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# Set partition patterns and the dimension index

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**Abstract**

The notion of containment and avoidance provides a natural partial ordering on set partitions. Work of Sagan and of Goyt has led to enumerative results on avoidance classes of set partitions, which were refined by Dahlberg et al. through the use of combinatorial statistics. We extend the work of the latter authors by computing the distribution of the dimension index (a statistic arising from the supercharacter theory of finite groups) across avoidance classes. In doing so we obtain a novel connection between noncrossing partitions and 321-avoiding permutations, as well as connections to many other combinatorial objects such as Motzkin and Fibonacci polynomials.

## 1 Introduction

Given a finite set $S$, a *set partition* of $S$ is a unordered collection of disjoint nonempty blocks $B_1, \ldots, B_k$ such that

$$\bigcup_{j=1}^{k} B_j = S.$$

We will write a set partition $\pi$ as

$$\pi = B_1/\ldots/B_k,$$

and use the notation $\pi \vdash S$ to mean $\pi$ is a partition of $S$. We will restrict our attention in this article to set partitions of $[n] = \{1, 2, \ldots, n\}$ for some positive integer $n$; in this case we will omit set braces in the blocks of a partition, and write the blocks in *standard order*, meaning

$$\min(B_1) < \cdots < \min(B_k).$$

We let $\Pi_n$ denote the collection of set partitions of $[n]$; thus $\Pi_3$ consists of the partitions

$$123, 1/23, 12/3, 13/2, \text{ and } 1/2/3.$$

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Set partitions form a foundational topic in combinatorics; see the books of Mansour [28] or Stanley [34] for a general reference. In addition to having a rich combinatorial history, the theory of set partitions arises in the study of stochastic processes [31], algebras [20], Hopf algebras [2], and many other areas.

The combinatorics of set partitions have recently been enhanced through the use of *patterns*. Given a set partition $\pi = B_1/\ldots/B_k$ of $[n]$ and a subset $S \subset [n]$, let $\pi \cap S$ be the partition of $S$ given by taking the nonempty intersections of the form $B_i \cap S$, $i = 1, 2, \ldots, k$. We *standardize* $\pi \cap S$ to obtain a partition $\mathrm{st}(\pi \cap S)$ of $[|S|]$ by replacing the $i$th smallest entry in $\pi \cap S$ with $i$. For example, if $n = 6$, $\pi = 12/345/6$, and $S = \{2, 3, 5\}$, then $\pi \cap S = 2/35$ and $\mathrm{st}(\pi \cap S) = 1/23$.

Given two set partitions $\sigma \vdash [k]$ and $\pi \vdash [n]$, we say that $\pi$ *contains* $\sigma$ as a pattern if there exists a subset $S \subset [n]$ with $\mathrm{st}(\pi \cap S) = \sigma$. If no such subset exists, then $\pi$ *avoids* $\sigma$. Continuing our previous example, $\pi = 12/345/6$ contains the pattern $\sigma = 1/23$, but $\pi$ avoids the pattern $\tau = 1/2/3/4$ because $\pi$ only contains 3 blocks.

The theory of set partition patterns can be traced in part back to work of Kreweras [26], and was developed more generally in work of Klazar [22],[23],[24]. A fundamental question in the area, mirroring the related question in permutation patterns, is to enumerate the number of partitions of $[n]$ which avoid a set of fixed patterns. Namely, given a collection of partitions $P$, let

$$\Pi_n(P) = \{\pi \in \Pi_n : \pi \text{ avoids every partition in } P\}.$$

Work of Klazar and Marcus [25] has given asymptotic formulae for the sizes of these sets, and exact enumerative results for small patterns have been provided by Sagan [32] and Goyt [18].

Again in analogy with permutation patterns, work has been devoted to refining these enumerative results through the use of combinatorial statistics. Often the statistic of interest is related to "four fundamental statistics" of Wachs and White [37]. This can be found in work of Simion [33], Goyt and Sagan [19], Dahlberg et al. [15], Lin and Fu [27], and Acharyya, Czajkowski, and Williams [1].

The purpose of this paper is to continue the above work using a statistic arising from the supercharacter theory of finite groups. Building on work of André [3], [4], [5] and Yan [38], Diaconis and Isaacs have commenced a study of the representation theory of finite algebra groups through the use of *supercharacters* and *superclasses* [17]. We will not give an introduction to this theory here, but instead will simply say that often such supercharacters can be indexed by set partitions with additional data; in particular, the *dimension index* of such a partition $\pi = B_1/\ldots/B_k$, given as a sum

$$\dim(\pi) = \sum_{i=1}^{k} (\max(B_i) - \min(B_i) + 1),$$

returns algebraic data regarding the corresponding supercharacter.

As a result of this theory, there has been recent interest in the combinatorics of the dimension index on set partitions. For instance, Chern, Diaconis, Kane, and

Rhoades have shown that this statistic satisfies an asymptotic central limit theorem [14], and that the average of this statistic (taken over $\Pi_n$) can be expressed quite cleanly in terms of the Bell numbers [13].

The purpose of this work is to combine the study of the dimension index and set partition patterns. In doing so, we give numerous refinements of enumerative results, and also find connections to other combinatorial objects. As a sample of one such connection, we are able to give a quick alternative proof of Theorem 7.4 of [12] regarding 321-avoiding permutations (definitions for those unfamiliar may be found in Section 4):

**Theorem.** *Let $I_n(q,t,x)$ be the generating function for left-to-right maxima, inversions, and fixed points, taken over the* 321*-avoiding permutations of length $n$:*

$$I_n(q,t,x) = \sum_{\pi \in Av_n(321)} q^{\mathrm{inv}(\pi)} t^{\mathrm{LRM}(\pi)} x^{\mathrm{fix}(\pi)}.$$

*Then $I_0(q,t,x) = 1$ and, for $n \geq 1$,*

$$I_n(q,t,x) = txI_{n-1}(q,t,x) + \sum_{j=2}^{n} q^{j-1} I_{j-2}(q,t,1)\big(I_{n-j+1}(q,t,x) - t(x-1)I_{n-j}(q,t,x)\big).$$

Of note is that the proof of this theorem in [12] requires algebraic manipulation of continued fractions, whereas our proof of this theorem is purely combinatorial. It relies only on a Catalan recursion and the Inclusion-Exclusion Principle.

To aid our study, we introduce the following generating functions: for a set of partitions $P$ and a variable $q$, we define

$$\mathrm{Dim}_n(P;q) = \sum_{\pi \in \Pi_n(P)} q^{\dim(\pi)}.$$

In fact, it will be simpler (and more interesting) to use two related statistics. Namely, for $\pi = B_1/\ldots/B_k$ a partition of $[n]$, define the *spread* and *block* statistics by

$$\begin{aligned} \mathrm{sp}(\pi) &= \sum_{i=1}^{k} (\max(B_i) - \min(B_i)) \\ \mathrm{bl}(\pi) &= k, \end{aligned}$$

so that $\dim(\pi) = \mathrm{sp}(\pi) + \mathrm{bl}(\pi)$. In particular, we will examine the joint distributions of spread and block by calculating the joint generating function

$$\mathrm{SB}_n(P;q,t) = \sum_{\pi \in \Pi_n(P)} q^{\mathrm{sp}(\pi)} t^{\mathrm{bl}(\pi)},$$

from which we can obtain the desired information on $\mathrm{Dim}_n(P;q)$ by equating the variables $q$ and $t$.

The outline of this paper is as follows. In Section 2, we start by recalling basic facts on set partitions which will be useful in our proofs, such as the connection between partitions and restricted growth functions. We then provide several introductory calculations regarding partitions avoiding a single pattern of length 3. Sections 3 and 4 contain the most prominent of our results. In Section 3 we examine partitions which avoid the pattern 13/24; these are the so called *noncrossing partitions*. We follow this up in Section 4 by connecting noncrossing partitions to 321-avoiding permutations. We end with ideas for future work in Section 5.

## 2 Background and Introductory Calculations

### 2.1 Preliminary Notions

We start this section by recalling several preliminary facts which will be useful in the study of set partition patterns. The first notion is that of a *restricted growth function*, or *RGF* for short; it allows us to frame containment questions in terms of words and subwords, which simplifies many arguments.

A restricted growth function is a sequence $w = a_1a_2\dots a_n$ of positive integers such that $a_1 = 1$ and, for $i \geq 2$,

$$a_i \leq 1 + \max\{a_1, \dots, a_{i-1}\}.$$

Let $R_n$ denote the set of length $n$ RGFs. It is straightforward to show that the following gives a bijection between $\Pi_n$ and $R_n$. Given $\pi = B_1/\dots/B_k \in \Pi_n$ written in standard order, we map $\pi$ to the RGF $w(\pi) = w_1 \dots w_n$ with

$$w_i = j$$

if $i \in B_j$. For example, the partition $\pi = 14/25/378/6 \vdash [8]$ maps to the RGF 12312433 under this bijection.

Mapping partitions to their associated RGFs often provides a useful characterization of avoidance classes of set partitions. For a set of partitions $P$, let

$$R_n(P) = \{w(\pi) \in R_n : \pi \in \Pi_n(P)\}.$$

Below we collect several results from Sagan's article [32], which provide characterizations of $R_n(P)$ where $P$ consists of a single pattern of length 3.

**Theorem 2.1** ([32]). *We have the following characterizations.*

1. $R_n(1/2/3) = \{w \in R_n : w \text{ consists of only } 1\text{s and } 2\text{s}\}$.
2. $R_n(1/23) = \left\{ w \in R_n : \begin{array}{l} w \text{ is obtained by inserting a single } 1 \text{ into a word} \\ \text{of the form } 1^l 23\dots m \text{ for some } l \geq 0 \text{ and } m \geq 1 \end{array} \right\}$.
3. $R_n(13/2) = \{w \in R_n : w \text{ is weakly increasing}\}$.

4. $R_n(12/3) = \{w \in R_n : w \text{ has initial run } 1 \ldots m \text{ and } a_{m+1} = \cdots = a_n \leq m\}$.

5. $R_n(123) = \{w \in R_n : w \text{ has no element repeated more than twice}\}$.

With this theorem in hand, we now explain how to read the block, spread, and dimension statistics of a partition from its RGF. To do so it will help to introduce the following notation. Given an RGF $w$ and a letter $l$ of $w$, let first$(l)$ and last$(l)$ be the indices of the first and last occurrence of $l$ in $w$, respectively. For example, if $w = 11231$, then first$(1) = 1$, last$(1) = 5$, and first$(3) = $ last$(3) = 4$.

**Lemma 2.2.** *Let $\pi \in \Pi_n$ with $w = w(\pi)$. Then*

1. $\mathrm{bl}(\pi) = \max(w)$,

2. $\mathrm{sp}(\pi) = \sum_{i=1}^{\max(w)} (\mathrm{last}(i) - \mathrm{first}(i))$,

3. $\dim(\pi) = \sum_{i=1}^{\max(w)} (\mathrm{last}(i) - \mathrm{first}(i) + 1)$.

*Proof.* The first equality follows directly from the bijection between $R_n$ and $\Pi_n$. The spread statistic is given by the blockwise sum of the difference between the maximum and minimum elements in each block. To track the maximum and minimum elements in a block $B_i$, we must find the first occurrence of $i$ in the RGF and the last occurrence of $i$ in the RGF, and take their difference. Hence,

$$\mathrm{sp}(w) = \sum_{i=1}^{\max(w)} (\mathrm{last}(i) - \mathrm{first}(i)).$$

The last statement follows from $\dim(w) = \mathrm{sp}(w) + \mathrm{bl}(w)$. □

For example, below we display the partitions of length three together with their corresponding RGFs and block, spread, and dimension statistics.

| $\pi$ | $w(\pi)$ | $\mathrm{bl}(\pi)$ | $\mathrm{sp}(\pi)$ | $\dim(\pi)$ |
|---|---|---|---|---|
| 1/2/3 | 123 | 3 | 0 | 3 |
| 1/23 | 122 | 2 | 1 | 3 |
| 12/3 | 112 | 2 | 1 | 3 |
| 13/2 | 121 | 2 | 2 | 4 |
| 123 | 111 | 1 | 2 | 3 |

In the remainder of this section we will combine Theorem 2.1 and Lemma 2.2 to study $\mathrm{SB}_n(P; q, t)$ and $\mathrm{Dim}(P; q)$ for patterns of length three. To simplify notation somewhat we will use $[n]_q$ to denote the $q$-analogue of $n$,

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}.$$

## 2.2 Patterns of Length 3

In this subsection we will exhibit several calculations regarding partitions which avoid a single pattern of length 3. This is meant to provide a mild precursor to Section 3, which will contain more involved calculations. We start by computing the distribution of spread, block, and dimension over partitions which avoid the pattern 12/3.

**Proposition 2.3.** *For $n \geq 1$ we have*

$$\mathrm{SB}_n(12/3; q, t) = t^n + \sum_{i=1}^{n-1} t^i q^{n-i}[i]_q$$

*and*

$$\mathrm{Dim}_n(12/3; q) = q^n \left(1 + \sum_{i=1}^{n-1}[i]_q\right).$$

*Proof.* From Theorem 2.1, every RGF in $R_n(12/3)$ is of the form $12\ldots mk\ldots k$ where $1 \leq k \leq m$. From this, we can then characterize every RGF first by its maximum element, then by the value of the constant string at the end.

If $w$ is strictly increasing, $w = 12\ldots n$, then the trailing constant string is empty, and hence $\mathrm{bl}(w) = n$, $\mathrm{sp}(w) = 0$. Alternatively, suppose an RGF has maximum value $m$, and a string of length $n - m$ and value $k$ at the end. Then it is clear that $\mathrm{bl}(\pi) = m$ and $\mathrm{sp}(w) = n - k$. We have

$$\sum_{w \in R_n(12/3)} q^{\mathrm{sp}(w)} t^{\mathrm{bl}(w)} = t^n + \sum_{m=1}^{n-1} \sum_{k=1}^{m} t^m q^{n-k}$$

and simplifying gives the desired result. □

Having displayed the approach one takes when calculating these generating functions in Proposition 2.3, we omit the proofs of the following calculations for brevity's sake. The proofs pass through Theorem 2.1 and straightforward combinatorial manipulations, as in Proposition 2.3.

**Proposition 2.4.** *For $n \geq 1$ we have*

$$\mathrm{SB}_n(1/23; q, t) = t^n + \sum_{i=1}^{n-1} t^i q^{n-i}[i]_q$$

*and*

$$\mathrm{Dim}_n(1/23; q) = q^n \left(1 + \sum_{i=1}^{n-1}[i]_q\right).$$

□

**Proposition 2.5.** *For $n \geq 1$ we have*

$$\mathrm{SB}_n(13/2; q, t) = (q+t)^{n-1}t$$

*and*

$$\mathrm{Dim}_n(13/2; q) = 2^{n-1}q^n.$$

□

**Proposition 2.6.** *We have* $\mathrm{SB}_1(1/2/3; q, t) = q$, $\mathrm{Dim}_1(1/2/3; q) = q$, *and for* $n \geq 2$

$$\mathrm{SB}_n(1/2/3; q, t) = q^{n-1}t + t^2\left((n-2)q^{n-1} + (n-1)q^{n-2} + \sum_{i=2}^{n-2}\sum_{j=i+1}^{n-1} 2^{j-i-1}q^{j-i}(1+q^{n-1})\right),$$

$$\mathrm{Dim}_n(1/2/3; q) = (n-2)q^{n+1} + nq^n + \sum_{i=2}^{n-2}\sum_{j=i+1}^{n-1} 2^{j-i-1}q^{j-i}(1+q^{n-1}).$$

□

The last results of this section deal with the pattern 123. RGFs of partitions avoiding 123 will contain no element more than twice. This restriction is much less rigid than the restrictions arising from other patterns of length 3, which makes working with these partitions much more difficult. Accordingly, we only provide partial information on the individual statistics over this class. Namely, our result describes the words in $R_n(123)$ which maximize the spread statistic.

**Proposition 2.7.** *Let $w$ be an RGF in $R_n(123)$ which maximizes the spread statistic. Then*

$$\mathrm{sp}(w) = \left\lfloor \tfrac{n}{2} \right\rfloor \left\lceil \tfrac{n}{2} \right\rceil$$

*and*

$$\mathrm{bl}(w) = \left\lceil \tfrac{n}{2} \right\rceil.$$

*Moreover, $w$ is of the form $12\ldots\lceil\frac{n}{2}\rceil\sigma$, where $\sigma$ is a permutation of the set $\{1, 2, \ldots, \lfloor\frac{n}{2}\rfloor\}$.*

*Proof.* A direct calculation shows any partition of the described form has the statistics listed in the theorem, so it suffices to show that this indeed gives the maximum spread. Let $\pi \in \Pi_n(123)$ be a partition which maximizes spread. Note that $\pi$ cannot contain two singleton blocks; if it did, we could replace the singletons with their union to increase spread. Translating into RGFs, a spread-maximizing RGF $w$ can contain at most one unique letter.

Next, suppose $w \in R_n(123)$ maximizes spread and that the first $\lceil\frac{n}{2}\rceil$ elements are not strictly increasing. By the definition of RGFs there must exist indices $i, j$ such that $i < j \leq \lceil\frac{n}{2}\rceil$ but $w_i = w_j$. Note that in this scenario this common value is strictly less than $\lceil\frac{n}{2}\rceil$. Importantly, in order for $w$ to be in $R_n(123)$, the letter $\lceil\frac{n}{2}\rceil$ *must* appear in $w$ (a word of length $n$ which can repeat values at most twice must use at least $\lceil\frac{n}{2}\rceil$ distinct letters). Combining this with the fact that $w$ is an RGF, we know the value $k = \max(w_1, \ldots, w_j) + 1$ is in $w$ as well. Transposing the first occurrence of $k$ in $w$ with $w_j$ will increase spread, contradicting our assumption.

From the previous two paragraphs, we know that if $w$ maximizes spread then it must contain at most one unique letter and must strictly increase up to the letter $\lceil \frac{n}{2} \rceil$. The only such words which also avoid the pattern 123 are those described in our theorem. □

As a corollary of the previous result we obtain the following well-known identities:

$$\begin{aligned} 1 + 3 + \cdots + 2k - 1 =& k^2, \\ 2 + 4 + \cdots + 2k =& k(k + 1). \end{aligned}$$

This can be seen by applying Proposition 2.7 to the following partitions:

$$1(2k)/2(2k-1)/\ldots/k(k+1) \quad \text{and} \quad 1(2k+1)/2(2k)/\ldots/k(k+2)/(k+1).$$

We end this section by noting that the calculations in Proposition 2.7 in fact carry over to all of $\Pi_n$.

**Proposition 2.8.** *The partitions appearing in the proof of Proposition 2.7 also maximize spread over $\Pi_n$. If $n$ is odd, $n = 2k + 1$, there is one other class of partitions which do so. Using the notation from the above proof, they are the partitions whose associated RGF has the form*

$$w = 12\ldots kj\sigma_1 \ldots \sigma_k$$

*where $1 \leq j \leq k$ and $\sigma = \sigma_1 \ldots \sigma_k$ is a permutation of $[k]$ (note that the letter $j$ will appear three times in such a word, hence it is not encompassed by Proposition 2.7).*

*Proof.* First note that a block of size at least 4 cannot appear in a set partition which maximizes spread; indeed if $B = \{i_1, i_2, \ldots, i_j\}$ is a block with $i_1 < i_2 < \cdots < i_j$ and $j \geq 4$, we may replace $B$ with two blocks,

$$B = \{i_1, i_j\} \cup \{i_2, \ldots, i_{j-1}\},$$

in order to strictly increase spread. This reduces the calculation to partitions with blocks of size at most 3. If we have two blocks of size 3, $B_1 = \{i_1, i_2, i_3\}$ and $B_2 = \{j_1, j_2, j_3\}$ with $i_1 < i_2 < i_3$ and $j_1 < j_2 < j_3$, then we may replace $B_1$ and $B_2$ with the three blocks

$$\{i_1, i_3\}, \{i_2, j_2\}, \{j_1, j_3\}$$

to again strictly increase spread. Combining these observations with Proposition 2.7 gives the desired result. □

## 3 Noncrossing Partitions

Now we consider the set of partitions that avoid the pattern 13/24. These partitions are called *noncrossing* and have a rich combinatorial and algebraic history. The size of the set $\Pi_n(13/24)$ is given by the $n$th Catalan number,

$$C_n = \frac{1}{2n+1}\binom{2n}{n}.$$

See the work of Armstrong [6] for more information on these partitions. We will continue our analysis of the spread, block, and dimension statistics over this avoidance class, and in doing so obtain $(q,t)$-analogs of the standard Catalan recursion. Our main result is Theorem 3.3; to prove this we need characterizations of $R_n(13/24)$, which were provided by Campbell et al. in [9]. We restate their results for completeness.

**Lemma 3.1** ([9], Proposition 5.1 and Lemma 5.2). *For a partition $\pi$, the following are equivalent:*

1. *$\pi$ avoids* 13/24*,*
2. *the RGF $w(\pi) = w_1 \dots w_n$ avoids 1212 (as a subword pattern),*
3. *there are no $xyxy$ subwords in $w(\pi)$,*
4. *if $w_i = w_{i'}$ for some $i < i'$ then, for all $j' > i'$, either $w_{j'} \le w_{i'}$ or $w_{j'} > \max\{w_1, \dots, w_{i'}\}$.* □

For the next corollary and the following theorem we need the following notation. For a word $w = w_1 \dots w_n$, let $1w$ denote the word obtained by prepending a 1 to $w$. For an integer $k$, let $(w+k)$ denote the word whose $i$th letter is $w_i + k$. For example, if $w = 12134$, then $1(w+1) = 123245$. Similarly, if $u$ and $v$ are words then $uv$ is $u$ prepended to $v$.

**Corollary 3.2** ([9], Corollary 5.3). *If $w$ is in $R_n(13/24)$ then both $1w$ and $1(w+1)$ are in $R_{n+1}(13/24)$.* □

The next theorem gives a recursive formula for $\mathrm{SB}_n(13/24; q,t)$. The structure of this proof will closely follow that of Theorem 5.4 in [9].

**Theorem 3.3.** *We have*

$$\mathrm{SB}_0(13/24; q,t) = 1$$
$$\mathrm{SB}_1(13/24; q,t) = t$$

*and for $n \ge 2$*

$$\mathrm{SB}_n(13/24;q,t) = t\,\mathrm{SB}_{n-1}(13/24;q,t) + \sum_{k=2}^{n} q^{k-1}\mathrm{SB}_{k-2}(13/24;q,t)\mathrm{SB}_{n-k+1}(13/24;q,t).$$

*Similarly,* $\mathrm{Dim}_0(13/24;q) = 1$, $\mathrm{Dim}_1(13/24;q) = q$, *and for* $n \geq 2$

$$\mathrm{Dim}_n(13/24;q) = q\,\mathrm{Dim}_{n-1}(13/24;q) + \sum_{k=2}^{n} q^{k-1}\mathrm{Dim}_{k-2}(13/24;q)\mathrm{Dim}_{n-k+1}(13/24;q).$$

*Proof.* The initial conditions are readily verified; we will focus on proving the recursion. To prove this theorem we divide the set $R_n(13/24)$ into disjoint subsets, then split our analysis of the block and spread statistics individually over these sets. The sets of interest are as follows:

$$X = \{w \in R_n(13/24) : w_1 = 1 \text{ and there are no other 1s in } w\}$$

and, for $k \geq 2$,

$$Y_k = \{w \in R_n(13/24) : w_1 = w_k = 1 \text{ and } w_j > 1 \text{ for } 1 < j < k\}.$$

That these sets partition $R_n(13/24)$ is clear.

We will begin by looking at $X$. We can describe $X$ as

$$X = \{w \in R_n(13/24) : w = 1(u+1) \text{ for some } u \in R_{n-1}(13/24)\},$$

and from this description we obtain a bijection $X \to R_{n-1}(13/24)$.

If $w = 1(u+1)$, then $\mathrm{bl}(w) = \mathrm{bl}(u) + 1$ and $\mathrm{sp}(w) = \mathrm{sp}(u)$. Thus

$$\sum_{w\in X} q^{\mathrm{sp}(w)}t^{\mathrm{bl}(w)} = t\,\mathrm{SB}_{n-1}(13/24, q, t).$$

Next, we examine the sets $Y_k$. For $w \in Y_k$, we claim that $w$ is of the form $w = 1(u+1)1v$ where $u \in R_{k-2}(13/24)$, $\mathrm{st}(1v) \in R_{n-k+1}(13/24)$, and if $v_i \neq 1$, then $v_i > \max(u) + 1$.

The first two requirements are clear. For the third, if there exists $v_i$ such that $1 < v_i \leq \max(u) + 1$ then there must exist $u_j$ such that $u_j + 1 = v_i$. However, then we will have an $xyxy$ subword with $x = 1$ and $y = u_j + 1 = v_i$, which would then imply that $w \notin R_n(13/24)$ by Lemma 3.1.

This establishes a bijection

$$Y_k \to R_{k-2}(13/24) \times R_{n-k+1}(13/24);$$

the inverse map sends $(u, v) \to 1(u+1)v'$, where

$$v'_i = \begin{cases} 1 \text{ if } v_i = 1 \\ v_i + \max(u) \text{ otherwise.} \end{cases}$$

Now we examine the behaviour of block and spread under this map. Take $w = 1(u+1)1v \in Y_k$. If $1v$ consists solely of 1s, then $\max(w) = 1+\max(u)$. Alternatively, the maximum value of $w$ will be found in $1v$; in any case,

$$\mathrm{bl}(w) = \mathrm{bl}(u) + \mathrm{bl}(\mathrm{st}(1v)).$$

Now consider the spread statistic. Since $(u+1)$ and $1v$ are disjoint, the spread of $w$ is *almost* the sum $\mathrm{sp}(u)+\mathrm{sp}(\mathrm{st}(1v))$; however, this forgets the impact of prepending the leading 1. As $u$ has length $k-2$, the effect of this is to increase spread by $k-1$. The correct formula is therefore

$$\mathrm{sp}(w) = \mathrm{sp}(u) + \mathrm{sp}(\mathrm{st}(1v)) + k - 1.$$

Hence summing over $Y_k$ results in

$$\begin{aligned}\sum_{w\in Y_k} q^{\mathrm{sp}(w)} t^{\mathrm{bl}(w)} &= q^{k-1}\left(\sum_{u\in R_{k-2}(13/24)} q^{\mathrm{sp}(u)} t^{\mathrm{bl}(u)}\right)\left(\sum_{\mathrm{st}(1v)\in R_{n-k+1}(13/24)} q^{\mathrm{sp}(v)} t^{\mathrm{bl}(1v)}\right)\\ &= q^{k-1}\,\mathrm{SB}_{k-2}(13/24;q,t)\,\mathrm{SB}_{n-k+1}(13/24;q,t).\end{aligned}$$

Summing over $k$, and remembering the contribution of $X$, gives our desired result. □

We end this section by examining $\Pi_n(123, 13/24)$; in doing so we obtain $q$-analogs of the Motzkin numbers. By combining Theorem 2.1 and Lemma 3.1, it is easy to characterize $\Pi_n(123, 13/24)$ as simply the noncrossing partitions in which each block has size at most 2. Following the proof of Theorem 3.3, we have the following $(q,t)$-Motzkin recursion:

**Theorem 3.4.** *We have* $\mathrm{SB}_0(123, 13/24; q, t) = 1$*,* $\mathrm{SB}_1(123, 13/24; q, t) = t$*, and for* $n \geq 2$*,*

$$\begin{aligned}\mathrm{SB}_n(123, 13/24; q,t) &= t\,\mathrm{SB}_{n-1}(123, 13/24; q, t)\\ &\quad + \sum_{k=2}^{n} tq^{k-1}\,\mathrm{SB}_{k-2}(123, 13/24; q, t)\,\mathrm{SB}_{n-k}(123, 13/24; q, t).\end{aligned}$$

*Proof.* We proceed as in Theorem 3.3, partitioning $\Pi_n(123, 13/24)$ into the sets

$$X = \{w \in R_n(123, 13/24) : w_1 = 1 \text{ and there are no other 1}s \text{ in } w\}$$

and, for $2 \leq k \leq n$,

$$Y_k = \{w \in R_n(123, 13/24) : w_1 = w_k = 1 \text{ and } w_j > 1 \text{ for } 1 < j < k\}.$$

Analyzing these sets using the ideas discussed in Theorem 3.3 shows that $X$ is in bijection with $R_{n-1}(123, 13/24)$. Similarly, $Y_k$ is in bijection with $R_{k-2}(123, 13/24) \times R_{n-k}(123, 13/24)$. Be careful to note the change in the second index compared to Theorem 3.3; the bijection in this context is $1(u+1)1v \to (u, \mathrm{st}(v))$, in other words the latter 1 *is not* included in the word $v$. This is because our partitions now avoid the pattern 123, so that the second 1 is the *last* occurrence of 1 in $w$. Analyzing the behavior of block and spread under these maps gives the desired recursion. □

Unsurprisingly, the generating function of Theorem 3.4 has many interesting connections to the literature. For instance, it arises by specializing the Motzkin Tunnel Polynomials of Barnabei, Bonetti, and Castronuovo at $z = x_i = t$, $y_i = q$; see Theorem 5 of [8]. Alternatively, we may specialize to $q = 1$ to obtain the Motzkin polynomials (OEIS A055151) which arise, among other places, in Marberg's study of the so called "poor" noncrossing partitions [29]. Specializing to $t = 1$ gives an alternate form of Motzkin polynomials which count Motzkin paths by area (OEIS A129181). It would be interesting to give bijective proofs of these results, in analogy to Theorem 5.10 of [9].

## 4 Connections to 321-Avoiding Permutations

In this section we show how the methods of Section 3 can be used to obtain results on 321-avoiding permutations. To do so, we briefly recall a few notions (for a more extended introduction to permutation patterns, we refer the reader to a reference such as [21]).

A *permutation* of $[n]$ is a bijection $\pi : [n] \to [n]$. We will write $\pi$ in one line form as $\pi = \pi_1 \dots \pi_n$, where $\pi_i = \pi(i)$. Let $\mathfrak{S}_n$ denote the permutations of length $n$; for example,

$$\mathfrak{S}_3 = \{123, 132, 213, 231, 312, 321\}.$$

As with set partitions there is a natural notion of containment in permutations. Namely, given a permutation $\pi$ of length $n$ and a permutation $\tau$ of length $m$, we say $\pi$ contains $\tau$ as a pattern if there is a subword $\pi_{i_1} \dots \pi_{i_m}$ in $\pi$ with $i_1 < \cdots < i_m$ such that

$$\pi_{i_j} < \pi_{i_k} \text{ if and only if } \tau_j < \tau_k.$$

We let $\mathrm{Av}_n(\pi)$ denote the length $n$ permutations which avoid the pattern $\pi$.

This section will concern itself with five statistics on permutations; they are of combinatorial interest but also arise in the algebraic theory of symmetric groups. Let $\pi = \pi_1 \dots \pi_n$ be a permutation. An index $i$ is a *left-to-right maximum* if $\pi_j < \pi_i$ for all $j < i$ (this condition is vacuously satisfied by the first index). An index $i$ is a *fixed point* if $i = \pi_i$. A pair of indices $(i, j)$ form an *inversion* if $i < j$ and $\pi_i > \pi_j$. Finally, an index $i$ is a *descent* if $\pi_i > \pi_{i+1}$. We may now define the following statistics:

$$\begin{aligned}
\mathrm{LRM}(\pi) &= \#\{i : i \text{ is a left-to-right maximum }\}\\
\mathrm{inv}(\pi) &= \#\{(i,j) : (i,j) \text{ is an inversion }\}\\
\mathrm{fix}(\pi) &= \#\{i : i \text{ is a fixed point }\}\\
\mathrm{des}(\pi) &= \#\{i : i \text{ is a descent }\}\\
\mathrm{maj}(\pi) &= \sum_{i \text{ a descent}} i.
\end{aligned}$$

Sparked by Conjecture 3.2 and Question 3.4 of [16], Cheng, Elizalde, Kasraoui, and Sagan have provided (among other things) Catalan recursions for the inversion polynomials for 321-avoiding permutations, as well as for the joint generating functions for descents and the major index [12]. Their proofs involve beautiful connections of 321-avoiding permutations with lattice paths and polyominoes.

The purpose of this section is to give a quick alternative proof of certain results of [12] using noncrossing partitions. Namely, we will present Catalan recursions for the generating functions

$$I_n(q,t,x) := \sum_{\pi \in Av_n(321)} q^{\mathrm{inv}(\pi)} t^{\mathrm{LRM}(\pi)} x^{\mathrm{fix}(\pi)}$$
$$M_n(q,t,x) := \sum_{\pi \in Av_n(321)} q^{\mathrm{maj}(\pi)} t^{\mathrm{des}(\pi)} x^{\mathrm{LRM}(\pi)}.$$

To do so, we take advantage of the following characterization of 321-avoiding permutations. Given $\pi \in \mathfrak{S}_n$, define binary vectors

$$\mathrm{pos}(\pi) = (p_1, \ldots, p_n)$$
$$\mathrm{val}(\pi) = (v_1, \ldots, v_n)$$

with

$$p_i = \begin{cases} 1 \text{ if } i \text{ is a left-right maximum in } \pi \\ 0 \text{ otherwise} \end{cases}$$

and

$$v_i = \begin{cases} 1 \text{ if there is a left-to-right maximum } j \text{ with } \pi_j = i \\ 0 \text{ otherwise.} \end{cases}$$

In other words, $\mathrm{val}(\pi)$ describes the values of left-right maxima in $\pi$, and $\mathrm{pos}(\pi)$ determines the positions of these maxima.

The following is a folklore lemma, documented in [12]:

**Lemma 4.1.** *[CEKS13, Lemma 2.1] The assignment $\pi \to (pos(\pi), val(\pi))$ induces a bijection between $Av_n(321)$ and the set of pairs of binary vectors $(p_1 \ldots p_n, v_1 \ldots v_n)$ satisfying*

- *The number of 1s in $p_1 \ldots p_n$ equals the number of 1s in $v_1 \ldots v_n$, and*
- *For any index $1 \leq i \leq n-1$, the number of 1s in $p_1 \ldots p_i$ is* strictly greater *than the number of 1s in $v_1 \ldots v_{i-1}$.* □

Note that for the second condition to apply for $i = 1$, we require that $p_1 = 1$. We will call such a pair of binary sequences a *ballot pair*. The bijection for $n = 3$ is reproduced below:

$123 \to (111, 111),\ 132 \to (110, 101),\ 213 \to (101, 011),\ 231 \to (110, 011),\ 312 \to (100, 001).$

The first step of this section is to establish a bijection between noncrossing partitions and ballot pairs; via Lemma 4.1, this will establish a bijection between noncrossing partitions and 321-avoiding permutations.

Let $w = w_1 \dots w_n$ be the RGF of a noncrossing partition. Call a letter $w_i$ a *first* if $w_j \neq w_i$ for $j < i$ and a *last* if $w_j \neq w_i$ for $j > i$. With this terminology, we can define a map $T$ from noncrossing partitions to ballot pairs by sending

$$w_1 \dots w_n \to (f_1 \dots f_n, l_1 \dots l_n),$$

with

$$f_i = \begin{cases} 1 \text{ if } w_i \text{ is a first,} \\ 0 \text{ otherwise} \end{cases}$$

and

$$l_i = \begin{cases} 1 \text{ if } w_i \text{ is a last,} \\ 0 \text{ otherwise.} \end{cases}$$

As an example, below we show the action of $T$ on $R_3(13/24)$:

$$111 \to (100, 001),\ 112 \to (101, 011),\ 122 \to (110, 101),\ 121 \to (110, 011),\ 123 \to (111, 111).$$

Using Lemma 3.1, it is not hard to show the following:

**Lemma 4.2.** *The map $T$ induces a bijection between noncrossing partitions of $[n]$ and ballot pairs of length $n$.*

*Proof.* We sketch how to reconstruct a noncrossing partition from its corresponding ballot pair. Let $(p_1 \dots p_n, v_1 \dots v_n)$ be a ballot pair of length $n$. We construct $w = w_1 \dots w_n \in R_n(13/24)$ iteratively as follows. Start with the empty RGF, and set $L_0 = \emptyset$. Here the sets $L_j$ denote the set of "available" letters at any given step, i.e. the letters whose first occurrences have been placed but whose last occurrences have not yet been established. Having constructed $w_1 \dots w_j$ and the set of available letters $L_j$, we determine $w_{j+1}$ and $L_{j+1}$ as follows.

- If $p_{j+1} = v_{j+1} = 1$, set $w_{j+1} = \max\{w_1, \dots, w_j\} + 1$ and $L_{j+1} = L_j$. In this case, the "first" letter is also a "last," so our available letters do not change.
- If $p_{j+1} = 1$ and $v_{j+1} = 0$, set $w_{j+1} = \max\{w_1, \dots, w_j\} + 1$ and $L_{j+1} = L_j \cup \{\max\{w_1, \dots, w_j\} + 1\}$.
- If $p_{j+1} = 0$ and $v_{j+1} = 1$, set $w_{j+1} = \max L_j$ and $L_{j+1} = L_{j+1} \setminus \max L_j$.
- If $p_{j+1} = v_{j+1} = 0$, set $w_{j+1} = \max L_j$ and $L_{j+1} = L_j$.

This process is well defined by the definitions of a ballot pair. It is easy to see that this process yields an RGF with no $xyxy$ patterns, and by Lemma 3.1 this implies $w$ is noncrossing. That this map is an inverse to $T$ follows from inspection. $\square$

By combining Lemmas 4.1 and 4.2, we have a bijection between noncrossing partitions of $[n]$ and 321-avoiding permutations of length $n$. We will call this map $\overline{T}$; for each $n$, we will have $\overline{T} : R_n(13/24) \to Av_n(321)$. Below we show $\overline{T}(111) = 312$:

$$\begin{tikzcd} 111 \arrow[r, "T"'] \arrow[rr, bend left, "\overline{T}"] & (100,001) \arrow[r] & 312 \end{tikzcd}$$

To keep track of fixed points, descents, and the major index on permutations, we introduce the following statistics on partitions.

Given a partition $\pi$, let $w = w_1 \dots w_n$ denote its associated RGF. Maintaining notation as for permutations, call an index $i$ a left-to-right maximum if $w_j < w_i$ for all $j < i$ (this condition is vacuously satisfied by the first index). We call an index $i$ a *checkpoint* if it is a left-to-right maximum and if $w_i < w_j$ for all $j > i$ (the second half of this condition is vacuously satisfied by the last index). Finally, an index $i$ is an *apex* if it is a left-to-right maximum in $w$ and if $w_i \geq w_{i+1}$ (the second half of this condition *is not* vacuous; the last index in a word is never an apex).

For example, in the word 12213454 the indices $1, 2, 5, 6, 7$ are left-to-right maxima, the index 5 is a checkpoint, and the indices 2 and 7 are apices.

**Lemma 4.3.** *Let $\pi$ be a noncrossing partition with corresponding RGF $w = w_1 \dots w_n$. We have the following relationship between set partition statistics and permutation statistics:*

- $\mathrm{sp}(w) = \mathrm{inv}(\overline{T}(w))$,
- $\mathrm{bl}(w) = \mathrm{LRM}(\overline{T}(w))$,
- *$i$ is a checkpoint in $w$ if and only if $i$ is a fixed point in $\overline{T}(w)$,*
- *$i$ is an apex in $w$ if and only if $i$ is a descent in $\overline{T}(w)$.*

*Proof.* Throughout this proof we will write the permutation $\overline{T}(w)$ as $\overline{T}(w) = a_1 \dots a_n$ and we will let $T(w) = (p_1 \dots p_n, v_1 \dots v_n)$ denote the corresponding ballot pair of $w$ and $\overline{T}(w)$.

For the first assertion, note that a LRM in an RGF is equivalently an index of a first occurrence of some letter. Thus

$$\mathrm{sp}(w) = \sum_{i \text{ a LRM of } w} (\mathrm{last}(w_i) - i).$$

Similarly, as $\overline{T}(w)$ is a 321-avoiding permutation we have

$$\mathrm{inv}(\overline{T}(w)) = \sum_{i \text{ a LRM of } \overline{T}(w)} (a_i - i).$$

This last equality is not necessarily trivial unless one has experience with permutation patterns; it follows from the fact that in a 321-avoiding permutation, the letters at indices which are not LRM must form a strictly increasing sequence.

Since the bijection $\overline{T}$ exchanges the set of last occurrences in $w$ with the set of LRM values in $\overline{T}(w)$, both sums are equal to

$$\left(\sum_{v_i=1} i\right) - \left(\sum_{p_j=1} j\right).$$

For the second assertion, we simply observe that both $\mathrm{bl}(w)$ and $\mathrm{LRM}(\overline{T}(w))$ count the number of 1s in $p_1 \dots p_n$.

For the third assertion, we claim that $w_i$ is a checkpoint if and only if $p_i = v_i = 1$ and the prefix pair $(p_1 \dots p_{i-1}, v_1 \dots v_{i-1})$ is a ballot pair. Indeed the first condition is implied by $w_i$ being unique; the condition that $w_i$ is a checkpoint is equivalent to having $\mathrm{last}(l) < i$ for every letter $l < w_i$, which implies the second. Translating this to permutations, the fact that $p_i = v_i = 1$ implies that the index $i$ is a LRM in $\overline{T}(w)$, and that the letter $i$ is a value of a LRM in $\overline{T}(w)$. The prefix condition assures us that these conditions imply $a_i = i$.

Finally, in a 321-avoiding permutation the letters at indices which are not left-to-right maxima must be strictly increasing. In particular, $a_i > a_{i+1}$ if and only if $i$ is a left-to-right maximum, but $i+1$ is not. Thus the descents in $\overline{T}(w)$ are precisely the indices $i$ such that $p_i = 1$ and $p_{i+1} = 0$. Thus in $w$, $w_i$ is the first occurrence of a letter, and $w_{i+1}$ is not. This implies $w_{i+1} \leq w_i$ by the growth restrictions of RGFs, i.e. $i$ must be an apex. $\square$

We now establish the main results of this section. Theorem 4.4 below should be compared to Theorem 7.4 of [12], which is proved using continued fractions.

**Theorem 4.4** ([12] Theorem 7.4). *The polynomials $I_n(q,t,x)$ satisfy $I_0(q,t,x) = 1$ and, for $n \geq 1$,*

$$I_n(q,t,x) = txI_{n-1}(q,t,x) + \sum_{j=2}^{n} q^{j-1} I_{j-2}(q,t,1)\big(I_{n-j+1}(q,t,x) - t(x-1)I_{n-j}(q,t,x)\big).$$

*Proof.* By the previous three lemmas, it suffices to work with the distribution of spread, block and checkpoints on noncrossing partitions of length $n$, defining

$$\mathrm{cp}(w) = \#\{i : i \text{ is a checkpoint in } w\}.$$

Our goal is to compute the generating function

$$\sum_{w \in R_n(13/24)} q^{\mathrm{sp}(w)} t^{\mathrm{bl}(w)} x^{\mathrm{cp}(w)}.$$

To do so, we use the recursive argument developed in Theorem 3.3. Recall the definition of the sets

$$X = \{w_1 \dots w_n \in R_n(13/24) : w_i > 1 \text{ for } i > 1\}$$

and, for $k = 2, 3, \dots, n$,

$$Y_k = \{w_1 \dots w_n \in R_n(13/24) : w_k = 1 \text{ and } w_j > 1 \text{ for } 1 < j < k\}.$$

As before, $X$ is in bijection with $R_{n-1}(13/24)$, with the map given by

$$u = u_1 \dots u_{n-1} \to 1(u+1).$$

Examining the behavior of the three statistics of interest under this map yields

$$\sum_{w \in X} q^{\mathrm{sp}(w)} t^{\mathrm{bl}(w)} x^{\mathrm{cp}(w)} = t \cdot x \cdot I_{n-1}(q,t,x).$$

Next, for $w \in Y_k$ let us write $w = 1(u+1)1v$, with $u \in R_{k-2}(13/24)$ and $\mathrm{st}(1v) \in R_{n-k+1}(13/24)$. As in Theorem 3.3, $\mathrm{sp}(w) = \mathrm{sp}(u) + \mathrm{sp}(\mathrm{st}(1v)) + k - 1$ and $\mathrm{bl}(w) = \mathrm{bl}(u) + \mathrm{bl}(\mathrm{st}(1v))$. The checkpoint statistic is slightly more subtle; the relation of $\mathrm{cp}(w)$ to $\mathrm{cp}(u)$ and $\mathrm{cp}(\mathrm{st}(1v))$ depends on whether or not there is a 1 in the word $v$.

We will get around the previous issues by an application of the Inclusion-Exclusion Principle. Let $V_k$ be the set

$$V_k = \{w \in R_{n-k+1}(13/24) : w \text{ contains a single } 1\ \}.$$

Writing $w \in Y_k$ as $w = 1(u+1)1v$ induces a bijection between $Y_k$ and the disjoint union

$$\Big(R_{k-2}(13/24) \times (R_{n-k+1}(13/24) \setminus V_k)\Big) \coprod \Big(R_{k-2}(13/24) \times V_k\Big).$$

If $w = 1(u+1)1v$, then $\mathrm{cp}(w) = \mathrm{cp}(\mathrm{st}(1v))$ if $\mathrm{st}(1v) \in R_{n-k+1}(13/24) \setminus V_k$, and $\mathrm{cp}(w) = \mathrm{cp}(\mathrm{st}(1v)) - 1$ if $\mathrm{st}(1v) \in V_k$. This is because if $\mathrm{st}(1v)$ has a unique 1 (necessarily at the first index), then the index 1 will be a checkpoint in $\mathrm{st}(1v)$ but will no longer be a checkpoint in $1(u+1)1v$.

A variant of Inclusion Exclusion gives that the generating function for the three statistics over $Y_k$ factors as a product of the polynomial

$$q^{k-1}\left(\sum_{u \in R_{k-2}(13/24)} q^{\mathrm{sp}(u)} t^{\mathrm{bl}(u)}\right)$$

with

$$\left(\sum_{\mathrm{st}(1v) \in R_{n-k+1}(13/24)} q^{\mathrm{sp}(\mathrm{st}(1v))} t^{\mathrm{bl}(\mathrm{st}(1v))} x^{\mathrm{cp}(\mathrm{st}(1v))} - (1-x^{-1}) \sum_{\mathrm{st}(1v) \in V_k} q^{\mathrm{sp}(\mathrm{st}(1v))} t^{\mathrm{bl}(\mathrm{st}(1v))} x^{\mathrm{cp}(\mathrm{st}(1v))}\right).$$

In other words, we have taken a naive count over all $u \in R_{k-2}(13/24)$ and $\mathrm{st}(1v) \in R_{n-k+1}(13/24)$, and then modified it with the appropriate correction where it is needed (i.e. with respect to $V_k$).

But by an argument that is now standard, we can put $V_k$ in bijection with $R_{n-k}(13/24)$ by sending $w \in R_{n-k}(13/24)$ to $1(w+1)$. Examining how our statistics are affected by this map, we obtain

$$\sum_{w \in Y_k} q^{\mathrm{sp}(w)} t^{\mathrm{bl}(w)} x^{\mathrm{cp}(w)} = q^{k-1} I_{k-2}(q,t,1)\big(I_{n-k+1}(q,t,x) - t(x-1)I_{n-k}(q,t,x)\big).$$

Summing over $k$ completes the proof. ☐

Of course, upon specialization of variables we recover, for instance, Theorem 1.1 of [12] as well, which provides a recursion for inversions and left-to-right maxima.

We end by showing how to mildly generalize part of Theorem 6.2 of [12], which examines descents and the major index. Cheng, Elizalde, Kasraoui, and Sagan prove their theorem using the theory of polyominoes.

**Theorem 4.5.** *We have $M_0(q,t,x) = 1$ and, for $n \geq 1$,*

$$\begin{aligned} M_n(q,t,x) &= xM_{n-1}(q,qt,x) \\ &\quad + \sum_{k=2}^{n} \Big(M_{k-1}(q,t,x) + x(q^{k-1}t - 1)M_{k-2}(q,t,x)\Big) M_{n-k}(q,q^k t,x). \end{aligned}$$

*Proof.* Define

$$\mathrm{ap}(w) = \#\{i : i \text{ is an apex in } w\}$$

and

$$\mathrm{maj}(w) = \sum_{i \text{ an apex}} i.$$

It suffices to determine the distribution of apices, major index, and block over $R_n(13/24)$, as by Lemma 4.3 we have

$$\sum_{w \in R_n(13/24)} q^{\mathrm{maj}(w)} t^{\mathrm{ap}(w)} x^{\mathrm{bl}(w)} = M_n(q,t,x).$$

This proof is similar in spirit to Theorems 3.3 and 4.4, but requires a different recursive argument. Partition $R_n(13/24)$ into the sets

$$R_n(13/24) = \coprod_{k=1}^{n} Y_k,$$

with

$$Y_k := \{w \in R_n(13/24) : \text{ the last occurrence of the letter 1 in } w \text{ has index } k\}.$$

Similarly to the proof of Theorem 3.3, $Y_1$ is in bijection with $R_{n-1}(13/24)$, and

$$\sum_{w \in Y_1} q^{\mathrm{maj}(w)} t^{\mathrm{ap}(w)} x^{\mathrm{bl}(w)} = x M_{n-1}(q, qt, x).$$

The sets $Y_k$, for $k \geq 2$, are in bijection with the Cartesian products

$$Y_k \leftrightarrow R_{k-1}(13/24) \times R_{n-k}(13/24);$$

the map exhibiting this sends $(u, v) \in R_{k-1}(13/24) \times R_{n-k}(13/24)$ to $w = u1(v + \max(u) + 1)$. Any index which is an apex in $u$ or $v$ will promote to an apex of $w$; additionally, if $u$ ends in a unique letter, then this will provide an additional apex of $w$ which was not an apex of $u$. Accordingly,

$$\mathrm{ap}(w) = \begin{cases} \mathrm{ap}(u) + \mathrm{ap}(v) + 1 & \text{if } u \text{ ends in a unique letter,} \\ \mathrm{ap}(u) + \mathrm{ap}(v) & \text{otherwise.} \end{cases}$$

Keeping track of the position of these apices gives

$$\mathrm{maj}(w) = \begin{cases} \mathrm{maj}(u) + \mathrm{maj}(v) + k\,\mathrm{ap}(v) + k - 1 & \text{if } u \text{ ends in a unique letter,} \\ \mathrm{maj}(u) + \mathrm{maj}(v) + k\,\mathrm{ap}(v) & \text{otherwise.} \end{cases}$$

Applying the same Inclusion-Exclusion argument as in Theorem 4.4 and summing over $k$ gives the desired result. □

Specializing the variable $x = 1$ yields the first recursive formula presented in Theorem 6.2 of [12].

## 5 Future Directions

As is evident, many interesting connections in combinatorics can be found by studying a combination of combinatorial statistics and combinatorial patterns. We end with several ideas one could examine in this area.

**Longer Patterns:** The most obvious extension one could make to this article is to continue studying the distribution of these statistics over avoidance classes of longer patterns. For example, Sagan provides closed formulae for the number of partitions avoiding $12/3/\ldots/m$ and for the number of partitions avoiding $1/23\ldots m$ in [32]. Can one generalize our arguments to those settings?

**Other Classes of Partitions:** There are several other natural classes of set partitions, which are not defined via the notion pattern avoidance defined above. What can one say about the distribution of dimension, spread, and block over these classes? For example, one could work with the notion of pattern avoidance in terms of restricted growth functions as is done in [9]. Alternatively, one could work with other combinatorially defined sets of partitions, such as the nonnesting partitions.

For an introduction to such objects and their relation to noncrossing partitions, see [7].

**Machine Learning:** Can machine learning be used to examine combinatorial patterns, in any context? Such computations have found use in computational algebraic geometry and theoretical physics [10]; analogs in the combinatorial setting could be useful in further developing combinatorial databases, such as Tenner's Database of Permutation Pattern Avoidance [35].

**Connections to Permutations:** With Section 4 in mind, can one find more connections between pattern avoidance in set partitions and pattern avoidance in permutations? Following a comment of Kyle Petersen on OEIS entry A055151, a potential start would be to connect Theorem 3.3 to descents and peaks in 231-avoiding permutations. One could also try to relate partitions in $\Pi_n(123, 13/24)$ to permutations avoiding 321 and the so called *barred pattern* $3\bar{1}24$; see [11] for more information.

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