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A group induced four-circulant construction for self-dual codes and new extremal binary self-dual codes

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Abstract

We introduce an altered version of the four-circulant construction over group rings for self-dual codes. We consider this construction over the binary field, the rings $\mathbb{F}_2 + u\mathbb{F}_2$ and $\mathbb{F}_4 + u\mathbb{F}_4$ using groups of orders 3, 7, 9, 13, and 15. Through these constructions and their extensions, we find binary self-dual codes of lengths 32, 40, 56, 64, 68 and 80, all of which are extremal or optimal. In particular, we find five new self-dual codes with parameters $[56, 28, 10]$, twenty-three extremal binary self-dual codes of length 68 with new weight enumerators and fifteen new self-dual codes with parameters $[80, 40, 14]$.

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1 Introduction

Constructions coming from group rings that have emerged in recent works have extended the tools for binary self-dual codes. Many of the classical constructions have been found to be related to certain groups and considering different group rings have brought new construction methods into the literature. This connection was first highlighted in [15], where a connection between certain group ring elements called unitary units and self-dual codes was established and the connection was used to produce many self-dual codes. The main idea is to take a matrix of the form $[I_n|A]$, where A is an $n \times n$ matrix that has a special structure depending on the group rings used.

Previously, group ring elements were used in a different way to construct certain special codes. In [1], an ideal of the group algebra $\mathbb{F}_2S_4$ was used to construct the well-known binary extended Golay code where S_4 is the symmetric group on 4 elements. In [21], an isomorphism between a group ring and a certain subring of the $n \times n$ matrices over the ring was established. This isomorphism was used to produce special self-dual codes in [22, 24].

The inspiration for this work comes from modifying the four-circulant construction, which was first introduced in [2]. Let G be the matrix

$$\left[\begin{array}{c|cc} I_{2n} & \begin{array}{cc} A & B \\ -B^T & A^T \end{array} \end{array}\right]$$

where A and B are circulant matrices. Then the code generated by G over $\mathbb{F}_p$ is self-dual if and only if $AA^T + BB^T = -I_n$. Note that when the alphabet is a ring of characteristic 2, then the matrix and the conditions can be written in an alternative form, where the negative signs disappear.

In this work, we will consider constructing self-dual codes from the following variation of the four-circulant matrix. Consider the matrix

$$\left[\begin{array}{c|cc} I_{2n} & \begin{array}{cc} A & B \\ B^T & A^T \end{array} \end{array}\right],$$

where both A and B are matrices that arise from group rings. Depending on the groups, the matrices will usually not be circulant matrices, which is a variation from the usual four-circulant construction. Under this construction, we establish the link between units/non-units in the group ring and corresponding self-dual codes. Using this connection for some particular examples of groups over the field $\mathbb{F}_2$ and the rings $\mathbb{F}_2+u\mathbb{F}_2$ and $\mathbb{F}_4+u\mathbb{F}_4$, we are able to construct many extremal and optimal binary self-dual codes of different lengths. In particular, we construct five new self-dual codes with parameters $[56, 28, 10]$, twenty-three extremal binary self-dual codes of length 68 with new weight enumerators, and fifteen new self-dual codes with parameters $[80, 40, 14]$.

The rest of the work is organized as follows. In Section 2, we give the necessary background on codes, the alphabets we use and the group rings. In Section 3, we give the constructions and the theoretical results about the group ring elements that lead to self-dual codes. In Sections 4 and 5, we apply the construction methods to produce the numerical results, using MAGMA [3]. The paper ends with concluding remarks and possible further research directions.

2 Preliminaries

In this section we will define self-dual codes over Frobenius rings of characteristic 2. We will recall some of the properties of the family of rings called R_k and the ring $\mathbb{F}_4 + u\mathbb{F}_4$. This section concludes with an introduction to group rings and an established isomorphism between a group ring and a certain subring of the $n \times n$ matrices over a ring.

2.1 Self-dual codes

Throughout this work, all rings are assumed to be commutative, finite Frobenius rings with a multiplicative identity.

A code over a finite commutative ring R is said to be any subset C of R^n. When the code is a submodule of the ambient space then the code is said to be linear. To the ambient space, we attach the usual inner-product, specifically $[\mathbf{v}, \mathbf{w}] = \sum v_i w_i$. The dual code with respect to this inner-product is defined as $C^\perp = \{\mathbf{w} \mid \mathbf{w} \in R^n,\ [\mathbf{w}, \mathbf{v}] = 0, \forall\ \mathbf{v} \in C\}$. Since the ring is Frobenius we have that for all linear codes over R, $|C||C^\perp| = |R|^n$. If a code satisfies $C = C^\perp$ then the code C is said to be self-dual. If $C \subseteq C^\perp$ then the code is said to be self-orthogonal.

For binary codes, a self-dual code where all weights are divisible by 4 is said to be Type II, and otherwise the code is said to be Type I. The bounds on the minimum distances for binary self-dual codes are given in the following theorem:

Theorem 2.1. ([25]) *Let $d(n)$ denote the minimum distance of a binary self-dual code of length n. Then we have*

$$d(n) \leq \begin{cases} 4\lfloor \frac{n}{24} \rfloor + 4 & \text{if } n \not\equiv 22 \pmod{24} \\ 4\lfloor \frac{n}{24} \rfloor + 6 & \text{if } n \equiv 22 \pmod{24}. \end{cases}$$

Self-dual codes that meet these bounds are called *extremal*. In Sections 4 and 5 we will construct extremal binary self-dual codes of different lengths.

2.2 R_k family of rings

An important class of rings that has been used extensively in constructing codes is the ring family of R_k, which has been introduced in [12]. We will be mainly using $R_0 = \mathbb{F}_2$ and $R_1 = \mathbb{F}_2 + u\mathbb{F}_2$ in this work; however, some of the theoretical results will be true for all R_k, which is why we would like to give a brief description of the rings, mainly from [12] and [13]. For $k \geq 1$, define

$$R_k = \mathbb{F}_2[u_1, u_2, \ldots, u_k]/\langle u_i^2, u_i u_j - u_j u_i \rangle,$$

which can also be defined recursively as

$$R_k = R_{k-1}[u_k]/\langle u_k^2, u_k u_j - u_j u_k \rangle = R_{k-1} + u_k R_{k-1}.$$

For any subset $A \subseteq \{1, 2, \ldots, k\}$ we will fix

$$u_A := \prod_{i \in A} u_i$$

with the convention that $u_\emptyset = 1$. Then any element of R_k can be represented as

$$\sum_{A \subseteq \{1,\ldots,k\}} c_A u_A, \qquad c_A \in \mathbb{F}_2.$$

With this representation of the elements, we have

$$u_A u_B = \begin{cases} 0 & \text{if } A \cap B \neq \emptyset \\ u_{A\cup B} & \text{if } A \cap B = \emptyset \end{cases}$$

and

$$\left(\sum_A c_A u_A\right)\left(\sum_B d_B u_B\right) = \sum_{A,B \subseteq \{1,\ldots,k\}, A\cap B=\emptyset} c_A d_B u_{A\cup B}.$$

It is shown in [12] that the ring family R_k is a commutative ring with $|R_k| = 2^{(2^k)}$.

A Gray map from R_k to $\mathbb{F}_2^{2^k}$ was defined inductively starting with the map on $R_1 : \phi_1(a + bu_1) = (b, a + b)$. We recall that if $c \in R_k$, then c can be written as $c = a + bu_{k-1}$, $a, b \in R_{k-1}$. Then

$$\phi_k(c) = (\phi_{k-1}(b), \phi_{k-1}(a + b)).$$

The map ϕ_k is a distance preserving map and the following is shown in [13].

Theorem 2.2. *Let C be a self-dual code over R_k. Then $\phi_k(R_k)$ is a binary self-dual code of length $2^k n$.*

The following lemma describes a property of units and non-units in R_k.

Lemma 2.3. ([12]) *For an element $\alpha \in R_k$ we have*

$$\alpha^2 = \begin{cases} 1 & \textit{if } \alpha \textit{ is a unit} \\ 0 & \textit{otherwise.} \end{cases}$$

The next result, which was introduced in [11], can easily be extended to be true for R_k as well.

Theorem 2.4. *Let $\mathcal{C}$ be a self-dual code over R_k of length n and $G = (r_i)$ be a $j \times n$ generator matrix for $\mathcal{C}$, where r_i is the i-th row of G, $1 \leq i \leq j$. Let c be a unit in R_k and X be a vector in ${R_k}^n$ with $\langle X, X \rangle = 1$. Let $y_i = \langle r_i, X \rangle$ for $1 \leq i \leq j$. Then the matrix*

$$\left(\begin{array}{cc|c} 1 & 0 & X \\ \hline y_1 & cy_1 & r_1 \\ \vdots & \vdots & \vdots \\ y_j & cy_j & r_j \end{array}\right)$$

generates a self-dual code $\mathcal{C}'$ over R_k of length $n + 2$.

2.3 The ring $\mathbb{F}_4 + u\mathbb{F}_4$

Let $\mathbb{F}_4 = \mathbb{F}_2(\omega)$ be the quadratic field extension of $\mathbb{F}_2$, where $\omega^2 + \omega + 1 = 0$. The ring $\mathbb{F}_4 + u\mathbb{F}_4$ is defined via $u^2 = 0$. Note that $\mathbb{F}_4 + u\mathbb{F}_4$ can be viewed as an extension of $R_1 = \mathbb{F}_2 + u\mathbb{F}_2$ and so we can describe any element of $\mathbb{F}_4 + u\mathbb{F}_4$ in the form $\omega a + \bar{\omega} b$ uniquely, where $a, b \in \mathbb{F}_2 + u\mathbb{F}_2$.

A linear code C of length n over $\mathbb{F}_4 + u\mathbb{F}_4$ is an $(\mathbb{F}_4 + u\mathbb{F}_4)$-submodule of $(\mathbb{F}_4 + u\mathbb{F}_4)^n$. In [14] and [8] the following Gray maps were introduced:

$$\begin{array}{l|l} \psi_{\mathbb{F}_4} : (\mathbb{F}_4)^n \to (\mathbb{F}_2)^{2n} & \varphi_{\mathbb{F}_2+u\mathbb{F}_2} : (\mathbb{F}_2 + u\mathbb{F}_2)^n \to \mathbb{F}_2^{2n} \\ a\omega + b\overline{\omega} \mapsto (a, b)\,, \;\; a, b \in \mathbb{F}_2^n & a + bu \mapsto (b, a + b)\,, \;\; a, b \in \mathbb{F}_2^n. \end{array}$$

Note that $\varphi_{\mathbb{F}_2+u\mathbb{F}_2}$ is the same map as ϕ_1 described before. Those were generalized to the following maps in [23]:

$$\begin{array}{l|l} \psi_{\mathbb{F}_4+u\mathbb{F}_4} : (\mathbb{F}_4 + u\mathbb{F}_4)^n \to (\mathbb{F}_2 + u\mathbb{F}_2)^{2n} & \varphi_{\mathbb{F}_4+u\mathbb{F}_4} : (\mathbb{F}_4 + u\mathbb{F}_4)^n \to \mathbb{F}_4^{2n} \\ a\omega + b\overline{\omega} \mapsto (a, b)\,, \;\; a, b \in (\mathbb{F}_2 + u\mathbb{F}_2)^n & a + bu \mapsto (b, a + b)\,, \;\; a, b \in \mathbb{F}_4^n \end{array}$$

These maps preserve orthogonality in the corresponding alphabets. The binary images $\varphi_{\mathbb{F}_2+u\mathbb{F}_2} \circ \psi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$ and $\psi_{\mathbb{F}_4} \circ \varphi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$ are equivalent. The Lee weight of an element is defined to be the Hamming weight of its binary image.

Proposition 2.5. ([23]) *Let C be a code over $\mathbb{F}_4 + u\mathbb{F}_4$. If C is self-orthogonal, so are $\psi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$ and $\varphi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$. Also C is a Type I (respectively, Type II) code over $\mathbb{F}_4 + u\mathbb{F}_4$ if and only if $\varphi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$ is a Type I (respectively, Type II) $\mathbb{F}_4$-code, if and only if $\psi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$ is a Type I (respectively, Type II) $\mathbb{F}_2 + u\mathbb{F}_2$-code. Furthermore, the minimum Lee weight of C is the same as the minimum Lee weight of $\psi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$ and $\varphi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$.*

Corollary 2.6. *Suppose that C is a self-dual code over $\mathbb{F}_4 + u\mathbb{F}_4$ of length n and minimum Lee distance d. Then $\varphi_{\mathbb{F}_2+u\mathbb{F}_2} \circ \psi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$ is a binary $[4n, 2n, d]$ self-dual code. Moreover, C and $\varphi_{\mathbb{F}_2+u\mathbb{F}_2} \circ \psi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$ have the same weight enumerator. If C is Type I (Type II), then so is $\varphi_{\mathbb{F}_2+u\mathbb{F}_2} \circ \psi_{\mathbb{F}_4+u\mathbb{F}_4}(C)$.*

2.4 Shorthand notation for elements of $\mathbb{F}_2 + u\mathbb{F}_2$ and $\mathbb{F}_4 + u\mathbb{F}_4$

In subsequent sections we will be writing tables in which vectors with elements from the rings $\mathbb{F}_2 + u\mathbb{F}_2$ and $\mathbb{F}_4 + u\mathbb{F}_4$ will appear. In order to avoid writing long vectors with elements that can be confused with other elements, we will be describing the elements of these rings in a shorthand way, which will make the tables more compact.

For elements of $\mathbb{F}_2 + u\mathbb{F}_2$, elements $0, 1, u$ will be written as they are, while $1 + u$ will be replaced by 3. So, for example, a vector of the form $(1, 1 + u, 0, 0, u, 1 + u)$ will be described as $(1300u3)$.

For the elements of $\mathbb{F}_4 + u\mathbb{F}_4$, we use the ordered basis $\{u\omega, \omega, u, 1\}$ to express the elements of $\mathbb{F}_4 + u\mathbb{F}_4$ as binary strings of length 4. Then we will use the hexadecimal number system to describe each element:

$0 \leftrightarrow 0000$, $1 \leftrightarrow 0001$, $2 \leftrightarrow 0010$, $3 \leftrightarrow 0011$, $4 \leftrightarrow 0100$, $5 \leftrightarrow 0101$, $6 \leftrightarrow 0110$, $7 \leftrightarrow 0111$, $8 \leftrightarrow 1000$, $9 \leftrightarrow 1001$, $A \leftrightarrow 1010$, $B \leftrightarrow 1011$, $C \leftrightarrow 1100$, $D \leftrightarrow 1101$, $E \leftrightarrow 1110$, $F \leftrightarrow 1111$.

For example, $1 + u\omega$ corresponds to 1001, which is represented by the hexadecimal 9, while $\omega + u\omega$ corresponds to 1100, which is represented by C.

2.5 Certain matrices and group rings

We start with a description of circulant matrices and block circulant matrices, the details for which can be found in [6].

Definition 1. *A circulant matrix over a ring R is a square $n \times n$ matrix, which takes the form*

$$\operatorname{circ}(a_1, a_2, \dots, a_n) = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \dots & a_1 \end{pmatrix}$$

where $a_i \in R$.

Definition 2. *A block circulant matrix over a ring R is a square $kn \times kn$ matrix, which takes the form*

$$\operatorname{CIRC}(A_1, A_2, \dots, A_n) = \begin{pmatrix} A_1 & A_2 & A_3 & \dots & A_n \\ A_n & A_1 & A_2 & \dots & A_{n-1} \\ A_{n-1} & A_n & A_1 & \dots & A_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & A_4 & \dots & A_1 \end{pmatrix}$$

where each A_i is a $k \times k$ matrix over R.

Let G be a finite group of order n. Then the group ring RG consists of $\sum_{i=1}^{n} \alpha_i g_i$, $\alpha_i \in R$, $g_i \in G$. Addition in the group ring is done by coordinate addition, namely

$$\sum_{i=1}^{n} \alpha_i g_i + \sum_{i=1}^{n} \beta_i g_i = \sum_{i=1}^{n} (\alpha_i + \beta_i) g_i.$$

The product of two elements in a group ring is given by

$$\left(\sum_{i=1}^{n} \alpha_i g_i \right) \left(\sum_{j=1}^{n} \beta_j g_j \right) = \sum_{i,j} \alpha_i \beta_j g_i g_j.$$

It follows that the coefficient of g_k in the product is $\sum_{g_i g_j = g_k} \alpha_i \beta_j$.

The following construction of a matrix was first given by Hurley in [21]. Let R be a finite commutative Frobenius ring of characteristic 2 and let $G = \{g_1, g_2, \dots, g_n\}$ be a group of order n. Let $v = \sum_{i=1}^{n} \alpha_{g_i} g_i \in RG$. Define the matrix $\sigma(v) \in M_n(R)$ to be $\sigma(v) = (\alpha_{g_i^{-1} g_j})$ where $i, j \in \{1, \dots, n\}$. We note that the elements $g_1^{-1}, g_2^{-1}, \dots, g_n^{-1}$ are the elements of the group G in a given order.

Recall the canonical involution $* : RG \to RG$ on a group ring RG is given by $v^* = \sum_g a_g g^{-1}$, for $v = \sum_g a_g g \in RG$. An important connection between v^* and v appears when we take their images under the σ map:

$$\sigma(v^*) = \sigma(v)^T.$$

We will now describe $\sigma(v)$ for the following group rings RG where $G \in \{C_n, C_m \times C_n, C_{m,n}\}$.

- Let $G = \langle x \mid x^n = 1 \rangle \cong C_n$. If $v = \sum_{i=0}^{n-1} \alpha_{i+1} x^i \in RC_n$, then $\sigma(v) = \mathrm{circ}(a_0, a_1, \dots, a_{n-1})$.
- Let $G = \langle x, y \mid x^n = y^m = 1,\ xy = yx \rangle \cong C_m \times C_n$. If
$$v = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} a_{1+i+mj} x^i y^j \in R(C_m \times C_n),$$
then $\sigma(v) = \mathrm{CIRC}(A_1, \dots, A_n)$ where $A_{j+1} = \mathrm{circ}(a_{1+mj}, a_{2+mj}, \dots, a_{m+mj})$, $a_i \in R$ and $m, n \geq 2$.
- Let $G = C_{m,n} = \langle x \mid x^{mn} = 1 \rangle$. If $v = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} a_{1+i+mj} x^{ni+j} \in RC_{m,n}$, then
$$\sigma(v) = \begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_{n-1} & A_n \\ A'_n & A_1 & A_2 & \cdots & A_{n-2} & A_{n-1} \\ A'_{n-1} & A'_n & A_1 & \cdots & A_{n-3} & A_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A'_2 & A'_3 & A'_4 & \cdots & A'_n & A_1 \end{pmatrix},$$
where $A_{j+1} = \mathrm{circ}(a_{1+mj}, a_{2+mj}, \dots, a_{m+mj})$, $A'_{j+1} = \mathrm{circ}(a_{m+mj}, a_{1+mj}, \dots, a_{(m-1)+mj})$, $a_i \in R$ and $m, n \geq 2$. Note that $C_{m,n}$ is the same as the cyclic group C_{mn}; however, we give a different labeling to the elements, which makes $\sigma(v)$ different from the matrix that we obtain from the standard labeling of cyclic groups.

3 The construction

Let $v_1, v_2 \in RG$ where R is a finite commutative Frobenius ring of characteristic 2, G is a finite group of order p (where p is odd) and $\gamma_i \in R$. Define the following matrix:

$$M_\sigma = \left[\begin{array}{c|c|c} I_{2p+2} & \left(\begin{array}{c|ccc} \gamma_1 & \gamma_2 & \cdots & \gamma_2 \\ \hline \gamma_2 & & & \\ \vdots & & \sigma(v_1) & \\ \gamma_2 & & & \end{array}\right) & \left(\begin{array}{c|ccc} \gamma_3 & \gamma_4 & \cdots & \gamma_4 \\ \hline \gamma_4 & & & \\ \vdots & & \sigma(v_2) & \\ \gamma_4 & & & \end{array}\right) \\ \hline & \left(\begin{array}{c|ccc} \gamma_3 & \gamma_4 & \cdots & \gamma_4 \\ \hline \gamma_4 & & & \\ \vdots & & \sigma(v_2) & \\ \gamma_4 & & & \end{array}\right)^T & \left(\begin{array}{c|ccc} \gamma_1 & \gamma_2 & \cdots & \gamma_2 \\ \hline \gamma_2 & & & \\ \vdots & & \sigma(v_1) & \\ \gamma_2 & & & \end{array}\right)^T \end{array}\right]$$

$$= \left[\begin{array}{c|c|c|c|c} I_{2p+2} & \gamma_1 & \gamma_2 \cdots \gamma_2 & \gamma_3 & \gamma_4 \cdots \gamma_4 \\ \hline & \gamma_2 & & \gamma_4 & \\ & \vdots & \sigma(v_1) & \vdots & \sigma(v_2) \\ & \gamma_2 & & \gamma_4 & \\ \hline & \gamma_3 & \gamma_4 \cdots \gamma_4 & \gamma_1 & \gamma_2 \cdots \gamma_2 \\ \hline & \gamma_4 & & \gamma_2 & \\ & \vdots & \sigma(v_2)^T & \vdots & \sigma(v_1)^T \\ & \gamma_4 & & \gamma_2 & \end{array}\right]$$

Let C_σ be a code that is generated by the matrix $M(\sigma)$. Then the code C_σ has length $4p+4$. We aim to establish some restrictions when this construction yields self-dual codes.

Theorem 3.1. *Let R be a finite commutative Frobenius ring of characteristic 2 and let $G = \{g_1, g_2, \ldots, g_p\}$ be a finite group of order p (where p is odd). If*

1. $v_1 v_2 = v_2 v_1$,
2. $\sum_{i=1}^{4} \gamma_i^2 = 1$,
3. $v_1 v_1^* + v_2 v_2^* + (\gamma_2 + \gamma_4)^2 \widehat{g} + 1 = 0$,
4. $v_1^* v_1 + v_2^* v_2 + (\gamma_2 + \gamma_4)^2 \widehat{g} + 1 = 0$, *and*
5. $\gamma_1 = \delta_1$ *and* $\gamma_3 = \delta_2$,

then C_σ is a self-dual code of length $4p+4$ where $\widehat{g} = \sum_{i=1}^{p} g_i$, $v_1 = \sum_{g \in G} \alpha_g g$, $v_2 = \sum_{g \in G} \beta_g g$, $\delta_1 = \sum_{g \in G} \alpha_g$, and $\delta_2 = \sum_{g \in G} \beta_g$.

Proof. Clearly, C_σ has free rank $2p+2$, because the left-hand side of the generator matrix is the $2p+2$ by $2p+2$ identity matrix. Let us consider $M(\sigma)M(\sigma)^T$. Let $M(\sigma) = \left(\begin{array}{c|cc} I_{2n} & A & B \\ & B^T & A^T \end{array}\right)$; then $M(\sigma)M(\sigma)^T = \begin{pmatrix} AA^T+BB^T+I & AB+BA \\ B^TA^T+A^TB^T & B^TB+A^TA+I \end{pmatrix}$ where

$$A = \left(\begin{array}{c|ccc} \gamma_1 & \gamma_2 & \cdots & \gamma_2 \\ \hline \gamma_2 & & & \\ \vdots & & \sigma(v_1) & \\ \gamma_2 & & & \end{array}\right) \text{ and } B = \left(\begin{array}{c|ccc} \gamma_3 & \gamma_4 & \cdots & \gamma_4 \\ \hline \gamma_4 & & & \\ \vdots & & \sigma(v_2) & \\ \gamma_4 & & & \end{array}\right).$$

Let $A = \begin{pmatrix} B_1 & B_2 \\ B_2^T & B_3 \end{pmatrix}$ and $B = \begin{pmatrix} B_4 & B_5 \\ B_5^T & B_6 \end{pmatrix}$ where $B_1 = \gamma_1,\ B_2 = (\gamma_2 \cdots \gamma_2),\ B_3 = \sigma(v_1)$, $B_4 = \gamma_3,\ B_5 = (\gamma_4 \cdots \gamma_4)$ and $B_6 = \sigma(v_2)$. Now,

$$AB + BA = \begin{pmatrix} B_1B_4 + B_2B_5^T + B_4B_1 + B_5B_2^T & B_1B_5 + B_2B_6 + B_4B_2 + B_5B_3 \\ B_2^TB_4 + B_3B_5^T + B_5^TB_1 + B_6B_2^T & B_2^TB_5 + B_3B_6 + B_5^TB_2 + B_6B_3 \end{pmatrix},$$

$$\begin{aligned} B_1B_4 + B_2B_5^T + B_4B_1 + B_5B_2^T &= \gamma_1\gamma_3 + \begin{pmatrix} \gamma_2 & \cdots & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_4 \\ \vdots \\ \gamma_4 \end{pmatrix} + \gamma_3\gamma_1 + \begin{pmatrix} \gamma_4 & \cdots & \gamma_4 \end{pmatrix} \begin{pmatrix} \gamma_2 \\ \vdots \\ \gamma_2 \end{pmatrix} \\ &= \gamma_1\gamma_3 + \gamma_2\gamma_4 + \gamma_3\gamma_1 + \gamma_4\gamma_2 = 0, \end{aligned}$$

$$\begin{aligned} &B_1B_5 + B_2B_6 + B_4B_2 + B_5B_3 \\ &\quad = \gamma_1 \begin{pmatrix} \gamma_4 & \cdots & \gamma_4 \end{pmatrix} + \begin{pmatrix} \gamma_2 & \cdots & \gamma_2 \end{pmatrix} \sigma(v_2) + \gamma_3 \begin{pmatrix} \gamma_2 & \cdots & \gamma_2 \end{pmatrix} + \begin{pmatrix} \gamma_4 & \cdots & \gamma_4 \end{pmatrix} \sigma(v_1) \\ &\quad = \gamma_1\gamma_4 \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} + \gamma_2\delta_2 \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} + \gamma_3\gamma_2 \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} + \gamma_4\delta_1 \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} \\ &\quad = (\gamma_1\gamma_4 + \gamma_2\delta_2 + \gamma_3\gamma_2 + \gamma_4\delta_1) \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} = 0, \end{aligned}$$

$$\begin{aligned} B_2^TB_4 + B_3B_5 + B_5^TB_1 + B_6B_2^T &= \begin{pmatrix} \gamma_2 \\ \vdots \\ \gamma_2 \end{pmatrix} \gamma_3 + \sigma(v_1) \begin{pmatrix} \gamma_4 \\ \vdots \\ \gamma_4 \end{pmatrix} + \begin{pmatrix} \gamma_4 \\ \vdots \\ \gamma_4 \end{pmatrix} \gamma_1 + \sigma(v_2) \begin{pmatrix} \gamma_2 \\ \vdots \\ \gamma_2 \end{pmatrix} \\ &= \gamma_2\gamma_3 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + \delta_1\gamma_4 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + \gamma_4\gamma_1 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + \delta_2\gamma_2 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \\ &= (\gamma_2\gamma_3 + \delta_1\gamma_4 + \gamma_4\gamma_1 + \delta_2\gamma_2) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 0, \end{aligned}$$

$$\begin{aligned} &B_2^TB_5 + B_3B_6 + B_5^TB_2 + B_6B_3 \\ &\quad = \begin{pmatrix} \gamma_2 \\ \vdots \\ \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_4 & \cdots & \gamma_4 \end{pmatrix} + \sigma(v_1)\sigma(v_2) + \begin{pmatrix} \gamma_4 \\ \vdots \\ \gamma_4 \end{pmatrix} \begin{pmatrix} \gamma_2 & \cdots & \gamma_2 \end{pmatrix} + \sigma(v_2)\sigma(v_1) \\ &\quad = \gamma_2\gamma_4 \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} + \sigma(v_1v_2) + \gamma_4\gamma_2 \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} + \sigma(v_2v_1) \\ &\quad = \sigma(v_1v_2) + \sigma(v_2v_1) = 0. \end{aligned}$$

Also,

$$AA^T + BB^T + I = \begin{pmatrix} B_1^2 + B_2B_2^T + B_4^2 + B_5B_5^T + 1 & B_1B_2 + B_2B_3^T + B_4B_5 + B_5B_6^T \\ B_2^TB_1 + B_3B_2^T + B_5^TB_4 + B_6B_5^T & B_2^TB_2 + B_3B_3^T + B_5^TB_5 + B_6B_6^T + I \end{pmatrix},$$

$$\begin{aligned} B_1^2 + B_2B_2^T + B_4^2 + B_5B_5^T + 1 &= \gamma_1^2 + \begin{pmatrix} \gamma_2 & \cdots & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_2 \\ \vdots \\ \gamma_2 \end{pmatrix} + \gamma_3^2 + \begin{pmatrix} \gamma_4 & \cdots & \gamma_4 \end{pmatrix} \begin{pmatrix} \gamma_4 \\ \vdots \\ \gamma_4 \end{pmatrix} = 0 \\ &= (\gamma_1 + \gamma_3)^2 + p(\gamma_2 + \gamma_4)^2 + 1 \\ &= i \sum_{i=1}^{4} \gamma_i^2 + 1, \end{aligned}$$

$$\begin{aligned}
&B_1B_2+B_2B_3^T+B_4B_5+B_5B_6^T\\
&\quad=\gamma_1\left(\gamma_2 \cdots \gamma_2\right)+\left(\gamma_2 \cdots \gamma_2\right)\sigma(v_1)^T+\gamma_3\left(\gamma_4 \cdots \gamma_4\right)+\left(\gamma_4 \cdots \gamma_4\right)\sigma(v_2)^T\\
&\quad=\left(\gamma_1\gamma_2 \cdots \gamma_1\gamma_2\right)+\left(\gamma_2\delta_1 \cdots \gamma_2\delta_1\right)+\left(\gamma_3\gamma_4 \cdots \gamma_3\gamma_4\right)+\left(\gamma_4\delta_2 \cdots \delta_2\gamma_4\right)\\
&\quad=(\gamma_1\gamma_2+\gamma_2\delta_1+\gamma_3\gamma_4+\gamma_4\delta_2)\left(1 \cdots 1\right)=0,
\end{aligned}$$

where $v_1=\sum_{g\in G}\alpha_g g$, $v_2=\sum_{g\in G}\beta_g g$, $\delta_1=\sum_{g\in G}\alpha_g$ and $\delta_2=\sum_{g\in G}\beta_g$, and

$$\begin{aligned}
&B_2^TB_2+B_3B_3^T+B_5^TB_5+B_6B_6^T+I\\
&\quad=\begin{pmatrix}\gamma_2\\ \vdots\\ \gamma_2\end{pmatrix}\left(\gamma_2 \cdots \gamma_2\right)+\sigma(v_1)\sigma(v_1)^T+\begin{pmatrix}\gamma_4\\ \vdots\\ \gamma_4\end{pmatrix}\left(\gamma_4 \cdots \gamma_4\right)+\sigma(v_2)\sigma(v_2)^T+I\\
&\quad=\sigma(v_1v_1^*)+\sigma(v_2v_2^*)+(\gamma_2+\gamma_4)^2\begin{pmatrix}1 & \cdots & 1\\ \vdots & \ddots & \vdots\\ 1 & \cdots & 1\end{pmatrix}+I.
\end{aligned}$$

Additionally,

$$\begin{aligned}
&B^TB+A^TA+I\\
&\quad=\begin{pmatrix}B_1^2+B_2B_2^T+B_4^2+B_5B_5^T+1 & B_1B_2+B_2B_3+B_4B_5+B_5B_6\\ B_2^TB_1+B_3^TB_2^T+B_5^TB_4+B_6^TB_5^T & B_2^TB_2+B_3^TB_3+B_5^TB_5+B_6^TB_6+I\end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
&B_1B_2+B_2B_3+B_4B_5+B_5B_6\\
&\quad=\gamma_1\left(\gamma_2 \cdots \gamma_2\right)+\left(\gamma_2 \cdots \gamma_2\right)\sigma(v_1)+\gamma_3\left(\gamma_4 \cdots \gamma_4\right)+\left(\gamma_4 \cdots \gamma_4\right)\sigma(v_2)\\
&\quad=\left(\gamma_1\gamma_2 \cdots \gamma_1\gamma_2\right)+\left(\gamma_2\delta_1 \cdots \gamma_2\delta_1\right)+\left(\gamma_3\gamma_4 \cdots \gamma_3\gamma_4\right)+\left(\gamma_4\delta_2 \cdots \gamma_4\delta_2\right)\\
&\quad=(\gamma_1\gamma_2+\gamma_2\delta_1+\gamma_3\gamma_4+\gamma_4\delta_2)\left(1 \cdots 1\right)=0,
\end{aligned}$$

and

$$B_2^TB_2+B_3^TB_3+B_5^TB_5+B_6^TB_6+I=\sigma(v_1^*v_1)+\sigma(v_2^*v_2)+(\gamma_2+\gamma_4)^2\begin{pmatrix}1 & \cdots & 1\\ \vdots & \ddots & \vdots\\ 1 & \cdots & 1\end{pmatrix}+I.$$

Finally, $M(\sigma)M(\sigma)^T$ is a symmetric matrix and C_σ is self-orthogonal when $\sigma(v_1v_2)=\sigma(v_2v_1)$, $\sum_{i=1}^4\gamma_i^2=1$, $\gamma_1=\delta_1$, $\gamma_3=\delta_2$, $v_1v_1^*+v_2v_2^*+(\gamma_2+\gamma_4)^2\widehat{g}+1=0$ and $v_1^*v_1+v_2^*v_2+(\gamma_2+\gamma_4)^2\widehat{g}+1=0$. □

Corollary 3.2. *Let $R=R_k$, G be a finite group of order p (where p is odd), $\gamma_2+\gamma_4$ be a non-unit in R_k and C_σ be self-dual. Then either*

- *$v_1\in RG$ is a unitary unit and v_2 is a non-unit, or*
- *$v_2\in RG$ is a unitary unit and v_1 is a non-unit.*

Proof. Let $v_1\in RG$ be a unitary unit and $\gamma_2+\gamma_4$ be a non-unit in R_k. Then $v_1v_1^*=1$ and $(\gamma_2+\gamma_4)^2=0$. Clearly $v_2v_2^*=0$, $\det(\sigma(v_2v_2^*))=0$ is a non-unit by Corollary 3 in [21]. Therefore v_2 is a non-unit in RG. Similarly, if $v_2\in RG$ is unitary and $\gamma_2+\gamma_4$ is a non-unit in R_k, then v_1 is a non-unit in RG. □

Corollary 3.3. *Let $R = R_k$, G be a finite group of order p (where p is odd), $\gamma_2 + \gamma_4$ be a non-unit in R_k and C_σ be self-dual. Then $v_1^* v_1 + v_2^* v_2$ is a non-unit in RG.*

Proof. Let $\gamma_2 + \gamma_4$; then $(\gamma_2 + \gamma_4)^2 = 1$. Now

$$\sigma(v_1^* v_1) + \sigma(v_2^* v_2) + (\gamma_2 + \gamma_4)^2 \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} + I = 0$$

$$\sigma(v_1^* v_1) + \sigma(v_2^* v_2) + \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} + I = 0$$

$$\sigma(v_1^* v_1 + v_2^* v_2) = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}.$$

Now,

$$\begin{aligned} \det \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix} &= \det \begin{pmatrix} p-1 & p-1 & p-1 & \cdots & p-1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix} \\ &= (p-1)(-1)^{p-1} = 0 \end{aligned}$$

since p is odd. Therefore $\sigma(v_1^* v_1 + v_2^* v_2) = 0$ and $v_1^* v_1 + v_2^* v_2$ is a non-unit by Corollary 3 in [21]. □

4 Extremal binary self-dual codes from the constructions

In this section, we will present the results obtained using the construction described in Section 3, to construct self-dual codes for certain groups over different alphabets.

4.1 Constructions coming from C_3

We apply the constructions over the rings $\mathbb{F}_4$ and $\mathbb{F}_4 + u\mathbb{F}_4$. The Gray images of the self-dual codes are binary self-dual codes of lengths 32 and 64 respectively. We only list the extremal ones of length 64.

Recall that the possible weight enumerators for a self-dual Type I $[64, 32, 12]$-code are given in [5, 10] as:

$$\begin{aligned} W_{64,1} &= 1 + (1312 + 16\beta)\, y^{12} + (22016 - 64\beta)\, y^{14} + \cdots, 14 \leq \beta \leq 284, \\ W_{64,2} &= 1 + (1312 + 16\beta)\, y^{12} + (23040 - 64\beta)\, y^{14} + \cdots, 0 \leq \beta \leq 277. \end{aligned}$$

With the most updated information, the existence of codes is known for β =14, 18, 22, 25, 29, 32, 35, 36, 39, 44, 46, 53, 59, 60, 64 and 74 in $W_{64,1}$ and for β =0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, . . . , 25, 28, 19, 30, 32, 33, 34, 36, 37, 38, 40, 41, 42, 44, 45, 48, 50, 51, 52, 56, 58, 64, 72, 80, 88, 96, 104, 108, 112, 114, 118, 120 and 184 in $W_{64,2}$.

Table 1: Extremal binary self-dual codes of length 64 from self-dual codes over $\mathbb{F}_4 + u\mathbb{F}_4$ of length 16 via C_3.

$\mathcal{C}_i$	(γ_1, γ_2)	v_1	(γ_3, γ_4)	v_2	$\lvert Aut(\mathcal{C}_i)\rvert$	$W_{64,2}$
1	$(1, 8)$	$(2, A, 9)$	$(6, 6)$	$(0, 9, F)$	$2^2 \cdot 3$	$\beta = 13$
2	$(0, A)$	$(2, 9, B)$	$(6, 5)$	$(8, B, 5)$	$2^3 \cdot 3$	$\beta = 13$
3	$(0, A)$	$(A, 2, 9)$	$(4, 7)$	$(9, 6, 1)$	$2^4 \cdot 3$	$\beta = 16$
4	$(0, A)$	$(A, 1, 6)$	$(6, 5)$	$(4, D, F)$	$2^2 \cdot 3$	$\beta = 19$
5	$(1, 8)$	$(B, 4, E)$	$(4, 4)$	$(0, 2, 6)$	$2^2 \cdot 3$	$\beta = 22$
6	$(9, 2)$	$(2, A, 1)$	$(6, 6)$	$(8, 3, D)$	$2^2 \cdot 3$	$\beta = 25$
7	$(1, 8)$	$(A, A, 1)$	$(4, 4)$	$(8, B, 7)$	$2^3 \cdot 3$	$\beta = 25$
8	$(1, 8)$	$(A, 6, D)$	$(4, 4)$	$(E, 5, F)$	$2^2 \cdot 3$	$\beta = 37$
9	$(2, 9)$	$(1, E, D)$	$(4, E)$	$(4, F, F)$	$2^3 \cdot 3$	$\beta = 37$
10	$(0, A)$	$(0, 9, 9)$	$(4, 7)$	$(0, 1, 5)$	$2^4 \cdot 3^2$	$\beta = 40$
11	$(0, A)$	$(0, 9, 9)$	$(4, 7)$	$(2, 9, F)$	$2^4 \cdot 3$	$\beta = 64$

4.2 Constructions coming from C_7

We apply the constructions coming from C_7 over the binary field and the ring $R_1 = \mathbb{F}_2 + u\mathbb{F}_2$, as a result of which we obtain extremal binary self-dual codes of lengths 32 and 64 respectively. We tabulate the ones of length 64:

Table 2: Extremal binary self-dual codes of length 64 from self-dual codes over $\mathbb{F}_2 + u\mathbb{F}_2$ of length 32 via C_7.

(γ_1, γ_2)	v_1	(γ_3, γ_4)	v_2	$\lvert Aut(\mathcal{C})\rvert$	$W_{64,2}$
(u, u)	$(u, 0, 0, u, 0, 1, 3)$	$(1, 1)$	$(u, 1, 1, 0, u, 3, 1)$	$2^2 \cdot 7$	$\beta = 16$
(u, u)	$(u, u, 0, 0, 0, 1, 3)$	$(1, 1)$	$(u, 1, 1, u, 0, 3, 1)$	$2^2 \cdot 7$	$\beta = 30$
(u, u)	$(u, 0, 1, 0, 1, 1, 1)$	$(u, 1)$	$(u, 1, 1, 3, 1, 1, 3)$	$2^2 \cdot 7$	$\beta = 37$
(u, u)	$(u, u, 1, u, 1, 1, 1)$	$(u, 1)$	$(u, 1, 1, 3, 3, 1, 1)$	$2^3 \cdot 3 \cdot 7$	$\beta = 37$
(u, u)	$(u, 0, u, 0, 0, 1, 3)$	$(1, 1)$	$(u, 1, 1, u, 0, 1, 3)$	$2^2 \cdot 7$	$\beta = 44$
(u, u)	$(u, u, 0, 0, u, 1, 1)$	$(u, 1)$	$(u, 1, 3, u, u, 1, 3)$	$2^3 \cdot 7$	$\beta = 44$
(u, u)	$(u, 0, 1, 0, 1, 3, 3)$	$(1, 1)$	$(u, 1, 1, 1, 3, 3, 1)$	$2^2 \cdot 7$	$\beta = 51$
(u, u)	$(u, u, u, u, u, 1, 1)$	$(u, 1)$	$(u, 1, 3, u, u, , 1)$	$2^4 \cdot 3 \cdot 7$	$\beta = 72$

4.3 Constructions coming from groups of order 9

We apply the constructions C_9, $C_{3,3}$ and $C_3 \times C_3$ over the binary field and the ring $R_1 = \mathbb{F}_2 + u\mathbb{F}_2$, as a result of which we obtain binary self-dual codes of lengths 40 and 80. For length 40 we get extremal self-dual codes, and for length 80 we get the best Type I codes, i.e., self-dual codes that have parameters $[80, 40, 14]$.

Table 3: Extremal binary self-dual codes of length 40 from C_9, $C_{3,3}$ and $C_3 \times C_3$

Const	(γ_1, γ_2)	v_1	(γ_3, γ_4)	v_2	$\lvert Aut(C)\rvert$	Type
C_9	$(0,0)$	$(0,0,0,0,0,0,0,1,1)$	$(0,1)$	$(0,0,1,1,1,0,1,1,1)$	$2^{11} \cdot 3^2$	$[40,20,8]_I$
C_9	$(0,0)$	$(0,0,0,0,1,0,1,1,1)$	$(0,1)$	$(0,0,1,0,1,0,0,1,1)$	$2^2 \cdot 3^2$	$[40,20,8]_I$
C_9	$(0,0)$	$(0,0,0,1,1,1,1,1,1)$	$(0,1)$	$(0,0,1,1,0,1,1,1,1)$	$2^2 \cdot 3^2$	$[40,20,8]_I$
C_9	$(1,0)$	$(0,0,0,0,0,0,1,1,1)$	$(1,1)$	$(0,0,0,0,1,0,0,1,1)$	$2^2 \cdot 3^2$	$[40,20,8]_{II}$
C_9	$(1,0)$	$(0,0,0,0,0,0,1,1,1)$	$(1,1)$	$(0,1,0,1,1,1,1,1,1)$	$2^{11} \cdot 3^2$	$[40,20,8]_{II}$
C_9	$(1,0)$	$(0,0,0,1,0,1,1,1,1)$	$(1,1)$	$(0,0,1,1,0,1,0,1,1)$	$2^3 \cdot 3^2 \cdot 5 \cdot 19$	$[40,20,8]_{II}$
$C_{3,3}$	$(0,0)$	$(0,0,0,0,0,1,0,0,1)$	$(0,1)$	$(0,1,1,0,1,1,1,0,1)$	$2^{11} \cdot 3^2$	$[40,20,8]_I$
$C_{3,3}$	$(0,0)$	$(0,0,0,0,1,1,0,1,1)$	$(0,1)$	$(0,0,1,0,1,1,1,0,0)$	$2^2 \cdot 3^2$	$[40,20,8]_I$
$C_{3,3}$	$(0,0)$	$(0,0,1,0,1,1,1,1,1)$	$(0,1)$	$(0,1,1,1,0,1,0,1,1)$	$2^2 \cdot 3^2$	$[40,20,8]_I$
$C_{3,3}$	$(1,0)$	$(0,0,0,0,0,1,0,1,1)$	$(1,1)$	$(0,0,1,0,1,0,0,0,1)$	$2^2 \cdot 3^2$	$[40,20,8]_{II}$
$C_{3,3}$	$(1,0)$	$(0,0,1,0,0,1,0,0,1)$	$(1,1)$	$(0,1,1,1,1,0,1,1,1)$	$2^{11} \cdot 3^2$	$[40,20,8]_{II}$
$C_{3,3}$	$(1,0)$	$(0,0,1,0,0,1,1,1,1)$	$(1,1)$	$(0,0,1,0,1,1,1,0,1)$	$2^3 \cdot 3^2 \cdot 5 \cdot 19$	$[40,20,8]_{II}$
$C_3 \times C_3$	$(0,0)$	$(0,0,0,0,1,1,0,1,1)$	$(0,1)$	$(0,0,1,0,0,1,1,1,0)$	$2^{15} \cdot 3^2 \cdot 5$	$[40,20,8]_I$
$C_3 \times C_3$	$(1,0)$	$(0,0,0,0,0,0,1,1,1)$	$(1,1)$	$(0,0,1,0,0,1,0,1,0)$	$2^4 \cdot 3^4$	$[40,20,8]_{II}$
$C_3 \times C_3$	$(1,0)$	$(0,0,1,0,0,1,1,1,1)$	$(1,1)$	$(0,0,1,1,1,0,1,1,0)$	$2^{15} \cdot 3^2 \cdot 5$	$[40,20,8]_{II}$

The possible weight enumerators for a self-dual Type I $[80, 40, 14]$-code are given in [26] as:

$$W_{80,2} = 1 + (3200 + 4\alpha)\, y^{14} + (47645 - 8\alpha + 256\beta)\, y^{16} + \cdots ,$$

where α and β are integers. An $[80, 40, 14]$ was constructed in [7] (its weight enumerator was not stated), and an $[80, 40, 14]$ code was constructed in [18] with $\alpha = -280$, $\beta = 10$. In [26], $[80, 40, 14]$ codes were constructed for $\beta = 0$ and $\alpha = -17k$ where $k \in \{2, \ldots , 25, 27\}$.

4.4 Constructions coming from C_{13}

The best known Type I binary codes of length 56 have minimum weight 10. The possible weight enumerators for such a $[56, 28, 10]$-code are given in [20] as:

$$\begin{aligned} W_{56,1} &= 1 + (308 + 4\alpha)\, y^{10} + (4246 - 8\alpha)\, y^{12} + (40852 - 28\alpha)\, y^{14} + \cdots , \\ W_{56,2} &= 1 + (308 + 4\alpha)\, y^{10} + (3990 - 8\alpha)\, y^{12} + (42900 - 28\alpha)\, y^{14} + \cdots \end{aligned}$$

where α is an integer. In [20], codes were constructed for the values of $\alpha = -18, -22, -24$ in $W_{56,1}$ and $\alpha = 0, -2, -4, -6, -8, -10, -12, -14, -16, -18, -20, -22$ and -24 in $W_{56,2}$.

Table 4: Binary $[80, 40, 14]$-codes from self-dual codes over $\mathbb{F}_2 + u\mathbb{F}_2$ via C_9, $C_{3,3}$, $C_3 \times C_3$.

Const	(γ_1, γ_2)	v_1	(γ_3, γ_4)	v_2	$\lvert Aut(\mathcal{C})\rvert$	$W_{80,2}$
C_9	(u, u)	$(u, u, 0, u, u, 0, u, 1, 1)$	$(0, 1)$	$(0, 0, 1, 1, 1, 0, 3, 1, 3)$	$2^2 \cdot 3^2$	$\alpha = -330,\ \beta = 10$
C_9	$(1, u)$	$(u, 0, 0, 3, u, 1, 3, 3, 3)$	$(1, 1)$	$(u, 0, 1, 3, 0, 3, u, 1, 1)$	$2^2 \cdot 3^2$	$\alpha = -258,\ \beta = 1$
C_9	$(u, 0)$	$(0, 0, 0, u, 1, u, 3, 3, 3)$	$(u, 1)$	$(0, u, 1, 0, 1, u, 0, 1, 3)$	$2^2 \cdot 3^2$	$\alpha = -240,\ \beta = 1$
C_9	(u, u)	$(u, 0, 0, 0, 1, 0, 1, 3, 3)$	$(0, 1)$	$(u, u, 1, u, 3, 0, u, 3, 1)$	$2^2 \cdot 3^2$	$\alpha = -204,\ \beta = 1$
C_9	(u, u)	$(0, 0, 0, u, 1, 0, 3, 3, 1)$	$(0, 1)$	$(0, u, 1, u, 3, 0, 0, 3, 1)$	$2^2 \cdot 3^2$	$\alpha = -186,\ \beta = 1$
C_9	$(u, 0)$	$(u, u, 0, u, 1, 0, 1, 1, 1)$	$(u, 1)$	$(u, u, 1, u, 1, u, u, 3, 3)$	$2^3 \cdot 3^2$	$\alpha = -168,\ \beta = 1$
C_9	$(u, 0)$	$(0, 0, 0, u, 1, 0, 3, 3, 1)$	$(u, 1)$	$(u, u, 1, 0, 3, 0, u, 3, 1)$	$2^2 \cdot 3^2$	$\alpha = -150,\ \beta = 1$
C_9	(u, u)	$(u, u, 0, u, 1, 0, 1, 1, 1)$	$(0, 1)$	$(0, 0, 1, 0, 1, 0, 0, 3, 3)$	$2^3 \cdot 3^2$	$\alpha = -96,\ \beta = 1$
$C_{3,3}$	$(u, 0)$	$(u, u, u, u, u, 1, 0, 0, 1)$	$(u, 1)$	$(u, 1, 1, u, 3, 1, 3, u, 1)$	$2^2 \cdot 3^2$	$\alpha = -366,\ \beta = 10$
$C_{3,3}$	$(u, 0)$	$(u, u, u, u, u, 1, u, 0, 3)$	$(u, 1)$	$(u, 1, 3, 0, 1, 3, 1, u, 3)$	$2^2 \cdot 3^2$	$\alpha = -348,\ \beta = 10$
$C_{3,3}$	$(1, u)$	$(0, u, u, 0, u, 3, u, 3, 1)$	$(1, 1)$	$(u, u, 1, u, 3, u, u, u, 3)$	$2^3 \cdot 3^2$	$\alpha = -312,\ \beta = 1$
$C_{3,3}$	$(0, u)$	$(0, 0, 0, u, u, 1, 0, 0, 1)$	$(u, 1)$	$(u, 1, 1, u, 3, 1, 3, u, 1)$	$2^2 \cdot 3^2$	$\alpha = -294,\ \beta = 10$
$C_{3,3}$	$(1, u)$	$(u, 0, u, 0, u, 3, u, 1, 3)$	$(1, 1)$	$(0, 0, 3, u, 1, u, u, u, 3)$	$2^2 \cdot 3^2$	$\alpha = -222,\ \beta = 1$
$C_{3,3}$	$(1, u)$	$(0, 0, u, 0, u, 3, 0, 3, 1)$	$(1, 1)$	$(0, 0, 3, u, 1, u, u, u, 3)$	$2^2 \cdot 3^2$	$\alpha = -168,\ \beta = 1$
$C_{3,3}$	$(0, u)$	$(0, 0, u, u, 1, 1, 0, 3, 3)$	$(u, 1)$	$(0, 0, 1, 0, 3, 1, 3, u, 0)$	$2^2 \cdot 3^2$	$\alpha = -186,\ \beta = 1$
$C_3 \times C_3$	(u, u)	$(0, u, 0, 0, 1, 1, 0, 3, 3)$	$(1, 1)$	$(u, u, 1, u, 0, 3, 3, 1, u)$	$2^2 \cdot 3^2$	$\alpha = -276,\ \beta = 10$
$C_3 \times C_3$	$(1, u)$	$(u, u, 3, 0, 0, 3, 1, 3, 3)$	$(1, 1)$	$(u, 0, 3, 3, 3, u, 1, 3, 0)$	$2^3 \cdot 3^2$	$\alpha = -276,\ \beta = 10$
$C_3 \times C_3$	$(1, u)$	$(u, u, u, u, 0, 0, 1, 1, 1)$	$(1, 1)$	$(u, 0, 1, u, 0, 1, 0, 1, 0)$	$2^3 \cdot 3^2$	$\alpha = -240,\ \beta = 1$
$C_3 \times C_3$	$(1, u)$	$(u, u, 3, 0, 0, 3, 1, 3, 3)$	$(1, 1)$	$(u, 0, 3, 3, 3, 0, 3, 1, u)$	$2^3 \cdot 3^2$	$\alpha = -204,\ \beta = 10$

In Table 5 we give a list of Type I self-dual codes with parameters $[56, 28, 10]$ by applying the construction C_{13} over the binary field. The codes listed below all have new weight enumerators.

Table 5: $[56, 28, 10]$ codes over $\mathbb{F}_2$ from C_{13}.

(γ_1, γ_2)	v_1	(γ_3, γ_4)	v_2	$\lvert Aut(C)\rvert$	α	$W_{56,i}$
$(0, 0)$	$(0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 1)$	$(0, 1)$	$(0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 1, 1, 1)$	$2 \cdot 13$	-51	1
$(0, 0)$	$(0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1)$	$(0, 1)$	$(0, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 1, 1)$	$2 \cdot 13$	-38	1
$(0, 0)$	$(0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1)$	$(0, 1)$	$(0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 1)$	$2 \cdot 13$	-25	1
$(0, 0)$	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1)$	$(0, 1)$	$(0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 1, 1)$	$2^2 \cdot 13$	-38	1
$(0, 0)$	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1)$	$(0, 1)$	$(0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 1, 0, 1)$	$2^2 \cdot 13$	-12	1
$(0, 0)$	$(0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 1)$	$(0, 1)$	$(0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1)$	$2^2 \cdot 3 \cdot 13$	-38	1
$(0, 0)$	$(0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1)$	$(0, 1)$	$(0, 1, 0, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1)$	$2^2 \cdot 3 \cdot 13$	-64	1

4.5 Constructions coming from C_{15}

We apply the constructions C_{15} and $C_3 \times C_5$ over the binary field to obtain a number of extremal binary self-dual codes of length 64. We only tabulate the codes obtained from construction C_{15}; however, we note that we have obtained the exact same codes from $C_3 \times C_5$ as well.

Table 6: Binary self-dual codes of length 64 from C_{15}.

(γ_1,γ_2)	v_1	(γ_3,γ_4)	v_2	$\|Aut(C)\|$	Type
$(0,0)$	$(0,0,0,0,0,0,0,0,0,1,0,1,0,1,1)$	$(0,1)$	$(0,0,0,0,1,0,1,0,1,0,0,0,1,1,1)$	$2\cdot 3\cdot 5$	$[64,32,12]_{II}$
$(0,0)$	$(0,0,0,0,0,0,0,0,0,0,0,1,0,0,1)$	$(0,1)$	$(0,0,0,0,1,1,0,1,1,0,0,1,1,1,1)$	$2^2\cdot 3\cdot 5$	$[64,32,12]_{II}$
$(0,0)$	$(0,0,0,0,1,1,0,1,1,0,0,1,1,1,1)$	$(0,1)$	$(0,0,0,1,0,0,0,1,0,0,0,1,1,1,1)$	$2^3\cdot 3\cdot 5$	$[64,32,12]_{II}$
$(0,0)$	$(0,0,0,0,1,0,0,1,1,0,1,0,0,1,1)$	$(0,1)$	$(0,0,0,1,1,0,1,0,1,1,0,1,0,1,1)$	$2^{12}\cdot 3\cdot 5$	$[64,32,12]_{II}$
$(1,0)$	$(0,0,0,0,0,0,0,0,0,1,0,1,1,1,1)$	$(1,1)$	$(0,0,0,1,0,0,1,0,1,0,1,1,1,0,1)$	$2\cdot 3\cdot 5$	$W_{64,1}$ $(\beta=14)$
$(1,0)$	$(0,0,0,0,0,0,0,1,0,1,1,0,0,1,1)$	$(1,1)$	$(0,0,0,0,1,0,0,1,1,0,1,1,0,1,1)$	$2^2\cdot 3\cdot 5$	$W_{64,1}$ $(\beta=14)$
$(1,0)$	$(0,0,0,0,0,0,0,0,0,0,1,0,0,1,1)$	$(1,1)$	$(0,0,0,0,1,1,1,0,1,1,1,0,1,1,1)$	$2^3\cdot 3\cdot 5$	$W_{64,1}$ $(\beta=14)$
$(1,0)$	$(0,0,0,0,0,0,0,0,0,1,0,1,1,1,1)$	$(1,1)$	$(0,0,0,0,0,1,0,1,0,0,1,1,1,1,1)$	$2\cdot 3\cdot 5$	$W_{64,1}$ $(\beta=29)$
$(1,0)$	$(0,0,0,0,0,0,0,0,1,0,1,0,1,1,1)$	$(1,1)$	$(0,0,0,1,0,1,0,0,0,1,0,1,1,1,1)$	$2\cdot 3\cdot 5$	$W_{64,1}$ $(\beta=44)$
$(1,0)$	$(0,0,0,0,0,0,0,0,0,0,0,0,1,1,1)$	$(1,1)$	$(0,0,0,1,0,1,1,1,1,1,0,1,0,1,1)$	$2^2\cdot 3\cdot 5$	$W_{64,1}$ $(\beta=44)$
$(1,0)$	$(0,0,0,0,0,1,0,0,0,1,0,1,0,1,1)$	$(1,1)$	$(0,0,1,1,1,1,0,1,1,0,1,1,1,1,1)$	$2\cdot 3\cdot 5$	$W_{64,1}$ $(\beta=59)$
$(1,0)$	$(0,0,0,0,0,0,0,0,0,0,1,0,0,1,1)$	$(1,1)$	$(0,0,0,1,0,0,1,1,1,1,1,1,0,1,1)$	$2^2\cdot 3\cdot 5$	$W_{64,1}$ $(\beta=74)$

5 New codes of length 68

In this section, we construct new extremal self-dual codes of length 68 by extending certain previously constructed codes of length 64 (using Theorem 2.4) from Table 1. In particular, we use $\mathcal{C}_{11}$ that was listed in Table 1.

5.1 New codes of length 68 from $(\mathbb{F}_4+u\mathbb{F}_4)C_4$

The possible weight enumerator of a self-dual $[68,34,12]_2$-code is in one of the following forms by [4, 19]:

$$W_{68,1} = 1+(442+4\beta)\,y^{12}+(10864-8\beta)\,y^{14}+\cdots,\ 104\le\beta\le 1358,$$
$$W_{68,2} = 1+(442+4\beta)\,y^{12}+(14960-8\beta-256\gamma)\,y^{14}+\cdots$$

where $0\le\gamma\le 9$. Recently, Yankov et al. constructed the first examples of codes with a weight enumerator for $\gamma=7$ in $W_{68,2}$. In [9] and [16], more unknown $W_{68,2}$ codes were constructed. Together with these, the existence of the codes in $W_{68,2}$ is known for:

$$\begin{aligned}
\gamma &= 0,\ \beta\in\{2m\,|\,m=0,7,11,14,17,21,\ldots,99,102,105,110,119,136,165\}\ \text{or}\\
\beta &\in \{2m{+}1\,|\,m=3,5,8,10,15,16,17,18,20,\ldots,82,87,93,94,101,104,110,115\};\\
\gamma &= 1,\ \beta\in\{2m\,|\,m=22,24,\ldots,99,102\}\ \text{or}\ \beta\in\{2m+1\,|\,m=24,\ldots,85,87\};\\
\gamma &= 2,\ \beta\in\{2m\,|\,m=29,\ldots,100,103,104\}\ \text{or}\ \beta\in\{2m+1\,|\,m=32,34,\ldots,79\};\\
\gamma &= 3,\ \beta\in\{2m\,|\,m=40,\ldots,98,101,102\}\ \text{or}\\
\beta &\in \{2m+1\,|\,m=41,43,\ldots,77,79,80,83,96\};\\
\gamma &= 4,\ \beta\in\{2m\,|\,m=43,44,48,\ldots,92,97,98\}\ \text{or}\\
\beta &\in \{2m+1\,|\,m=48,\ldots,55,58,60,\ldots,78,80,83,84,85\};\\
\gamma &= 5\ \text{with}\ \beta\in\{m\,|\,m=113,116,\ldots,181\};\\
\gamma &= 6\ \text{with}\ \beta\in\{2m\,|\,m=69,77,78,79,81,88\};\\
\gamma &= 7\ \text{with}\ \beta\in\{7m\,|\,m=14,\ldots,39,42\}.
\end{aligned}$$

Recall that the previously constructed codes of length 64 (from Table 1) are codes over $\mathbb{F}_4 + u\mathbb{F}_4$. In order to apply Theorem 2.4, it requires the codes to be over $\mathbb{F}_2 + u\mathbb{F}_2$. Before considering extensions of these codes, we need to use the Gray map $\psi_{\mathbb{F}_4+u\mathbb{F}_4}$ to convert them to a code over $\mathbb{F}_2 + u\mathbb{F}_2$. The following table details the new extremal self-dual codes of length 68. For each new code constructed we note the original code of length 64 from Table 1, the unit $c \in \mathbb{F}_2 + u\mathbb{F}_2$, the vector X required to apply Theorem 2.4. The codes listed all have new weight enumerators.

Table 7: Type I Extremal Self-dual code of length 68 from C_3 over $\mathbb{F}_4 + u\mathbb{F}_4$.

$\mathcal{C}_{68,i}$	$\mathcal{C}_i$	c	X	γ	β
$\mathcal{C}_{68,1}$	11	1	$(1, u, u, 3, 3, 0, 1, 3, u, 3, 0, 1, 0, 0, 0, 1, u, 0, 3, 3, 0, 1, 1, u, u, u, 3, 3, 0, u, u, 3)$	**4**	**190**
$\mathcal{C}_{68,2}$	11	1	$(0, 1, 0, 1, 3, 1, 0, 0, u, u, 1, u, u, 0, 1, 1, 1, 0, u, 1, u, 1, 1, 0, 1, 0, 3, 3, u, 1, u, u)$	**4**	**192**
$\mathcal{C}_{68,3}$	11	$u+1$	$(1, u, u, 3, 3, 0, 1, 3, u, 3, 0, 3, u, 0, u, 3, u, 0, 3, 1, 0, 1, 3, 0, 0, u, 1, 3, 0, u, u, 1)$	**4**	**204**
$\mathcal{C}_{68,4}$	11	$u+1$	$(u, 1, 0, 3, 0, 0, 0, u, 1, u, u, 0, 0, 0, 3, 3, 1, 3, u, 0, 0, u, 3, 1, 0, 0, u, 0, 0, 0, 1, 3)$	**4**	**208**
$\mathcal{C}_{68,5}$	11	1	$(0, 3, u, 3, 0, 0, u, 0, 1, u, u, 0, 0, u, 3, 3, 1, 3, 0, 0, u, u, 1, 3, u, u, 0, u, u, 0, 3, 1)$	**4**	**210**
$\mathcal{C}_{68,6}$	11	$u+1$	$(u, 1, u, 1, 0, 0, u, 0, 3, 0, u, 0, 0, u, 1, 1, 3, 3, 0, 0, 0, u, 3, 3, 0, 0, 0, 0, u, 0, 1, 1)$	**4**	**214**

5.2 New self-dual codes of length 68 from neighboring construction

Two self-dual binary codes of dimension k are said to be neighbors if their intersection has dimension $k-1$. Without loss of generality, we consider the standard form of the generator matrix of C. Let $x \in \mathbb{F}_2^n - C$ then $D = \left\langle \langle x \rangle^{\perp} \cap C, x \right\rangle$ is a neighbor of C. The first 34 entries of x are set to be 0, the rest of the vectors are listed in Table 8. As neighbors of codes in Table 7 we obtain 17 new codes with weight enumerators in $W_{68,2}$, which are listed in Table 8. All the codes have an automorphism group of order 2.

Table 8: New codes of length 68 as neighbors of codes in Table 7

$\mathcal{N}_{68,i}$	$\mathcal{C}_{68,i}$	$(x_{35}, x_{36}, ..., x_{68})$	γ	β	$\mathcal{N}_{68,i}$	$\mathcal{C}_{68,i}$	$(x_{35}, x_{36}, ..., x_{68})$	γ	β
1	6	(1111101001010000101110100001111010)	3	165	2	6	(0011101000011001110010111010000011)	3	169
3	6	(0110001110010110101000100011111101)	3	171	4	6	(0100010010100011000110000110001010)	3	173
5	6	(0110010001110000000011011110010100)	4	163	6	6	(1110111111010101100001011001111011)	4	165
7	6	(1000101111011011101011010101110100)	4	173	8	6	(0100101010011010111001000111111100)	4	177
9	6	(1101110100111100110010000111001100)	4	179	10	6	(1001010100010110110000010011000000)	4	181
11	2	(1000101100010110000101111000010010)	4	183	12	6	(0010111011011111100101111101000100)	4	185
13	6	(1011011110010100011001011011001111)	4	187	14	6	(0010010001110100011000001010000110)	4	188
15	6	(1001100011010110110101011110010001)	4	189	16	6	(0111110011011110010101111010001100)	4	193
17	5	(0101101101011000110011101010001000)	5	201					

6 Conclusion

In this work, we have introduced a new construction for self-dual codes using group rings. We have provided certain conditions when this construction produces self-dual codes and have established a link between units/non-units and self-dual codes. We have demonstrated the relevance of this new construction by constructing many binary self-dual codes, including new self-dual codes of lengths 56, 68 and 80. The following is a summary of our results:

- **Codes of length 56:** $[56, 28, 10]$ codes with new weight enumerators in $W_{56,1}$ with $\alpha \in \{-12, -25, -38, -51, -64\}$.
- **Codes of length 68:** Extremal binary self-dual codes with new weight enumerators in $W_{68,2}$:

$$\begin{aligned}
&(\gamma = 3, \quad \beta \in \{165, 169, 171, 173\}),\\
&(\gamma = 4, \quad \beta \in \{163, 165, 173, 177, 179, 181, 183, 185, 187, 188, 189, 190, 192, 193,\\
&\qquad\qquad 204, 208, 210, 214\}),\\
&(\gamma = 5, \quad \beta = 201).
\end{aligned}$$

  The binary generator matrices of these codes are available online at [17].

- **Codes of length 80:** $[80, 40, 14]$-codes with new weight enumerators in $W_{80,2}$:

$$\begin{aligned}
&(\beta = 1, \quad \alpha \in \{-96, -150, -168, -186, -204, -222, -240, -258, -312\}),\\
&(\beta = 10, \quad \alpha \in \{-204, -276, -294, -330, -348, -366\}).
\end{aligned}$$

We should point out that we have considered specific lengths that are suitable for the constructions over some special alphabets. A possible direction for future research could be considering other families of rings or groups of higher orders, which might require a considerable computational power.

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