union of graphs.

For two graphs A and B, their *lexicographic product* $A \otimes B$ has vertex set $V(A \otimes B) = V(A) \times V(B)$ and edge set $E(A \otimes B) = \{(a_1, b_1)(a_2, b_2) | a_1 a_2 \in E(A)$ or $a_1 = a_2$ and $b_1 b_2 \in E(B)\}$. Similarly, the *tensor product* $A \times B$ of two graphs A and B has vertex set $V(A) \times V(B)$ and edge set $E(A \times B) = \{(a_1, b_1)(a_2, b_2) \mid a_1 a_2 \in E(A)$ and $b_1 b_2 \in E(B)\}$. One can easily observe that $K_u \otimes \bar{K}_g \cong K_{g,g,\ldots,g}$, the complete u-partite graph in which each partite set has g vertices. Hereafter we denote a *complete u-partite graph*, with g vertices in each partite set, as $K_u \otimes \bar{K}_g$. It is clear that $(K_u \otimes \bar{K}_g) - gK_u \cong K_u \times K_g$, where gK_u denotes g disjoint copies of K_u. For more details about product graphs, the reader is referred to [11].

We say that the graph G has an H-*decomposition* if G can be partitioned into $H_1, H_2, \ldots, H_r$ for some integer $r \geq 1$ and each $H_i \cong H$ where $H_1, H_2, \ldots, H_r$ are

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pairwise edge-disjoint subgraphs of G. A C_k-*decomposition* of H is a partition of H into edge-disjoint cycles of length k, and the existence of such a decomposition is denoted as $C_k|H$. A k-*factor* (respectively, *near* k-*factor*) of H is a k-regular spanning subgraph of H (respectively, $H\backslash\{v\}$, for some $v \in V(H)$). A k-*factorization* (respectively, near k-factorization) of H is a partition of H into edge-disjoint k-factors (respectively, near k-factors). Note that a 2-*factor* (respectively, *near* 2-*factor*) of H can also be called a C_k-factor of H (respectively, $H \setminus \{v\}$, for some $v \in V(H)$), when the components are cycles of length k. A C_k-*factorization* of H is a partition of H into edge-disjoint C_k-factors, denoted by $C_k\|H$. A near C_k-*factorization* of H is a partition of H into edge-disjoint *near* C_k-*factors*.

A partial k-factor of $(K_u \otimes \bar{K}_g)(\lambda)$ is a k-factor of $(K_u \otimes \bar{K}_g)(\lambda) \setminus V_i$, for some $i \in \{1, 2, 3, \ldots, u\}$, where $V_1, V_2, V_3, \ldots, V_u$ are the partite sets of $(K_u \otimes \bar{K}_g)(\lambda)$. A partial k-factorization (respectively, partial C_k-factorization) of $(K_u \otimes \bar{K}_g)(\lambda)$ is a decomposition of $(K_u \otimes \bar{K}_g)(\lambda)$ into partial k-factors (respectively, partial C_k-factors).

Let K be a set of integers. A resolvable K-cycle decomposition, briefly K-RCD (respectively, almost resolvable K-cycle decomposition, briefly K-ARCD) of $(K_u \otimes \bar{K}_g)(\lambda)$ is a decomposition of $(K_u \otimes \bar{K}_g)(\lambda)$ into 2-factors (respectively, partial 2-factors) consisting of cycles of lengths from K. When $K = \{k\}$, we write K-RCD as k-RCD, and K-ARCD as k-ARCD. A (k, λ)-modified cycle frame, briefly (k, λ)-MCF, of $(K_u \otimes \bar{K}_g)(\lambda)$ is a decomposition of $(K_u \otimes \bar{K}_g)(\lambda) - gK_u(\lambda)$ into partial C_k-factors. It is appropriate to mention that a k-ARCD of $(K_u \times K_g)(\lambda)$ is nothing but a (k, λ)-modified cycle frame.

Studies on RCD/ARCD have a direct relationship with various kinds of cycle frames. Cycle frames have been studied by many researchers (e.g. Stinson [18], Cao et al. [6], Niu et al. [17], Chitra et al. [7], Muthusamy et al. [16], Buratti et al. [4]), due to their applicability in many well-known combinatorial problems such as the Oberwolfach problem, the Hamilton-Waterloo problem, etc. The above facts motivated us to do some work on RCD/ARCD in the present paper.

Cao et al. [6] proved that there exists a 3-ARCD of $(K_u \times K_g)(\lambda)$. Duraimurugan et al. [8] proved that there exists a k-ARCD of $(K_u \times K_g)(\lambda)$ for all even $k \geq 6$ with a few possible exceptions. In this paper we prove that, for all odd $k \geq 15$, $u \geq 4$ and $g \geq 3$, there exists a k-ARCD of $(K_u \times K_g)(\lambda)$ if and only if $\lambda(g-1) \equiv 0 \pmod 2$ and $g(u-1) \equiv 0 \pmod k$, except possibly for $(\lambda, u, g) \in \{(2m, u, kx),\ (2m, 5, g) \mid x \equiv 2 \pmod 4 \text{ and } m \geq 1\}$, and $(\lambda, u) \in \{(2m+1, \{16, 2r+1, 4(2s+1), 4t+2, kx+1\}) \mid x \in \{2t+1, 4, 6\},\ m, t \geq 0, \text{ for even } r, s \text{ and odd } s < 15\}$.

For all odd $k \geq 3$, the necessary conditions for the existence of a k-ARCD of $(K_u \times K_g)(\lambda)$ are shown in the following theorem.

Theorem 1.1. *For all odd integers $k \geq 3$, if $(K_u \times K_g)(\lambda)$ has a k-ARCD, then*

(i) $u \geq 4$ *and* $g \geq 3$,

(ii) $g(u-1) \equiv 0 \pmod k$,

(iii) $\lambda(g-1) \equiv 0 \pmod 2$.

Proof. Since $k \geq 3$ is an odd integer, it is clear from the definition of a k-ARCD that

$u \geq 4$ and $g \geq 3$. As the existence of k-ARCD gives the edge disjoint union of partial C_k-factors of $(K_u \times K_g)(\lambda)$, the number of vertices in $(K_u \times K_g)(\lambda) \setminus V_i$, for some $i \in \{1, 2, \ldots, u\}$, where $V_1, V_2, \ldots, V_u$ are the partite sets of $(K_u \times K_g)(\lambda)$, must be divisible by k; so $g(u-1) \equiv 0 \pmod{k}$. Since each partial C_k-factor of $(K_u \times K_g)(\lambda)$ consists of $g(u-1)$ edges, the number of partial C_k-factors in $(K_u \times K_g)(\lambda)$ is

$$\lambda \frac{\frac{u(u-1)}{2} g^2 - \frac{u(u-1)}{2} g}{(u-1)g} = \lambda \frac{u(g-1)}{2}.$$

Hence there are precisely $\frac{\lambda(g-1)}{2}$ partial C_k-factors corresponding to each missing partite set V_i, $i \in \{1, 2, \ldots, u\}$. □

2 Preliminaries

To prove our results we need the following:

Theorem 2.1. *[2] For any odd integer $t \geq 3$, if $u \equiv t \pmod{2t}$, then $C_t || K_u$.*

Theorem 2.2. *[9] For any odd $m \geq 3$, there exists a near C_m-factorization of $K_{2m+1}(2)$.*

Theorem 2.3. *[2] Let k and t be odd integers such that $3 \leq k \leq t$. Then $C_t \| C_k \otimes \bar{K}_t$.*

Theorem 2.4. *[17] There exists a near $\{C_3, C_5\}$-factorization of $K_u(2)$ for $u \geq 4$ and $u \neq 5, 8$.*

Theorem 2.5. *[13] For $m \neq 2$, odd integers $k \geq 5$ and $r \geq 3$, we have $C_k \| C_k \times K_m$, $C_k \| K_k \times C_5$ and $C_r \| K_r \times C_3$.*

Theorem 2.6. *[9] For any odd $m \geq 3$ and for any $s > 0$, there exists a near C_m-factorization of $K_{ms+1}(2)$.*

Theorem 2.7. *[12] The graph $C_m \otimes \bar{K}_n$ has a Hamilton decomposition.*

Theorem 2.8. *[15] If $C_k \| G$ and $n | m$, then $C_{kn} \| G \times K_m$, where $m \not\equiv 2 \pmod{4}$, when k is odd.*

Theorem 2.9. *[4] Let g be an even integer and let $k \geq 15$ be a divisor of g. Then there exists a k-ARCD of $K_u \otimes \bar{K}_g$ for any $u \geq 4$.*

Theorem 2.10. *[4] There exists an r-ARCD of $K_{s+1} \otimes \bar{K}_4$, $s \in \{r, 2r\}$, for all odd $r \geq 15$.*

Theorem 2.11. *[10] $K_{t,t,t}$ has a C_t-factorization*

3 Basic Constructions

Theorem 3.1. *([1] Walecki's Construction.) There exists a Hamilton cycle decomposition of K_k for all $k \geq 3$.*

Proof. We break this theorem into two cases.

Case (i): $k = 2t + 1,\ t \geq 1$.
Let $V(K_{2t+1}) = \{y_0, y_1, \ldots, y_{2t}\}$ and $H = (y_0 y_1 y_2 y_{2t} y_3 y_{2t-1} y_4 y_{2m-2} \cdots y_{t+3} y_t y_{t+2} y_{t+1})$ be the Hamilton cycle. Let σ be the permutation $(y_0)(y_1 y_2 y_3 \cdots y_{2t-1} y_{2t})$. Then $H_0 = H$, $H_1 = \sigma(H)$, $H_2 = \sigma^2(H), \ldots, H_{t-1} = \sigma^{t-1}(H)$ is a Hamilton cycle decomposition of K_{2t+1}.

Case (ii): $k = 2t,\ t \geq 2$.
By using a similar procedure to the previous case, we can get $t - 1$ edge disjoint Hamilton cycles $H_0 = H,\ H_1 = \sigma(H),\ H_2 = \sigma^2(H),\ \ldots,\ H_{t-2} = \sigma^{t-2}(H)$. The remaining edges $y_0 y_t,\ y_{t-1} y_{t+1},\ y_{t-2} y_{t+2}, \ldots,\ y_1 y_{2t-1}$ form a 1-factor of K_k. ☐

Lemma 3.1. *There exists a C_{ks}-factorization of $C_k \times C_s$, for all odd integers $s, k \geq 3$.*

Proof. Let $V(C_k) = \{y_0, y_1, \ldots, y_{k-1}\}$. Then $V(C_k \times C_s) = \bigcup_{i \in \mathbb{Z}_k} Y_i$, where $Y_i = \{y_i^j \mid j \in \mathbb{Z}_s\}$. Let

(i) $\mathcal{C}^1 = \bigcup_{i=0}^{\frac{k-1}{2}} F_1(Y_{2i}, Y_{2i+1}) \oplus \bigcup_{i=0}^{\frac{k-3}{2}} F_{s-1}(Y_{2i+1}, Y_{2i+2})$ and

(ii) $\mathcal{C}^2 = \bigcup_{i=0}^{\frac{k-1}{2}} F_{s-1}(Y_{2i}, Y_{2i+1}) \oplus \bigcup_{i=0}^{\frac{k-3}{2}} F_1(Y_{2i+1}, Y_{2i+2})$,

where the subscripts of Y are taken modulo k. One can check that both $\mathcal{C}^1$ and $\mathcal{C}^2$ are C_{ks}-factors of $C_k \times C_s$. ☐

Lemma 3.2. *There exists a C_3-factorization of $C_3 \times K_{2t+1}$, for all $t \geq 1$.*

Proof. Let $V(C_3) = \{y_0, y_1, y_2\}$ and $V(C_3 \times K_{2t+1}) = \cup_{i \in \mathbb{Z}_3} Y_i$, where $Y_i = \{y_i^j \mid j \in \mathbb{Z}_{2t+1}\}$. Let $G_i = F_i(Y_0, Y_1) \oplus F_i(Y_1, Y_2) \oplus F_{2(t-i)+1}(Y_2, Y_0),\ 1 \leq i \leq 2t$, where the subscripts of F are taken modulo $(2t+1)$. One can check that each G_i, $1 \leq i \leq 2t$, is a C_3-factor of $C_3 \times K_{2t+1}$ and $\bigcup_{i=1}^{2t} G_i$ gives a C_3-factorization of $C_3 \times K_{2t+1}$. ☐

Lemma 3.3. *There exists a C_k-factorization of $C_s \times K_k$, for all odd integers s, k with $k \geq s \geq 3$.*

Proof. We break this lemma into two cases.

Case (i): $s = k$
Let $V(C_k) = \{y_0, y_1, \ldots, y_{k-1}\}$ and $V(C_k \times K_k) = \cup_{i \in \mathbb{Z}_k} Y_i$, where $Y_i = \{y_i^j \mid j \in \mathbb{Z}_k\}$. Let $G_j = \bigcup_{i \in \mathbb{Z}_k} F_j(Y_i, Y_{i+1})$, $1 \leq j \leq k-1$, where the subscripts of Y are taken modulo k. One can check that each G_j, $1 \leq j \leq k-1$, is a C_k-factor of $C_k \times K_k$ and $\bigcup_{j=1}^{k-1} G_j$ gives a C_k-factorization of $C_k \times K_k$.

Case (ii): $s < k$
We can write

$$\begin{aligned} C_s \times K_k &\cong C_s \times \{C_k^1 \oplus C_k^2 \oplus \cdots \oplus C_k^{\frac{k-1}{2}}\}, \text{ by Theorem 3.1} \\ &\cong \oplus_{i=1}^{\frac{k-1}{2}} (C_s \times C_k^i) \\ &\cong \oplus_{i=1}^{\frac{k-1}{2}} (C_k^i \times C_s). \end{aligned} \tag{1}$$

Now we consider $C_k^i \times C_s \cong C_k \times C_s$ and find its C_k-factors as follows:

Let $V(C_k) = \{y_0, y_1, \ldots, y_{k-1}\}$ and $V(C_k \times C_s) = \bigcup_{i \in \mathbb{Z}_k} Y_i$, where $Y_i = \{y_i^j \mid j \in \mathbb{Z}_s\}$. Let

(1) $\mathcal{C}^1 = \bigcup_{i=0}^{\frac{k+s}{2}-1} F_1(Y_i, Y_{i+1}) \oplus \bigcup_{i=0}^{\frac{k-s}{2}-1} F_{s-1}(Y_{\frac{k+s}{2}+i}, Y_{\frac{k+s}{2}+i+1})$ and

(2) $\mathcal{C}^2 = \bigcup_{i=0}^{\frac{k+s}{2}-1} F_{s-1}(Y_i, Y_{i+1}) \oplus \bigcup_{i=0}^{\frac{k-s}{2}-1} F_1(Y_{\frac{k+s}{2}+i}, Y_{\frac{k+s}{2}+i+1})$,

where the subscripts of Y are taken modulo k. One can check that both $\mathcal{C}^1$ and $\mathcal{C}^2$ are C_k-factors of $C_k \times C_s$ and together give a C_k-factorization of $C_k^i \times C_s$. Thus each $C_s \times C_k^i$ has two C_k-factors and hence together they give a C_k-factorization of $C_s \times K_k$. □

Lemma 3.4. *There exists a C_k-factorization of $C_k \times K_s$, for all odd integers $s, k \geq 3$.*

Proof. Let $V(C_k) = \{y_0, y_1, \ldots, y_{k-1}\}$. Then $V(C_k \times K_s) = \bigcup_{i \in \mathbb{Z}_k} Y_i$, where $Y_i = \{y_i^j \mid j \in \mathbb{Z}_s\}$.
Case (i): $k = s$ and $s < k$.
The proof follows from Lemma 3.3.
Case (ii): $s > k$.
Let

(1) $G_i = \bigcup_{j=0}^{\frac{k-3}{2}} F_i(Y_{2j}, Y_{2j+1}) \oplus \bigcup_{j=0}^{\frac{k-5}{2}} F_{s-i}(Y_{2j+1}, Y_{2j+2}) \oplus F_i(Y_{k-2}, Y_{k-1}) \oplus F_{s-2i}(Y_{k-1}, Y_0)$
and

(2) $H_i = \bigcup_{j=0}^{\frac{k-3}{2}} F_{s-i}(Y_{2j}, Y_{2j+1}) \oplus \bigcup_{j=0}^{\frac{k-5}{2}} F_i(Y_{2j+1}, Y_{2j+2}) \oplus F_{s-i}(Y_{k-2}, Y_{k-1}) \oplus F_{2i}(Y_{k-1}, Y_0)$,

where $1 \leq i \leq \frac{s-1}{2}$. One can check that both G_i and H_i, $1 \leq i \leq \frac{s-1}{2}$, are C_k-factors of $C_k \times K_s$. Thus $\bigcup_{i=1}^{\frac{s-1}{2}} (G_i \oplus H_i)$ together gives a C_k-factorization of $C_k \times K_s$. □

Lemma 3.5. *There exists a partial C_5-factorization of $K_7 \times K_5$.*

Proof. To prove this lemma we consider the near C_3-factorization of $K_7(2)$, which exists by Theorem 2.2. Hence we write

$$\begin{aligned} K_7(2) \times K_5 &\cong (G_0 \oplus G_1 \oplus \cdots \oplus G_6) \times K_5 \text{ by Theorem 2.2} \\ &\cong \oplus_{i \in \mathbb{Z}_7} (G_i \times K_5), \end{aligned}$$

where each $G_i \cong \{(x_{6+i}\ x_{5+i}\ x_{3+i}),\ (x_{1+i}\ x_{2+i}\ x_{4+i})\}$, $i \in \mathbb{Z}_7$, (where the subscripts of x are taken modulo 7) is a near C_3-factor of $K_7(2)$.

Now we write

$$\begin{aligned} G_i \times K_5 &\cong (C_3 \oplus C_3) \times K_5 \\ &\cong (C_3 \times K_5) \oplus (C_3 \times K_5) \\ &\cong C_3 \times (C_5 \oplus C_5) \oplus C_3 \times (C_5 \oplus C_5) \text{ by Theorem 3.1} \\ &\cong (C_3 \times C_5) \oplus (C_3 \times C_5) \oplus (C_3 \times C_5) \oplus (C_3 \times C_5). \end{aligned}$$

From the proof of case (ii) of Lemma 3.3, each $C_3 \times C_5$ ($\cong C_5 \times C_3$) has two C_5-factors, namely $\mathcal{C}^1$ and $\mathcal{C}^2$, and hence each $G_i \times K_5$ has a C_5-factorization. Now the collection of all the $\mathcal{C}^1$ from the C_5-factorization of each $G_i \times K_5$, $i \in \mathbb{Z}_7$, together gives two partial C_5-factors of $K_7 \times K_5$; see Figures 1–4. Also, the collection of all the $\mathcal{C}^2$ from the C_5-factorization of each $G_i \times K_5$, $i \in \mathbb{Z}_7$, together gives another two partial C_5-factors of $K_7 \times K_5$; see Figures 1–4.

Finally, the collection of either $\mathcal{C}^1$ (or $\mathcal{C}^2$) gives the required partial C_5-factorization of $K_7 \times K_5$; see Figure 5. □

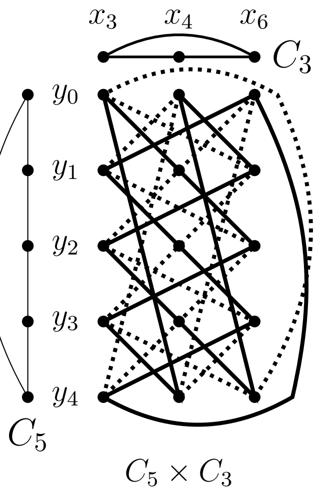


Fig. 1.

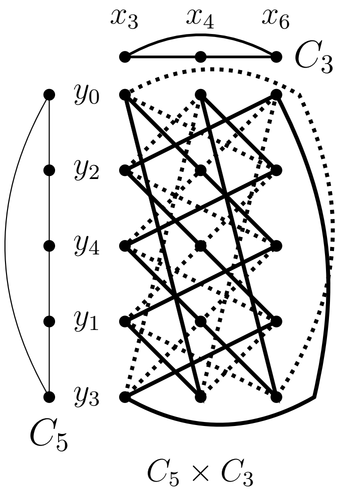


Fig. 2.

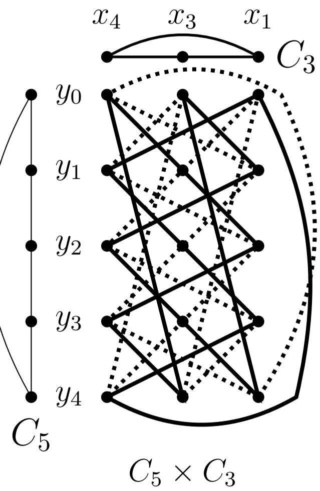


Fig.3.

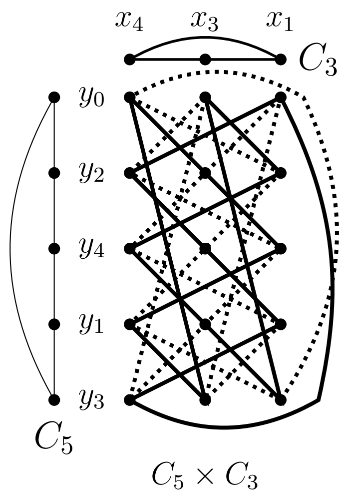
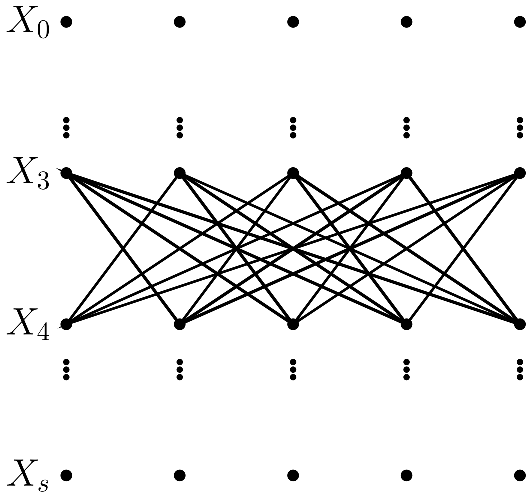


Fig.4.

— denotes the edge set of $\mathcal{C}^1$



$K_{2s+1} \times K_5$
Fig.5.

…denotes the edge set of $\mathcal{C}^2$

Theorem 3.2. *There exists a partial C_k-factorization of $K_4 \times K_k$ for all odd $k \geq 3$.*

Proof. Let $V(K_4) = \{x_0, x_1, x_2, x_3\}$. The near C_3-factor $G_i = (x_{1+i}x_{2+i}x_{3+i})$, with missing vertex x_i, $0 \leq i \leq 3$, where the subscripts of x are taken modulo 4, generate the near C_3-factorization of $K_4(2)$. Now we can write

$$\begin{aligned} K_4(2) \times K_k &\cong \{\mathcal{C}_3^0 \oplus \cdots \oplus \mathcal{C}_3^3\} \times K_k \\ &\cong (\mathcal{C}_3^0 \times K_k) \oplus \cdots \oplus (\mathcal{C}_3^3 \times K_k) \\ &\cong \oplus_{i \in \mathbb{Z}_3}(\mathcal{C}_3^i \times K_k), \end{aligned} \tag{2}$$

where each $\mathcal{C}_3^i$ is a near C_3-factor of $K_4(2)$. Now we consider $\mathcal{C}_3^i \times K_k \cong C_3 \times K_k$ and find its C_3-factors as follows:

Case (i): $k = 3$

By the proof of case (i) of Lemma 3.3, the collection of all C_3-factors of $C_3 \times K_3$ generated by G_1 (respectively, G_2) together give the required partial C_3-factorization of $K_4 \times K_3$.

Case (ii) $k > 3$

Using (1) of Lemma 3.3, we have $\mathcal{C}_3^i \times K_k \cong \oplus_{j=1}^{\frac{k-1}{2}}(C_k^j \times C_3)$, $i \in \mathbb{Z}_3$. By the proof of case (ii) of Lemma 3.3, each $C_k^j \times C_3$ has two C_k-factors, namely $\mathcal{C}^1$ and $\mathcal{C}^2$. Now the collection of all the $\mathcal{C}^1$ from $\mathcal{C}_3^i \times K_k,\ i = 0, 2$, and the collection of all the $\mathcal{C}^2$ from $\mathcal{C}_3^i \times K_k,\ i = 1, 3$ together give a partial C_k-factorization of $K_4 \times K_k$. In a similar manner, we can also have another partial C_k-factorization of $K_4 \times K_k$ by taking all the $\mathcal{C}^1$ from $\mathcal{C}_3^i \times K_k,\ i = 1, 3$, and $\mathcal{C}^2$ from $\mathcal{C}_3^i \times K_k,\ i = 0, 2$. One of the above gives the required partial C_k-factorization of $K_4 \times K_k$. □

Theorem 3.3. *There exists a partial C_3-factorization of $K_4 \times K_k$ for all odd $k \geq 3$.*

Proof. Using (2) of Theorem 3.2, we have $K_4(2) \times K_k \cong \oplus_{i \in \mathbb{Z}_3}(\mathcal{C}_3^i \times K_k)$, where each $\mathcal{C}_3^i$, $0 \leq i \leq 3$, is a near C_3-factor of $K_4(2)$. Hence we write $\mathcal{C}_3^i \times K_k \cong C_3 \times K_k$ and by the proof of Lemma 3.2, the collection of all C_3-factors of $C_3 \times K_k$ generated by G_i, $1 \leq i \leq \frac{k-1}{2}$ (respectively, G_i, $\frac{k+1}{2} \leq i \leq k-1$) together give a required partial C_3-factorization of $K_4 \times K_k$. □

Theorem 3.4. *There exists a partial C_k-factorization of $K_{2s+1} \times K_k$, for all odd integers s, k with $k \geq s \geq 3$.*

Proof. By Theorem 2.2, $K_{2s+1}(2)$ has a near C_s-factorization. Now we write

$$\begin{aligned} K_{2s+1}(2) \times K_k &\cong \{\mathcal{C}_s^0 \oplus \mathcal{C}_s^1 \oplus \cdots \oplus \mathcal{C}_s^{2s}\} \times K_k, \quad \text{by Theorem 2.2} \\ &\cong (\mathcal{C}_s^0 \times K_k) \oplus \cdots \oplus (\mathcal{C}_s^{2s} \times K_k) \\ &\cong \oplus_{i \in \mathbb{Z}_{2s+1}}(\mathcal{C}_s^i \times K_k), \end{aligned} \tag{3}$$

where each $\mathcal{C}_s^i$, $0 \leq i \leq 2s$, is a near C_s-factor of $K_{2s+1}(2)$, and each near C_s-factor contains two cycles of length s.

Hence we write $\mathcal{C}_s^i \times K_k \cong 2(C_s \times K_k)$ and find its C_k-factors as follows:

Case (i): $s = k$

By the proof of case (i) of Lemma 3.3, the collection of all C_k-factors of $\mathcal{C}_k^i \times K_k$ generated by G_l, $1 \leq l \leq \frac{k-1}{2}$ (respectively, G_l, $\frac{k+1}{2} \leq l \leq k-1$) together give a required partial C_k-factorization of $K_{2k+1} \times K_k$.

Case (ii): $s < k$

By the proof of case (ii) of Lemma 3.3, the collection of all $\mathcal{C}^1$ (respectively, $\mathcal{C}^2$) from $\mathcal{C}_s^i \times K_k$ together give a required partial C_k-factorization of $K_{2s+1} \times K_k$. □

Theorem 3.5. *There exists a partial C_k-factorization of $K_{2k+1} \times K_s$, for all odd integers $s, k \geq 3$.*

Proof. Using (3) of Theorem 3.3, we have $K_{2k+1}(2) \times K_s \cong \oplus_{i \in \mathbb{Z}_{2k+1}}(\mathcal{C}_k^i \times K_s)$, where each $\mathcal{C}_k^i$, $0 \leq i \leq 2k$, is a near C_k-factor of $K_{2k+1}(2)$, and each near C_k-factor contains two cycles of length k.

Now we consider $\mathcal{C}_k^i \times K_s \cong 2(C_k \times K_s)$, and find its C_k-factors as follows:

Case (i): $k = s$ and $s < k$.

The proof follows from Theorem 3.4.

Case (ii): $s > k$.

By the proof of case (ii) of Lemma 3.4, the collection of all C_k-factors of $\mathcal{C}_k^i \times K_s$ generated by G_l (respectively, H_l), $1 \leq l \leq \frac{s-1}{2}$, together give a required partial C_k-factorization of $K_{2k+1} \times K_s$. □

Theorem 3.6. *There exists a partial C_{ks}-factorization of $K_{2k+1} \times K_s$, for all odd integers $s, k \geq 3$.*

Proof. Using (3) of Theorem 3.3, we have $K_{2k+1}(2) \times K_s \cong \oplus_{i \in \mathbb{Z}_{2k+1}} (\mathcal{C}_k^i \times K_s)$, where each $\mathcal{C}_k^i$, $0 \leq i \leq 2k$, is a near C_k-factor of $K_{2k+1}(2)$, and each near C_k-factor contains two cycles of length k.

Now we consider $\mathcal{C}_k^i \times K_s \cong 2(C_k \times K_s)$, and find its C_{ks}-factors as follows:

By using (1) of Lemma 3.3, we can write $C_k \times K_s \cong \oplus_{j=1}^{\frac{s-1}{2}} (C_k \times C_s^j)$. By the proof of Lemma 3.1, each $C_k \times C_s^j$ ($\cong C_k \times C_s$) has two C_{ks}-factors, namely $\mathcal{C}^1$ and $\mathcal{C}^2$. Now the collection of all the $\mathcal{C}^1$ from $\mathcal{C}_k^i \times C_s$ together gives a required partial C_{ks}-factorization of $K_{2k+1} \times K_s$. Similarly, the collection of all the $\mathcal{C}^2$ from $\mathcal{C}_k^i \times C_s$ together also gives another partial C_{ks}-factorization of $K_{2k+1} \times K_s$. □

Construction 1. *If there exists a near* 2*-factorization of* $K_u(2)$ *(a near* 2*-factor* F *can contain different cycle lengths with* $V(F) = u - 1$*, and the set of cycle lengths is denoted by* J*), a* C_t*-factorization of* $C_t \times K_x$*, a* C_k*-factorization of* $C_t \otimes \bar{K}_k$ *and a* C_k*-factorization of* $C_t \times K_k$ *for any* $t \in J$*, then there exists a partial* C_k*-factorization of* $(K_u \times K_{kx})(2)$*.*

4 k-ARCD of $(K_u \times K_g)(\lambda)$

In this section we investigate the existence of a k-ARCD of the tensor product of complete graphs with edge multiplicity λ.

Theorem 4.1. *For all odd* $k \geq 5$*,* $u \geq 4$ *and* $g \equiv 0 \pmod{k}$*, there exists a* k*-ARCD of* $(K_u \times K_g)(2)$*, except possibly when* $(u, g) \in \{(u, kx), (5, g) \mid x \equiv 2 \pmod{4}\}$*.*

Proof. Let $g = kx$, where $x \geq 1$.

By Theorems 2.4 and 2.6, let $\mathcal{F} = \{F_0, F_2, \ldots, F_{u-1}\}$ be the near 2-factorization of $K_u(2)$ (each near 2-factor F_i, $0 \leq i \leq u - 1$, may contain different cycle lengths with $V(F_i) = u - 1$, and the set of all of cycle lengths is denoted by $J = \{3, 5, 7\}$). A C_t-factorization of $C_t \times K_x$ and a C_k-factorization $C_t \otimes \bar{K}_k$, for any $t \in J$, can be obtained by Theorems 2.8 and 2.3, respectively. Furthermore, $C_t \times K_k$ has a C_k-factorization by Theorem 2.5 and Lemma 3.3. Then by using construction 1, we get a required partial C_k-factorization of $(K_u \times K_g)(2)$. Therefore a k-ARCD of $(K_u \times K_g)(2)$ exists. □

Theorem 4.2. *For all odd integers* s, k *with* $3 \leq s \leq k$*,* $u = 2s + 1$ *and* $g \equiv k \pmod{2k}$*, there exists a* k*-ARCD of* $K_u \times K_g$*.*

Proof. Let $g = k(2t + 1)$, $t \geq 1$. We can write

$$\begin{aligned} K_u \times K_g &\cong K_{2s+1} \times K_{k(2t+1)} \\ &\cong \{(K_{2s+1} \times K_{2t+1}) \otimes \bar{K}_k\} \oplus (2t + 1)(K_{2s+1} \times K_k). \end{aligned} \tag{4}$$

The right-hand side of (4) can be obtained by making $2t + 1$ holes of type $K_{2s+1} \times K_k$ and identifying each k-subset of $K_{k(2t+1)}$ (in the resulting graph) into a single vertex, with two of them being adjacent if the corresponding k-subsets form a $K_{k,k}$

in $K_{2s+1} \times K_{k(2t+1)}$. The resulting graph is isomorphic to $K_{2s+1} \times K_{2t+1}$. Expand the identified vertices into k-subsets and two k-subsets, to form a $K_{k,k}$ whenever their corresponding vertices are adjacent in $K_{2s+1} \times K_{2t+1}$. Thus the resulting expanded graph will be isomorphic to the first graph of (4).

Now we construct the partial C_k-factors of the right-hand side of (4) as follows:

$$\text{Consider} \quad (K_{2s+1} \times K_{2t+1}) \otimes \bar{K}_k \cong \{\mathcal{C}_s^0 \oplus \cdots \oplus \mathcal{C}_s^{2s}\} \otimes \bar{K}_k, \text{ by Theorem 3.5}$$
$$\cong (\mathcal{C}_s^0 \otimes \bar{K}_k) \oplus \cdots \oplus (\mathcal{C}_s^{2s} \otimes \bar{K}_k)$$
$$\cong \oplus_{i \in \mathbb{Z}_{2s+1}} (\mathcal{C}_s^i \otimes \bar{K}_k).$$

Note that each $\mathcal{C}_s^i$, $0 \le i \le 2s$, contains t partial C_s-factors of $K_{2s+1} \times K_{2t+1}$, and each partial C_s-factor contains $2(2t+1)$ cycles of length s. Hence we write $\mathcal{C}_s^i \otimes \bar{K}_k \cong t\{2(2t+1)(C_s \otimes \bar{K}_k)\}$ and by Theorem 2.3, each $C_s \otimes \bar{K}_k$ has k C_k-factors. Thus we have obtained kt partial C_k-factors of $(K_{2s+1} \times K_{2t+1}) \otimes \bar{K}_k$ corresponding to each $\mathcal{C}_s^i \otimes \bar{K}_k$. When i varies, we get $(2s+1)kt$ partial C_k-factors of $(K_{2s+1} \times K_{2t+1}) \otimes \bar{K}_k$.

Furthermore, by Theorem 3.4, the second graph $(2t+1)(K_{2s+1} \times K_k)$ has $(2s+1)\frac{k-1}{2}$ partial C_k-factors. Finally, the partial C_k-factors of the two graphs of the right-hand side of (4) obtained above together give a required partial C_k-factorization of $K_{2s+1} \times K_{k(2t+1)} \cong K_u \times K_g$. □

Theorem 4.3. *For all odd integers r, k with $15 \le r \le k$, $u = 4(s+1)$, $s \in \{r, 2r\}$ and $g \equiv k \pmod{2k}$, there exists a k-ARCD of $K_u \times K_g$.*

Proof. Let $g = k(2t+1)$, $t \ge 1$. We can write

$$K_u \times K_g \cong K_{4(s+1)} \times K_g$$
$$\cong \{(K_{s+1} \otimes \bar{K}_4) \oplus (s+1)K_4\} \times K_g$$
$$\cong \{(K_{s+1} \otimes \bar{K}_4) \times K_g\} \oplus (s+1)(K_4 \times K_g). \qquad (5)$$

Now we construct the partial C_k-factors of the right-hand side of (5) as follows:

$$\text{Consider} \quad (K_{s+1} \otimes \bar{K}_4) \times K_g \cong \{\mathcal{C}_r^0 \oplus \cdots \oplus \mathcal{C}_r^s\} \times K_g, \text{ by Theorem 2.10}$$
$$\cong (\mathcal{C}_r^0 \times K_g) \oplus \cdots \oplus (\mathcal{C}_r^s \times K_g)$$
$$\cong \oplus_{i \in \mathbb{Z}_{s+1}} (\mathcal{C}_r^i \times K_g).$$
$$\text{We know that} \quad \mathcal{C}_r^i \times K_g \cong \mathcal{C}_r^i \times K_{k(2t+1)}, \ i \in \mathbb{Z}_{s+1}$$
$$\cong 2\Big\{\frac{4s}{r}(C_r \times K_{k(2t+1)})\Big\},$$

where each $\mathcal{C}_r^i$, $0 \le i \le s$, contains two partial C_r-factors of $K_{s+1} \otimes \bar{K}_4$, in which each partial C_r-factor contains $4s/r$ cycles of length r.

$$\text{We write} \quad C_r \times K_{k(2t+1)} \cong C_r \times \{K_{2t+1} \otimes \bar{K}_k \oplus (2t+1)K_k\}$$
$$\cong \{(C_r \times K_{2t+1}) \otimes \bar{K}_k\} \oplus (2t+1)(C_r \times K_k). \qquad (6)$$

By applying a similar procedure to (4) we get the right-hand side of (6). By Theorems 2.5 and 2.3, $(C_r \times K_{2t+1}) \otimes \bar{K}_k$ has $\frac{4tk}{2}$ C_k-factors of $C_r \times K_{k(2t+1)}$. By

Theorem 3.3, $(2t+1)(C_r \times K_k)$ has $(k-1)$ C_k-factors of $C_r \times K_{k(2t+1)}$. Put together we get $\frac{2(g-1)}{2}$ C_k-factors of $C_r \times K_{k(2t+1)}$. Thus we have obtained $\frac{4(g-1)}{2}$ partial C_k-factors of $(K_{s+1} \otimes \bar{K}_4) \times K_g$ corresponding to each missing partite set of size $4g$ which makes a hole $K_4 \times K_g$ of $K_u \times K_g$. Note that corresponding to each partite set of $(K_{s+1} \otimes \bar{K}_4) \times K_g$ we have a hole $K_4 \times K_g$ of $K_u \times K_g$ in (5). To complete the proof, it is enough to find the partial C_k-factors of the hole $K_4 \times K_g$ corresponding to the missing partite set, and this put together to get the required partial C_k-factors of $K_u \times K_g$.

Now consider the second graph of (5), that is,

$$\begin{aligned} K_4 \times K_g &\cong K_4 \times K_{k(2t+1)} \\ &\cong K_4 \times \{K_{2t+1} \otimes \bar{K}_k \oplus (2t+1)K_k\} \\ &\cong \{(K_4 \times K_{2t+1}) \otimes \bar{K}_k\} \oplus (2t+1)(K_4 \times K_k). \end{aligned} \tag{7}$$

By applying a similar procedure to (4) we get the right-hand side of (7).

Now we construct the partial C_k-factors of the right-hand side of (7) as follows:

By Theorem 3.3, $K_4 \times K_{2t+1}$ has $\frac{4(2t)}{2}$ partial C_3-factors, and by Theorem 2.11, each $C_3 \otimes \bar{K}_k$ ($\cong K_{k,k,k}$) has k C_k-factors. Thus we have obtained $\frac{4(2tk)}{2}$ partial C_k-factors of $K_4 \times K_g$, corresponding to the first graph $(K_4 \times K_{2t+1}) \otimes \bar{K}_k$. Furthermore, by Theorem 3.2, $(2t+1)(K_4 \times K_k)$ has $\frac{4(k-1)}{2}$ partial C_k-factors of $K_4 \times K_g$. Put together, we have $\frac{4(g-1)}{2}$ partial C_k-factors of $K_4 \times K_g$. Finally, the combination of the $\frac{4(g-1)}{2}$ partial C_k-factors of the first graph of the right-hand side of (5) corresponds to one missing partite set of size $4g$ and the $\frac{4(g-1)}{2}$ partial C_k-factors of the hole $K_4 \times K_g$, which correspond to that missing partite set, together gives $\frac{4(g-1)}{2}$ partial C_k-factors of $K_u \times K_g$. Repeat the process $s+1$ times (as there are $s+1$ holes) to get $4(s+1)\frac{g-1}{2}$ partial C_k-factors of $K_u \times K_g$. Thus a k-ARCD of $K_u \times K_g$ exists. □

Theorem 4.4. *For all odd $k \geq 5$, $u \equiv 1 \pmod{k}$ and $g \geq 3$, there exists a k-ARCD of $(K_u \times K_g)(2)$.*

Proof. Let $u = kx + 1$, $x \geq 1$. We can write

$$\begin{aligned} (K_u \times K_g)(2) &\cong K_u(2) \times K_g \\ &\cong (\mathcal{C}_k^0 \oplus \mathcal{C}_k^1 \oplus \cdots \oplus \mathcal{C}_k^{u-1}) \times K_g, \text{ by Theorem 2.6} \\ &\cong (\mathcal{C}_k^0 \times K_g) \oplus (\mathcal{C}_k^1 \times K_g) \oplus \cdots \oplus (\mathcal{C}_k^{u-1} \times K_g) \\ &\cong \oplus_{i \in \mathbb{Z}_u} (\mathcal{C}_k^i \times K_g), \end{aligned}$$

where each $\mathcal{C}_k^i$, $0 \leq i \leq u-1$ is a near C_k-factor of $K_u(2)$, and each near C_k-factor contains x cycles of length k. Hence we write $\mathcal{C}_k^i \times K_g \cong x(C_k \times K_g)$. By Theorem 2.5, each $C_k \times K_g$ has $g-1$ C_k-factors. The collection of all C_k-factors from $\mathcal{C}_k^i \times K_g$ together gives some partial C_k-factors of $(K_u \times K_g)(2)$. Thus, in total, we get $u(g-1)$ partial C_k-factors of $(K_u \times K_g)(2)$. Therefore a k-ARCD of $(K_u \times K_g)(2)$ exists. □

Theorem 4.5. *For all $u = 2k+1$, with odd integers $k, g \geq 3$, there exists a k-ARCD of $K_u \times K_g$.*

Proof. The proof follows from Theorem 3.5. □

Theorem 4.6. *For all $u = 2kx + 1$, with odd integers k and g with $k \geq 5$, $g \geq 3$, and for $x \geq 4$, there exists a k-ARCD of $K_u \times K_g$.*

Proof. Given that $u = 2kx + 1$ and $g \geq 3$ is odd, where $x \geq 4$, we can write

$$\begin{aligned} K_{2kx+1} \times K_g &\cong \{(K_x \otimes \bar{K}_{2k}) \oplus xK_{2k+1}\} \times K_g & (8)\\ &\cong \{(K_x \otimes \bar{K}_{2k}) \times K_g\} \oplus x(K_{2k+1} \times K_g). & (9) \end{aligned}$$

The right-hand side of (8) can be obtained by making x holes of type K_{2k} in K_{2kx+1} and adjoining the omitted vertex, say ∞, to each hole K_{2k} to get x copies of K_{2k+1}.

Now we construct the partial C_k-factors of the right-hand side of (9) as follows. Consider

$$\begin{aligned} (K_x \otimes \bar{K}_{2k}) \times K_g &\cong \{\mathcal{C}_k^0 \oplus \cdots \oplus \mathcal{C}_k^{x-1}\} \times K_g, \text{ by Theorem 2.9} & (10)\\ &\cong (\mathcal{C}_k^0 \times K_g) \oplus \cdots \oplus (\mathcal{C}_k^{x-1} \times K_g), \end{aligned}$$

where each $\mathcal{C}_k^i$, $0 \leq i \leq x-1$, contains $\frac{2k}{2}$ partial C_k-factors of $(K_x \otimes \bar{K}_{2k})$, and each partial C_k-factor contains $2(x-1)$ cycles of length k. Hence we write $\mathcal{C}_k^i \times K_g \cong \frac{2k}{2}\{2(x-1)(C_k \times K_g)\}$ and each $C_k \times K_g$ has $g-1$ C_k-factors by Theorem 2.5. Thus we have obtained $\frac{2k(g-1)}{2}$ partial C_k-factors of $(K_x \otimes \bar{K}_{2k}) \times K_g$ corresponding to each missing partite set of size $2kg$ which makes a hole $K_{2k+1} \times K_g$ of $K_u \times K_g$. Note that corresponding to each partite set of $(K_x \otimes \bar{K}_{2k}) \times K_g$ we have a hole $K_{2k+1} \times K_g$ of $K_u \times K_g$ in (9). To complete the proof, it is enough to find the partial C_k-factors of the hole $K_{2k+1} \times K_g$ corresponding to the missing partite set and put together to get the required partial C_k-factors of $K_u \times K_g$.

By Theorem 4.5, $K_{2k+1} \times K_g$ has $\frac{2k(g-1)}{2}$ partial C_k-factors with missing partite sets corresponding to the vertices of K_{2k} and $\frac{g-1}{2}$ partial C_k-factors with missing partite set corresponding to the vertex ∞. The combination of the $\frac{2k(g-1)}{2}$ partial C_k-factors of $(K_x \otimes \bar{K}_{2k}) \times K_g$ corresponding to one missing partite set of size $2kg$ together with $\frac{2k(g-1)}{2}$ partial C_k-factors of $K_{2k+1} \times K_g$, which correspond to that missing partite set of $(K_x \otimes \bar{K}_{2k}) \times K_g$, gives $\frac{2k(g-1)}{2}$ partial C_k-factors of $K_{2kx+1} \times K_g$. As there are x holes, repeating the above process x times, we get $\frac{2kx(g-1)}{2}$ partial C_k-factors of $K_{2kx+1} \times K_g$. Further, the collection of all $\frac{g-1}{2}$ partial C_k-factors of each of the x copies of $K_{2k+1} \times K_g$ with missing partite set that corresponds to the vertex ∞, together gives $\frac{g-1}{2}$ partial C_k-factors of $K_{2kx+1} \times K_g$. Therefore a k-ARCD of $K_u \times K_g$ exists. □

Theorem 4.7. *Let $k = q_1q_2 \ldots q_k \geq 9$ be odd, where $q_1, q_2, \ldots, q_k \geq 3$ are odd and not necessarily distinct. If $u \equiv 1 \pmod{q_1q_2 \ldots q_i}$ and $g = (q_{i+1}q_{i+2} \ldots q_k)y$, $1 \leq i \leq k-1$, then there exists a k-ARCD of $(K_u \times K_g)(2)$, except possibly when $y \equiv 2 \pmod 4$.*

Proof. Let $u = mx+1$, $g = ny$ and $k = mn$ where $m = q_1q_2\dots q_i$, $n = q_{i+1}q_{i+2}\dots q_k$, and $x, y \geq 1$. We can write

$$\begin{aligned}(K_u \times K_g)(2) &\cong K_{mx+1}(2) \times K_{ny} \\ &\cong \{\mathcal{C}_m^0 \oplus \mathcal{C}_m^1 \oplus \cdots \oplus \mathcal{C}_m^{mx}\} \times K_{ny} \\ &\cong (\mathcal{C}_m^0 \times K_{ny}) \oplus \cdots \oplus (\mathcal{C}_m^{mx} \times K_{ny}) \\ &\cong \oplus_{j\in\mathbb{Z}_{mx+1}}(\mathcal{C}_m^j \times K_{ny}),\end{aligned}$$

where each $\mathcal{C}_m^j$, $0 \leq j \leq mx$, is a near C_m-factor of $K_{mx+1}(2)$, and each near C_m-factor contains x cycles of length m. Hence we write $\mathcal{C}_m^j \times K_{ny} \cong x(C_m \times K_{ny})$ and by applying a similar procedure to (4) we get the following:

$$\begin{aligned}C_m \times K_{ny} &\cong C_m \times \{(K_y \otimes \bar{K}_n) \oplus yK_n\} \\ &\cong \{(C_m \times K_y) \otimes \bar{K}_n\} \oplus y(C_m \times K_n). \qquad (11)\end{aligned}$$

By Theorem 2.8, $C_m \times K_y$ has $(y-1)$ C_m-factors, and $C_m \otimes \bar{K}_n$ has n C_{mn}-factors by Theorem 2.7. Thus we have obtained $n(y-1)$ C_{mn}-factors of $C_m \times K_{ny}$ corresponding to the first graph $(C_m \times K_y) \otimes \bar{K}_n$. Furthermore, by Theorem 3.1, $y(C_m \times K_n)$ has $n-1$ C_{mn}-factors. Finally, the C_{mn}-factors of the two graphs of the right-hand side of (11) together give $(ny-1)$ C_{mn}-factors of $C_m \times K_{ny}$. The collection of all C_{mn}-factors of $\mathcal{C}_m^j \times K_{ny}$ together gives a required partial C_{mn}-factorzation of $(K_u \times K_g)(2)$. □

Theorem 4.8. *Let $k = q_1q_2\dots q_k \geq 9$ be odd, where $q_1, q_2, \dots, q_k \geq 3$ are odd and not necessarily distinct. If $u = 2(q_1q_2\dots q_i) + 1$ and $g \equiv (q_{i+1}q_{i+2}\dots q_k)$ (mod $2q_{i+1}q_{i+2}\dots q_k$), $1 \leq i \leq k-1$, then there exists a k-ARCD of $K_u \times K_g$.*

Proof. Let $u = 2m+1$, $g = (2x+1)n$, and $k = mn$, where $m = q_1q_2\dots q_i$, $n = q_{i+1}q_{i+2}\dots q_k$, and $x \geq 1$. We can write

$$K_{2m+1} \times K_{(2x+1)n} \cong \{(K_{2m+1} \times K_{2x+1}) \otimes \bar{K}_n\} \oplus (2x+1)(K_{2m+1} \times K_n). \qquad (12)$$

By applying a similar procedure to (4) we get the right-hand side of (12).

We construct a partial C_{mn}-factorization of the right-hand side of (12) as follows. First we consider

$$\begin{aligned}(K_{2m+1} \times K_{2x+1}) \otimes \bar{K}_n &\cong \{\mathcal{C}_m^0 \oplus \cdots \oplus \mathcal{C}_m^{2m}\} \otimes \bar{K}_n, \text{ by Theorems 3.5} \\ &\cong (\mathcal{C}_m^0 \otimes \bar{K}_n) \oplus \cdots \oplus (\mathcal{C}_m^{2m} \otimes \bar{K}_n) \\ &\cong \oplus_{i\in\mathbb{Z}_{2m+1}}(\mathcal{C}_m^i \otimes \bar{K}_n),\end{aligned}$$

where each $\mathcal{C}_m^j$, $0 \leq j \leq 2m$, contains x partial C_m-factors of $K_{2m+1} \times K_{2x+1}$, in which each partial C_m-factor contains $2(2x+1)$ cycles of length m. Hence we write $\mathcal{C}_m^j \times \bar{K}_n \cong x\{2(2x+1)(C_m \otimes \bar{K}_n)\}$, and by Theorem 2.7, each $C_m \otimes \bar{K}_n$ has n C_{mn}-factors. Thus we have obtained xn partial C_{mn}-factors of $(K_{2m+1} \times K_{2x+1}) \otimes \bar{K}_n$ corresponding to one missing partite set of size $(2x+1)n$. By taking the union of all partial C_{mn}-factors corresponding to all the missing partite sets together gives $(2m+1)xn$ partial C_{mn}-factors of $(K_{2m+1} \times K_{2x+1}) \otimes \bar{K}_n$.

Furthermore, by Theorem 3.6 we get $\frac{n-1}{2}$ partial C_{mn}-factors of the second graph $(2x+1)(K_{2m+1} \times K_n)$ corresponding to one missing partite set. Taking the union of all the partial C_{mn}-factors together corresponding to all the missing partite sets, we get $(2m+1)\frac{n-1}{2}$ partial C_{mn}-factors of $(2x+1)(K_{2m+1} \times K_n)$. Finally, the partial C_{mn}-factors of the two graphs of the right-hand side of (12) obtained above together gives a required partial C_{mn}-factorization of $K_u \times K_g$. □

Theorem 4.9. *Let $k = q_1 q_2 \dots q_k \geq 45$ be odd, where $q_1, q_2, \dots, q_i \geq 15$, $q_{i+1}, q_{i+2}, \dots, q_k \geq 3$ and $q_1, q_2, \dots, q_k$ are odd and not necessarily distinct. If $u = 2(q_1 q_2 \dots q_i)x + 1$, where $x \geq 4$, and $g \equiv (q_{i+1} q_{i+2} \dots q_k) \pmod{2 q_{i+1} q_{i+2} \dots q_k}$, $1 \leq i \leq k-1$, then there exists a k-ARCD of $K_u \times K_g$.*

Proof. Let $u = 2mx+1$, $g = (2y+1)n$ and $k = mn \geq 45$, where $m = q_1 q_2 \dots q_i \geq 15$, $n = q_{i+1} q_{i+2} \dots q_k \geq 3$, and $x \geq 4$, $y \geq 1$. By using (9), we can write

$$\begin{aligned} K_{2mx+1} \times K_g &\cong \{(K_x \otimes \bar{K}_{2m}) \oplus x K_{2m+1}\} \times K_g && (13) \\ &\cong \{(K_x \otimes \bar{K}_{2m}) \times K_g\} \oplus x(K_{2m+1} \times K_g). && (14) \end{aligned}$$

Now, we construct the partial C_{mn}-factors of the right-hand side of (14) as follows. First we consider

$$\begin{aligned} (K_x \otimes \bar{K}_{2m}) \times K_g &\cong \{\mathcal{C}_m^0 \oplus \cdots \oplus \mathcal{C}_m^{x-1}\} \times K_g, \quad \text{by Theorem 2.9} \\ &\cong (\mathcal{C}_m^0 \times K_g) \oplus \cdots \oplus (\mathcal{C}_m^{x-1} \times K_g) \\ &\cong \oplus_{l \in \mathbb{Z}_x} (\mathcal{C}_m^l \times K_g) && (15) \end{aligned}$$

where each $\mathcal{C}_m^l$, $0 \leq l \leq x-1$, contains $\frac{2m}{2}$ partial C_k-factors of $K_x \otimes \bar{K}_{2m}$, and each partial C_k-factor contains $2(x-1)$ cycles of length k. Hence we write $\mathcal{C}_m^i \times K_g \cong \frac{2m}{2}\{2(x-1)(C_m \times K_{(2y+1)n})\}$.

$$\begin{aligned} C_m \times K_{(2y+1)n} &\cong C_m \times \{\mathbb{C}_n^1 \oplus \cdots \oplus \mathbb{C}_n^{\frac{g-1}{2}}\}, \text{ by Theorem 2.1} \\ &\cong (C_m \times \mathbb{C}_n^1) \oplus \cdots \oplus (C_m \times \mathbb{C}_n^{\frac{g-1}{2}}) \\ C_m \times \mathbb{C}_n^j &\cong (2y+1)(C_m \times C_n), \end{aligned}$$

where each $\mathbb{C}_n^j$, $1 \leq j \leq \frac{g-1}{2}$, is a C_n-factor of $K_{(2y+1)n}$, and each C_n-factor contains $(2y+1)$ cycles of length n. By Theorem 3.1, each $C_m \times C_n$ has two C_{mn} factors. Thus we have obtained $\frac{2m(g-1)}{2}$ partial C_{mn}-factors of $(K_x \otimes \bar{K}_{2m}) \times K_g$ corresponding to one missing partite set of size $2mg$ which makes a hole $K_{2m+1} \times K_g$ of $K_u \times K_g$. Note that corresponding to each partite set of $(K_x \otimes \bar{K}_{2m}) \times K_g$ we have a hole $K_{2m+1} \times K_g$ of $K_u \times K_g$ in (14). To complete the proof, it is enough to find the partial C_{mn}-factors of the hole $K_{2m+1} \times K_g$ corresponding to the missing partite set and put together to get the required partial C_{mn}-factors of $K_u \times K_g$.

Now we consider

$$\begin{aligned} K_{2m+1} \times K_g &\cong K_{2m+1} \times K_{(2y+1)n} \\ &\cong \{(K_{2m+1} \times K_{2y+1}) \otimes \bar{K}_n\} \oplus (2y+1)(K_{2m+1} \times K_n) && (16) \end{aligned}$$

By Theorems 3.5 and 2.7, $(K_{2m+1} \times K_{2y+1}) \otimes \bar{K}_n$ has $2m(yn)$ partial C_{mn}-factors with missing partite sets corresponding to the vertices of K_{2m} and yn partial C_{mn}-factors with missing partite sets corresponding to the vertex ∞. Furthermore, by Theorem 3.6, $(2y+1)(K_{2m+1} \times K_n)$ has $(2m)(\frac{n-1}{2})$ partial C_{mn}-factors with missing partite sets corresponding to the vertices of K_{2m}, and $\frac{n-1}{2}$ partial C_{mn}-factors with missing partite sets corresponding to the vertex ∞. Adding all the partial C_{mn}-factors of the two graphs in the right-hand side of (16), we get $\frac{(2m)(g-1)}{2}$ partial C_{mn}-factors of $K_{2m+1} \times K_g$ with missing partite sets corresponding to the vertices of K_{2m}, and $\frac{g-1}{2}$ partial C_{mn}-factors with missing partite sets corresponding to the vertex ∞.

Finally, the combination of $\frac{2m(g-1)}{2}$ partial C_{mn}-factors of $(K_x \otimes \bar{K}_{2m}) \times K_g$ corresponding to the one missing partite set of size $2mg$, together with the $\frac{2m(g-1)}{2}$ partial C_{mn}-factors of $K_{2m+1} \times K_g$ with missing partite sets corresponding to the vertices of K_{2m}, gives $\frac{2mx(g-1)}{2}$ partial C_{mn}-factors of $K_{2mx+1} \times K_g$. As there are x holes, repeat the process x times, and we get $\frac{2mx(g-1)}{2}$ partial C_{mn}-factors of $K_u \times K_g$. Furthermore, the collection of all $\frac{g-1}{2}$ partial C_{mn}-factors of each of the x copies of $K_{2m+1} \times K_g$ with missing partite set corresponding to the vertex ∞, together gives $\frac{g-1}{2}$ partial C_{mn}-factors of $K_{2mx+1} \times K_g$. Therefore a k-ARCD of $K_u \times K_g$ exists. □

Theorem 4.10. *For all odd $k \geq 15$, there exists a k-ARCD of $(K_u \times K_g)(\lambda)$ if and only if $u \geq 4$, $g \geq 3$, $\lambda(g-1) \equiv 0 \pmod 2$, $g(u-1) \equiv 0 \pmod k$, except possibly for $(\lambda, u, g) \in \{(2m, u, kx), (2m, 5, g) \mid x \equiv 2 \pmod 4$ and $m \geq 1\}$, and $(\lambda, u) \in \{(2m+1, \{16, 2r+1, 4(2s+1), 4t+2, kx+1\}) \mid x \in \{2t+1, 4, 6\},\ m, t \geq 0$, for even r, s and odd $s < 15\}$.*

Proof. Necessity follows from Theorem 1.1. Sufficiency can be divided into two cases.
Case (i): When $\lambda = 1$, the values of u and g fall into one of the following cases:

(a) $u = 2s+1$, $g \equiv k \pmod{2k}$, where $s \geq 3$ is odd and $s \leq k$;
(b) $u = 4(s+1)$, $s \in \{r, 2r\}$, $g \equiv k \pmod{2k}$, where $r, k \geq 15$ are odd integers and $r \leq k$;
(c) $u = 2k+1$, and $g \geq 3$ is odd;
(d) $u = 2kx+1$, $x \geq 4$ and $g \geq 3$ is odd.

Case (ii): When $\lambda = 2$, the values of u and g fall into one of the following cases:

(e) $u \geq 4$, $u \neq 5$ and $g \equiv 0 \pmod k$;
(f) $u = kx+1$, $x \geq 1$ and $g \geq 3$.

The proofs for (a), (b), (c), (d), (e), and (f) follow from Theorems 4.2, 4.3, 4.5, 4.6, 4.1, and 4.4, respectively. If $\lambda > 2$ is even (respectively, odd), then the values for u and g are the same as in cases (i) and (ii) (respectively, case (i)). Hence a k-ARCD of $(K_u \times K_g)(\lambda)$ exists. □

Theorem 4.11. *Let $k = q_1q_2\dots q_k \geq 9$ be odd, where $q_1, q_2, \dots, q_k \geq 3$ are odd integers and not necessarily distinct. There exists a k-ARCD of $(K_u \times K_g)(\lambda)$ if and only if $u \geq 4$, $g \geq 3$, $\lambda(g-1) \equiv 0 \pmod 2$, $g(u-1) \equiv 0 \pmod k$, except possibly for $(\lambda, u, g, k) \in \{(2m, rx+1, sy, rs) \mid m \geq 1,\ y \equiv 2 \pmod 4\}$ and $(\lambda, u, g, k) \in \{(2m+1, rx+1, s(2t+1), rs),\ (2m+1, u, g, n) \mid x \in \{4, 6, 2t+1\}, m, t \geq 0, n < 45$ is odd$\}$.*

Proof. Necessity follows from Theorem 1.1. We prove the sufficiency as follows:

The values of u, g and λ fall into one of the following cases:

(a) $u = 2(q_1q_2\dots q_i) + 1$, $g \equiv (q_{i+1}q_{i+2}\dots q_k) \pmod{q_{i+1}q_{i+2}\dots q_k}$, $1 \leq i \leq k-1$, when $\lambda = 1$;

(b) $u = 2(q_1q_2\dots q_i)x + 1$, $g \equiv (q_{i+1}q_{i+2}\dots q_k) \pmod{q_{i+1}q_{i+2}\dots q_k}$, $1 \leq i \leq k-1$ for any $x \geq 4$ and $k \geq 45$, when $\lambda = 1$;

(c) $u \equiv 1 \pmod{q_1q_2\dots q_i}$, $g \equiv 0 \pmod{q_{i+1}q_{i+2}\dots q_k}$, $1 \leq i \leq k-1$, when $\lambda = 2$.

The proofs for (a), (b), and (c) follow from Theorems 4.8, 4.9, and 4.7, respectively. If $\lambda > 2$ is even (respectively, odd), the values for u and g are the same as in (a), (b), and (c) (respectively, (a) and (b)). Hence a k-ARCD of $(K_u \times K_g)(\lambda)$ exists. □

5 Conclusion

In this paper, we have established the existence of a k-ARCD of $(K_u \times K_g)(\lambda)$, for all odd $k \geq 15$ with a few possible exceptions. Our results also provide a partial solution to the existence of modified cycle frames of complete multipartite multigraphs.

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References

[1] B. Alspach, J. C. Bermond and D. Sotteau, Decomposition into cycles I: Hamilton decompositions, in: *Cycles and Rays* (Eds. G. Hahn et al.), Kluwer Academic Publishers, 1990, pp. 9–18.

[2] B. Alspach, P. J. Schellenberg, D. R. Stinson and D. Wagner, The Oberwolfach problem and factors of uniform odd length cycles, *J. Combin. Theory Ser A.* 52 (1) (1989), 20–43.

[3] J. C. Bermond, Hamilton decomposition of graphs, directed graphs and hypergraphs, *Ann. Discrete Math.* 3 (1978), 21–28.

[4] M. Buratti, H. Cao, D. Dai and T. Traetta, A complete solution to the existence of (k, λ)-cycle frames of type g^u, *J. Combin. Des.* 25 (5) (2016), 197–230.

[5] J. Burling and K. Heinrich, Near 2-factorizations of 2kn : cycles of even length, *Graphs Combin.* 5 (1989), 213–221.

[6] H. Cao, Y. Wu and H. Zhou, Modified frames with block size 3, *J. Combin. Des.* 16 (5) (2008), 351–363.

[7] V. Chitra, A. Shanmuga Vadivu, and A. Muthusamy, C_{4p}-frame of complete multipartite multigraphs, *Aequat. Math.* 85 (2013), 563–579.

[8] S. Duraimurugan, A, Shanmuga Vadivu and A. Muthusamy, Almost resolvable even cycle systems of $(K_u \times K_g)(\lambda)$, (submitted).

[9] K. Heinrich, C. C. Lindner and C. A. Rodger, Almost resolvable decompositions of $2K_n$ into cycles of odd length, *J. Combin. Theory Ser A.* 49 (1998), 218–232.

[10] J. D. Horton, B. K. Roy, P. J. Schellenberg and D. R. Stinson, On decomposing graphs into isomorphic uniform 2-factors, *Ann. Discrete Math.* 27 (1985), 297–319.

[11] W. Imrich and S. Klavzar, *Product graphs-structure and recognition*, Wiley Interscience Series in Discrete Mathematics and Optimization, 2000.

[12] R. Laskar, Decomposition of some composite graphs into hamiltonian cycles, Proc. 5th Hungarian Coll. Kesathely 1976, North Holland (1978), 705–716.

[13] R. S. Manikandan and P. Paulraja, C_p-decompositions of some regular graphs, *Discrete Math.* 306 (2006), 429–451.

[14] R. S. Manikandan and P. Paulraja, C_5-decompositions of the tensor product of complete graphs, *Australas. J. Combin.* 37 (2007), 285–293.

[15] A. Muthusamy and P. Paulraja, Factorization of product graphs into cycles of uniform length, *Graphs Combin.* 11 (1995), 69–90.

[16] A. Muthusamy and A. Shanmuga Vadivu, Cycle frames of complete multipartite multigraphs III, *J. Combin. Des.* 22 (11) (2014), 473–487.

[17] M. Niu and H. Cao, More results on cycle frames and almost resolvable cycle systems, *Discrete Math.* 312 (2012), 3392–3405.

[18] D. R. Stinson, Frames for Kirkman triple systems, *Discrete Math.* 65 (1987), 289–300.

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