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# A question of Björner from 1981: Infinite geometric lattices of finite rank have matchings

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**Abstract**

It is proven that every geometric lattice of finite rank greater than 1 has a matching between the points and hyperplanes. This answers a question of Anders Björner from the 1981 Banff Conference on Ordered Sets.

## 1 Introduction

At the famous 1981 Banff Conference on Ordered Sets—such luminaries as Erdős, Birkhoff, Dilworth, Scott (the Turing Award-winner), Daykin, Garsia, Graham, Greene, Jónsson, Milner, Stanley, and Priestley attended—Björner asked if every geometric lattice $L$ of finite rank at least 2 had a matching between the points or atoms and the hyperplanes or co-atoms [13, pp. xi, xii, and 799].

Greene had proven this for finite lattices [7, Corollary 3]. Björner had proven this in special cases [4, Theorems 3 and 4]—for modular lattices and for "equicardinal lattices," i.e., lattices whose hyperplanes contained the same number of atoms.

In 1976, Björner wrote, "It would be interesting to know if the result of our theorems 3 and 4 can be extended to all infinite geometric lattices, or at least to some classes of such lattices other than the modular and the equicardinal." [4, p. 10]. In 1977, he proved it for lattices of rank 3 and for lattices of cardinality less than $\aleph_\omega$, the smallest singular cardinal [5, Theorems 3 and 6]; his argument essentially only worked for regular cardinals.

We answer Björner's 1976 question about matchings. The main new contribution of the present work is Proposition 10, the rest of the argument being derived from Björner's prior writings.

We selectively use some of the notation and terminology from [6] and [3, Chapter II, §8 and Chapter IV].

Let $P$ be a poset. Let $x, y \in P$ be such that $x \leq y$. The *closed interval* $[x, y]$ is $\{z \in P : x \leq z \leq y\}$. If $|[x, y]| = 2$, we say $x$ is a *lower cover* of $y$ and $y$ is an *upper*

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*cover* of $x$ and denote it $x \lessdot y$.

Let $P$ be a poset with least element 0. An *atom* or *point* is a cover of 0. The set of atoms is $\mathcal{A}(P)$. If $P$ is a poset with greatest element 1, a *co-atom*, *co-point*, or *hyperplane* is a lower cover of 1. The set of hyperplanes is $\mathcal{H}(P)$.

A poset is *semimodular* if, for all $a, b, c \in P$, $a \lessdot b, c$ and $b \neq c$ imply there exists $d \in P$ such that $b, c \lessdot d$. A *geometric lattice of finite height* is a semimodular lattice $L$ with no infinite *chains* (totally ordered subsets)—implying $L$ has a 0 and a 1—such that every element is a join of a subset of atoms. It is known [12, Theorem 9.4], [3, Chapter II, §8, Theorem 14] that such an $L$ is a complete lattice with a finite maximal chain and all maximal chains have the same size $r+1$, where $r$ is the *height* or *rank* of $L$. Moreover, every element is a join of a finite set of atoms and a meet of a finite subset of $\mathcal{H}(L)$ (see [4, Lemma 1]). Every interval is a geometric lattice [15, §3.3, Lemma]. For $x \in L$, the rank of $\downarrow x := [0, x]$ is the *rank* $r(x)$ of $x$. For $x, y \in L$, $r(x \vee y) + r(x \wedge y) \leq r(x) + r(y)$ [12, Theorem 9.5], [3, Chapter II, §8, Theorem 15]. For $x \in L$, let $\underline{x} := \mathcal{A}(L) \cap \downarrow x$ and let $\overline{x} := \mathcal{H}(L) \cap [x, 1]$.

Let $L$ be a geometric lattice of finite height. Let $a, b \in L$ be such that $a \leq b$. An element $x \in [a, b]$ has a *modular complement* $y$ in $[a, b]$ if $x \wedge y = a$, $x \vee y = b$, and $r(x) + r(y) = r(a) + r(b)$.

The following is a basic fact (see [4, p. 3]).

**Lemma 1** *Let $L$ be a geometric lattice of finite height. Let $a, b \in L$ be such that $a \leq b$. Then any $x \in [a, b]$ has a modular complement in $[a, b]$.*

*Proof.* If $x = c_0 \lessdot c_1 \lessdot \cdots \lessdot c_k = b$, find $a_i \in \mathcal{A}(L) \cap \downarrow c_i \setminus \downarrow c_{i-1}$ for $i = 1, \ldots, k$; we know $a_i$ exists for $i = 1, \ldots, k$, since $c_i$ is a join of atoms but if every atom less than or equal to $c_i$ were also less than or equal to $c_{i-1}$, then $c_i$ would be less than or equal to $c_{i-1}$, a contradiction. Let $y = a \vee a_1 \vee \cdots \vee a_k$. Clearly $r(y) - r(a) = k = r(b) - r(x)$, $x \vee y = b$, and $x \wedge y \geq a$. As $r(a) \leq r(x \wedge y) \leq r(x) + r(y) - r(x \vee y) = r(a) + r(b) - r(b) = r(a)$, we have $x \wedge y = a$. □

See [8, Chapters 2, 3, 5 and 8] and [9, Appendix 2, §3] for basic facts about ordinals and cardinals. If $\kappa$ is a regular cardinal, a subset $\Omega \subseteq \kappa$ is *closed in* $\kappa$ if for every non-empty subset $A \subseteq \Omega$, the supremum of $A$ is $\kappa$ or in $\Omega$; it is *unbounded in* $\kappa$ if the supremum of $\Omega$ is $\kappa$; it is a *club in* $\kappa$ if it is both. A subset $\Omega \subseteq \kappa$ is *stationary in* $\kappa$ if it intersects every club in $\kappa$; note that $|\Omega| = \kappa$.

We take our notation from [2, §§2, 4, and 6]. A *society* is a triple $\Lambda = (M_\Lambda, W_\Lambda, K_\Lambda)$ where $M_\Lambda \cap W_\Lambda = \emptyset$ and $K_\Lambda \subseteq M_\Lambda \times W_\Lambda$. If $A \subseteq M_\Lambda$ and $X \subseteq W_\Lambda$, then $K_\Lambda[A] := \{w \in W_\Lambda : (a, w) \in K_\Lambda \text{ for some } a \in A\}$, and $\Lambda[A, X] := \big(A, X, K_\Lambda \cap (A \times X)\big)$ is a *subsociety* of $\Lambda$. If $B \subseteq M_\Lambda$, then $\Lambda - B := \Lambda[M_\Lambda \setminus B, W_\Lambda]$. If $\Pi$ is a subsociety, then $\Lambda/\Pi := \Lambda[M_\Lambda \setminus M_\Pi, W_\Lambda \setminus W_\Pi]$. We call a subsociety $\Pi$ of $\Lambda$ *saturated* if $K_\Lambda[M_\Pi] \subseteq W_\Pi$ and we denote this situation by $\Pi \lhd \Lambda$.

An *espousal* for $\Lambda$ is an injective function $E : M_\Lambda \to W_\Lambda$ such that $E \subseteq K_\Lambda$. A society is *critical* if it has an espousal and every espousal is surjective.

If $I$ is a set and $\bar{\Pi} = (\Pi_i : i \in I)$ is a family of subsocieties of $\Lambda$, then $\bigcup \bar{\Pi} :=$

$(\bigcup_{i\in I} M_{\Pi_i}, \bigcup_{i\in I} W_{\Pi_i}, \bigcup_{i\in I} K_{\Pi_i})$. Assume $I$ is an ordinal. If $\theta \leq I$, then $\bar{\Pi}_\theta$ denotes $(\Pi_i : i < \theta)$. The sequence $\bar{\Pi}$ is *non-descending* if $\Pi_i$ is a subsociety of $\Pi_j$ whenever $i < j < I$; it is *continuous* if, in addition, $\bigcup \bar{\Pi}_\theta = \Pi_\theta$ for every limit ordinal $\theta < I$. If $I = J + 1$, $\bar{\Pi}$ is a *$J$-tower in* $\Lambda$ if $\bar{\Pi}$ is a continuous family of saturated subsocieties of $\Lambda$ such that $\Pi_0 = (\emptyset, \emptyset, \emptyset)$.

Let $\Pi$ be a subsociety of $\Lambda$. Assume $1 \leq \kappa \leq \aleph_0$. Then $\Pi$ is a *$\kappa$-obstruction in* $\Lambda$ if $\Pi \lhd \Lambda$ and $\Pi - A$ is critical for some $A \subseteq M_\Pi$ such that $|A| = \kappa$.

Now assume $\kappa$ is a regular, uncountable cardinal. We say $\Pi$ is a *$\kappa$-obstruction in* $\Lambda$ if $\Pi = \bigcup \bar{\Sigma}$ for an obstructive $\kappa$-tower $\bar{\Sigma}$ in $\Lambda$, where a $\kappa$-tower $\bar{\Sigma}$ in $\Lambda$ is *obstructive* when, for each $\alpha < \kappa$, $\Sigma_{\alpha+1}/\Sigma_\alpha$ is either (a) a $\mu$-obstruction in $\Lambda/\Sigma_\alpha$ for some $\mu < \kappa$ or (b) $(\emptyset, \{w\}, \emptyset)$ for some $w \in W_\Lambda$, and $\{\alpha < \kappa : \text{(a) holds at } \alpha\}$ is stationary in $\kappa$. By [2, Lemmas 4.2 and 4.3], $\Pi \lhd \Lambda$.

For a society $\Lambda$, $\delta(\Lambda)$ is the minimum of $\{|B| : B \subseteq M_\Lambda \text{ such that } \Lambda - B \text{ has an espousal}\}$.

We will use the following theorems of Aharoni, Nash-Williams, and Shelah:

**Theorem 2** (from [2, Lemma 4.2 and Corollary 4.9a]) *If* $\Pi$ *is a* $\kappa$*-obstruction, then* $\delta(\Pi) = \kappa$. □

**Theorem 3** [2, Theorem 5.1] *A society* $\Lambda$ *has an espousal if and only if it has no obstruction.* □

We will say that a geometric lattice of finite rank $r \geq 3$ has a *matching* if the society $\Big(\mathcal{A}(L), \mathcal{H}(L), \leq \cap \big(\mathcal{A}(L) \times \mathcal{H}(L)\big)\Big)$ has an espousal. (Since $\mathcal{A}(L) = \mathcal{H}(L)$ in geometric lattices of rank 2, we could say they also have a *matching*.)

This generalizes the notion of "matching" for a finite lattice of rank at least 3, which is this: a one-to-one map from the set of atoms of the lattice to the set of co-atoms of the lattice, sending each atom $p$ to a co-atom $h$ lying above the atom, i.e., $p \leq h$, or, showing the connection with the previous paragraph, $(p, h) \in \leq$. (A matching in this paper is a matching in the sense of graph theory, for the graph consisting of the atoms and co-atoms, such that all atoms are matched.)

Greene proved:

**Theorem 4** [7, Corollary 3] *Every finite geometric lattice of rank at least* 2 *has a matching.* □

Björner proved:

**Theorem 5** [5, Theorems 3 and 6] *Every geometric lattice of rank* 3*, or of finite height and infinite cardinality less than* $\aleph_\omega$*, has a matching.* □

We use the following results of Björner:

**Lemma 6** ([5, Lemma 1] and [4, Theorem 1]) *Let $L$ be a geometric lattice of finite height. (a) Let $p \in \mathcal{A}(L)$, $h \in \mathcal{H}(L)$ and assume $p \nleq h$. Then $|\underline{h}| \leq |\overline{p}|$. (b) If $L$ is infinite, then $|\mathcal{A}(L)| = |\mathcal{H}(L)| = |L|$.* □

**Theorem 7** [5, Theorem 4] *Let $L$ be an infinite geometric lattice of finite height such that $|\downarrow x| < |L|$ for every $x \in L$ of rank 2. If $|L|$ is a regular cardinal, then $L$ has a matching.* □

Björner also uses this theorem of Milner and Shelah:

**Theorem 8** [14, Theorem] *Let $\Gamma = (M, W, K)$ be a society such that $K[m] \neq \emptyset$ for all $m \in M$ and such that $(m, w) \in K$ implies $|K^{-1}[w]| \leq |K[m]|$. Then $\Gamma$ has an espousal.* □

## 2 Answering Björner's Question

We are ready to begin answering Björner's question.

As a referee noted, the following exemplifies a standard technique for geometric lattices [3, Chapter IV, §3, Lemma 3].

**Lemma 9** *Let $L$ be a geometric lattice of finite height. Let $B \subseteq \mathcal{A}(L)$. Let $\mathcal{L}(B)$ be the subposet $\big\{ \bigvee_L \{b_1, \dots, b_n\} : n \in \mathbb{N}_0, b_1, \dots, b_n \in B \big\}$.*
*Then $\mathcal{L}(B)$ is a geometric lattice of finite height with rank $r_L(\bigvee_L B)$, and $\mathcal{A}\big(\mathcal{L}(B)\big) = B$. The inclusion map is order- and cover-preserving. Also, $0_{\mathcal{L}(B)} = 0_L$ and $|\mathcal{L}(B)|$ is either finite or $|B|$. If $1_{\mathcal{L}(B)} = 1_L$, then $\mathcal{H}\big(\mathcal{L}(B)\big) \subseteq \mathcal{H}(L)$.*

*Proof.* Note that $0_L$ is the join of the empty set. Since $\mathcal{L}(B)$ is closed under arbitrary joins, it is a complete lattice (e.g., [6, Theorems 2.31 and 2.41]). Letting $n$ equal 0 or 1, we get $\{0_L\} \cup B \subseteq \mathcal{L}(B)$ and so $B \subseteq \mathcal{A}\big(\mathcal{L}(B)\big)$. But for $n \geq 2$, $b_1 \vee b_2 \vee \cdots \vee b_n \geq b_1$, so $\mathcal{A}\big(\mathcal{L}(B)\big) \subseteq B$. Clearly every element of $\mathcal{L}(B)$ is a join of atoms. Let $m, n \in \mathbb{N}_0$ and let $b_1, \dots, b_n, c_1, \dots, c_m \in B$. Assume $b_1 \vee \cdots \vee b_n \lessdot_{\mathcal{L}(B)} c_1 \vee \cdots \vee c_m$. Then $m \geq 1$. Pick $r \in \{1, \dots, m\}$ such that $b_1 \vee \cdots \vee b_n < b_1 \vee \cdots \vee b_n \vee c_r \in \mathcal{L}(B)$. Then $b_1 \vee \cdots \vee b_n < b_1 \vee \cdots \vee b_n \vee c_r \leq b_1 \vee \cdots \vee b_n \vee c_1 \vee \cdots \vee c_r \vee \cdots \vee c_m = c_1 \vee \cdots \vee c_m$. As $c_1 \vee \cdots \vee c_m$ covers $b_1 \vee \cdots \vee b_n$ in $\mathcal{L}(B)$, we conclude $b_1 \vee \cdots \vee b_n \vee c_r = c_1 \vee \cdots \vee c_m$.

By semimodularity in $L$, $b_1 \vee \cdots \vee b_n \lessdot_L b_1 \vee \cdots \vee b_n \vee c_r = c_1 \vee \cdots \vee c_m$. Now let $k \in \mathbb{N}_0$ and let $d_1, \dots, d_k \in B$. Assume that $b_1 \vee \cdots \vee b_n \lessdot_{\mathcal{L}(B)} d_1 \vee \cdots \vee d_k$ and $c_1 \vee \cdots \vee c_m \neq d_1 \vee \cdots \vee d_k$. As before, for some $s \in \{1, \dots, k\}$, $b_1 \vee \cdots \vee b_n \lessdot_L b_1 \vee \cdots \vee b_n \vee d_s = d_1 \vee \cdots \vee d_k$. Thus $c_r \nleq d_1 \vee \cdots \vee d_k$ and $d_s \nleq c_1 \vee \cdots \vee c_m$. By semimodularity, $c_1 \vee \cdots \vee c_m \lessdot_L c_1 \vee \cdots \vee c_m \vee d_s = b_1 \vee \cdots \vee b_n \vee c_r \vee d_s = d_1 \vee \cdots \vee d_k \vee c_r$ and $d_1 \vee \cdots \vee d_k \lessdot_L d_1 \vee \cdots \vee d_k \vee c_r$; hence $c_1 \vee \cdots \vee c_m$, $d_1 \vee \cdots \vee d_k \lessdot_{\mathcal{L}(B)} b_1 \vee \cdots \vee b_n \vee c_r \vee d_s$. This shows that $\mathcal{L}(B)$ is a geometric lattice, of finite height since $L$ has no infinite chains, with $1_{\mathcal{L}(B)} = \bigvee_L B$. As $\bigvee_L B = \bigvee_L \{b_1, \dots, b_n\}$ for some $n \in \mathbb{N}_0$ and some $b_1, \dots, b_n \in B$, picking the smallest such $n$ and using semimodularity in $L$ and $\mathcal{L}(B)$, we see that $r_L(\bigvee_L B) = r_{\mathcal{L}(B)}(\bigvee_L B)$, namely $n$. If $1_{\mathcal{L}(B)} = 1_L$, then the hyperplanes of $L$ and $\mathcal{L}(B)$ have the same rank; thus $\mathcal{H}(\mathcal{L}(B)) \subseteq \mathcal{H}(L)$.

The cardinality of $\mathcal{L}(B)$ follows from standard arguments (or see [4, Theorem 1]). $\square$

**Proposition 10** *Let $\lambda$ be a singular cardinal. Assume that every geometric lattice of finite rank at least 2 and of cardinality less than $\lambda$ has a matching. Then every geometric lattice of finite rank at least 2 of cardinality $\lambda$ has a matching.*

*Proof.* (Compare this with the proof of [2, Theorem 6.4].) Assume not, for a contradiction. Then, by Theorem 3, there is a geometric lattice $L$ (of rank at least three) such that the society $\Gamma = \Big(\mathcal{A}(L), \mathcal{H}(L), \leq \cap \big(\mathcal{A}(L) \times \mathcal{H}(L)\big)\Big)$ has a $\kappa$-obstruction $\Pi = (M, W, K)$. Since $|M| \leq \lambda$, then by Theorem 2, we have $\kappa \leq \lambda$; indeed $\kappa < \lambda$, since $\kappa$ is finite or a regular cardinal. By Theorem 2, there exists $A \subseteq M$ such that $|A| = \kappa$ and $\Pi - A$ has an espousal, $E$. Let $R \subseteq \mathcal{A}(L)$ be a finite subset such that $1_L = \bigvee R$.

Let $B_0 = A \cup R$, and, for $n < \omega$, if $B_n$ is defined, let $B_{n+1} = B_n \cup E^{-1}\Big(\mathcal{H}\big(\mathcal{L}(B_n)\big)\Big)$. Note that $R \subseteq B_n$ for all $n < \omega$, so the rank of $\mathcal{L}(B_n)$ is the rank of $L$ and $\mathcal{H}\big(\mathcal{L}(B_n)\big) \subseteq \mathcal{H}(L)$ by Lemma 9.

Let $B = \bigcup_{n<\omega} B_n \subseteq M \cup R$. Now $|B_0| \leq \max\{\kappa, \aleph_0\} < \lambda$. If $n < \omega$ and $|B_n| \leq \max\{\kappa, \aleph_0\}$, then $|\mathcal{H}\big(\mathcal{L}(B_n)\big)| \leq \max\{\kappa, \aleph_0\}$, so $|B_{n+1}| \leq \max\{\kappa, \aleph_0\} + \max\{\kappa, \aleph_0\} = \max\{\kappa, \aleph_0\}$. Hence $|B| \leq \aleph_0 \max\{\kappa, \aleph_0\} = \max\{\kappa, \aleph_0\} < \lambda$.

As $R \subseteq B$, Lemma 9 shows that $|\mathcal{L}(B)| < \lambda$ and $\mathcal{H}\big(\mathcal{L}(B)\big) \subseteq \mathcal{H}(L)$, so $\mathcal{L}(B)$ has a matching. Let $G$ be the espousal. Since

$$E[(M \setminus A) \setminus (M \setminus A) \cap B] \cap \mathcal{H}\big(\mathcal{L}(B)\big) = \emptyset$$

and $A \subseteq B \cap M$, so that $M = [(M \setminus A) \setminus (M \setminus A) \cap B] \cup (B \cap M)$, we know $E|_{(M\setminus A)\setminus(M\setminus A)\cap B} \cup G|_{B\cap M}$ is an espousal of $\Pi$, as $\Pi \lhd \Gamma$, contradicting Theorem 2. $\square$

With Theorem 5, Proposition 10 extends Björner's work to $\aleph_\omega$. But using the argument of [5, Theorem 6] almost verbatim, we can settle Björner's first question from the 1981 Banff Conference on Ordered Sets. Björner already did the heavy lifting in proving Theorem 5, but to make it clear that his proof is what we need, we include it.

We feel it is important to go through the argument, since one of the proofs we are copying is of this theorem [5, page 9]:

*Let $L$ be an infinite geometric lattice such that the cardinality of the set of rank 2 intervals $[x, y]$ with rank $x \geq 4$ is strictly less than $|L|$. Assume further that $L$ fulfills either condition (i) $|L|$ is a regular cardinal or (ii) $L$ has a rank 2 element such that the set of atoms below it has cardinality $|L|$. Then $L$ has a matching.*

If we merely told a reader to trust that the proof does what we want, a reader might be sceptical.

**Theorem 11** *Every geometric lattice of finite rank greater than* 1 *has a matching.*

*Proof.* The proof is drawn from [5, pp. 10–13]. Assume we have a counterexample $L$ of smallest cardinality, and, among those counterexamples, one of smallest rank. By Theorems 4 and 5 and Proposition 10, we can assume $|L|$ is a regular cardinal and that $L$ has rank at least 4. By Theorem 7, there is $\ell_0 \in L$ of rank 2 such that $|\downarrow \ell_0| = |L|$.

Assume that $|\overline{p}| = |L|$ for all $p \in \underline{\ell_0}$. Consider any $q \in \mathcal{A}(L) \setminus \underline{\ell_0}$ and consider the rank 3 geometric lattice $\downarrow (q \vee \ell_0)$. By Lemma 6(b),

$$|L| = |\downarrow \ell_0| = |\underline{\ell_0}| \leq |\{c \in \downarrow (q \vee \ell_0) : q \lessdot c\}|.$$

The reason for the inequality is that $q$ is an atom not below $\ell_0$; therefore $q \vee \ell_0$ has rank 3 (since $\ell_0$ has rank 2) and, in the geometric lattice $\downarrow (q \vee \ell_0)$, $\ell_0$ is a hyperplane. Every atom below $\ell_0$ in $L$ is an atom below $\ell_0$ in this smaller geometric lattice. The hyperplanes greater than or equal to $q$ in this smaller geometric lattice have rank 2, so must cover $q$; and any cover of $q$ in this smaller geometric lattice has rank 2, so is a hyperplane in the smaller geometric lattice. By Lemma 6(a), we get the inequality.

The right-hand side equals $|\overline{q}|$ by Lemma 6(b); hence $|L| = |\overline{q}|$. Hence $|\overline{p}| = |L|$ for all $p \in \mathcal{A}(L)$. By Theorem 8, $L$ has a matching.

So now assume $|\overline{q}| < |L|$ for some $q \in \underline{\ell_0}$.

**Case 1.** *Every cover of $q$ except $\ell_0$ covers only one other atom.*

Then define $s : \mathcal{A}(L) \setminus \underline{\ell_0} \to \{x \in L : q \lessdot x\}$ by $s(p) = p \vee q$ for all $p \in \mathcal{A}(L) \setminus \underline{\ell_0}$. In this case, $s$ is one-to-one. By the minimality of $L$, the geometric lattice $\uparrow q$ has a matching $t : \{x \in L : q \lessdot x\} \to \overline{q}$. We will define a matching $f$ for $L$.

Let $f(p) := t\big(s(p)\big)$ for all $p \in \mathcal{A}(L) \setminus \underline{\ell_0}$ and let $f(q) := t(\ell_0)$; we just need to define $f$ on $\underline{\ell_0} \setminus \{q\}$. Pick $h_0 \in \overline{\ell_0}$ and let $z$ be a modular complement of $\ell_0$ in $\downarrow h_0$. It exists by Lemma 1. Since the rank of $L$ is at least 4, the rank of $h_0$ is at least 3. Since $\ell_0$ has rank 2, the modular complement property of $z$ means that the rank of $z$ is the rank of $h_0$ minus 2, so at least 1. Define $R : \underline{\ell_0} \to \{x \in L : z \lessdot x \lessdot h_0\}$ by $R(p) = p \vee z$ for all $p \in \underline{\ell_0}$. This function is well-defined, since the modular complement property implies that $\ell_0 \wedge z = 0_L$. Hence, if $p$ is an atom below $\ell_0$, then $p \not\leq z$, and thus $p \vee z$ covers $z$; both $p$ and $z$ are less than or equal to $h_0$, so $p \vee z \leq h_0$, and because the rank of $z$ is the rank of $h_0$ minus 2, anything covering $z$ that is less than or equal to $h_0$ must be covered by $h_0$. This function $R$ is one-to-one: If $p, p' \in \underline{\ell_0}$ but $p \neq p'$ and $p \vee z = p' \vee z$, then $p \vee z = p \vee p' \vee z = \ell_0 \vee z = h_0$, a contradiction.

If $p \in \underline{\ell_0} \setminus \{q\}$, then $q \not\leq R(p)$ (or else $R(p) = p \vee q \vee z = \ell_0 \vee z = h_0$, a contradiction), so $R(p)$ is covered by exactly one hyperplane in $\overline{q}$, namely $q \vee R(p)$, and this is $h_0$. Since $f\big[\big(\mathcal{A}(L) \setminus \underline{\ell_0}\big) \cup \{q\}\big] \subseteq \overline{q}$, if $p \in \underline{\ell_0} \setminus \{q\}$, we can let $f(p)$ be any hyperplane covering $R(p)$ except $h_0$; there is one, since $R(p)$ is a meet of co-atoms, by a statement made three paragraphs before Lemma 1. If $p_1, p_2 \in \underline{\ell_0} \setminus \{q\}$ but $p_1 \neq p_2$ and $f(p_1) = f(p_2)$, then $f(p_1)$ covers $R(p_1) = p_1 \vee z$ and covers $R(p_2) = p_2 \vee z$, so $f(p_1) = p_1 \vee p_2 \vee z = \ell_0 \vee z = h_0$ (remember that $R$ is one-to-one, so $f(p_1)$ covers the distinct elements $p_1 \vee z$ and $p_2 \vee z$, and so is their join), a contradiction. Thus $f$ is a matching, as we will see:

The set of atoms is a disjoint union of the sets $\underline{\ell_0} \setminus \{q\}$, $\{q\}$, and $\mathcal{A}(L) \setminus \underline{\ell_0}$. If $p \in \underline{\ell_0} \setminus \{q\}$, then $f(p) \geq R(p) \geq p$. Also, $f(q) \geq \ell_0 \geq q$. Further, if $p \in \mathcal{A}(L) \setminus \underline{\ell_0}$, then $f(p) \geq s(p) \geq p$. In each of the three cases, $f(p)$ is a hyperplane.

Let us see that $f$ is one-to-one when restricted to each of the three sets in the disjoint union. Obviously it is one-to-one on $\{q\}$. We just showed $f$ is one-to-one on $\underline{\ell_0} \setminus \{q\}$. On $\mathcal{A}(L) \setminus \underline{\ell_0}$, it is a composition of two one-to-one functions, by the first paragraph of Case 1.

Since $f(p_1)$ is not in $\overline{q}$ if $p_1 \in \underline{\ell_0} \setminus \{q\}$, it cannot equal $f(p_2)$ if $p_2 \in \big(\mathcal{A}(L) \setminus \underline{\ell_0}\big) \cup \{q\}$.

If $f(q) = f(p)$ for some $p \in \mathcal{A}(L) \setminus \underline{\ell_0}$, then $t\big(s(p)\big) = t(\ell_0)$. Since $t$ is one-to-one, that means $s(p) = \ell_0$. That means $p \leq \ell_0$, a contradiction.

**Case 2.** *There exists $\ell_1 \in L \setminus \{\ell_0\}$ such that $q \lessdot \ell_1$ and $|\underline{\ell_1}| \geq 3$.*

Let $p_1, p_2 \in \underline{\ell_1}$ be such that $|\{p_1, p_2, q\}| = 3$. Since $q = \ell_0 \wedge \ell_1$, we have $p_1, p_2 \nleq \ell_0$. Let $h_0 \in \overline{\ell_0}$ be such that $p_1 \nleq h_0$. (Pick a modular complement of $\ell_0 \vee p_1$ in $\uparrow \ell_0$.) If $p_2 \leq h_0$, then $q \leq \ell_0 \leq h_0$ implies $\ell_1 = p_2 \vee q \leq h_0$, and so $p_1 \leq h_0$, a contradiction. Hence $p_2 \nleq h_0$.

By the minimality of $L$, $\downarrow h_0$ has a matching $g : \underline{h_0} \to C := \{x \in L : x \lessdot h_0\}$. Let $C_2 := \{c \in C : |\overline{c}| = 2\}$ and let $C_3 := C \setminus C_2$. We will show that $|C_3| = |L|$.

Because $\{p_1, p_2\} \in \mathcal{A}(L) \setminus \underline{h_0}$, we have that $q \leq \ell_1 = p_1 \vee p_2 \leq \bigvee \mathcal{A}(L) \setminus \underline{h_0} =: y$.

*Claim. For $x \in C$, $x \in C_2$ if and only if $x = h_0 \wedge h$ for some $h \in \overline{y}$.*

*Proof of claim.* We have a partition of $\mathcal{A}(L) \setminus \underline{x}$: $\{\underline{k} \setminus \underline{x} : x \lessdot k \in L\}$.

If $x \in C_2$, then $x \lessdot h$ for some $h \in \mathcal{H}(L) \setminus \{h_0\}$ and so $x = h_0 \wedge h$. If $w \in \mathcal{A}(L) \setminus \underline{h_0}$, then $w \notin \underline{x}$, so $x \lessdot w \vee x \in \mathcal{H}(L)$ but $w \vee x \neq h_0$, so $w \vee x = h$. Hence $w \leq h$. Therefore $y \leq h$.

Conversely, if $x = h_0 \wedge h$ for some $h \in \overline{y}$, then $h_0 \neq h$. If there exists $h' \in \overline{x} \setminus \{h_0, h\}$, then, for some $a \in \underline{h'} \setminus \underline{x}$, $h' = a \vee x$. Hence $a \notin \underline{h_0} \setminus \underline{x}$, and thus $a \leq y$, so $a \leq h$ and thus $a \leq h \wedge h' = x$, a contradiction. Hence $x \in C_2$. $\square$

By the claim, $|C_2| \leq |\overline{y}|$: send every $x$ in $C_2$ to the $h$ in the statement of the claim; since $x = h_0 \wedge h$, the map is one-to-one. But $q \leq y$ implies that $\overline{y} \subseteq \overline{q}$ and, since $|\overline{q}| < |L|$, we conclude $|C_2| < |L|$.

Remember that the set of elements less than or equal to $\ell_0$ has cardinality $|L|$, and $h_0 \geq \ell_0$, so the set of elements less than or equal to $h_0$ has cardinality $|L|$. This set is a geometric lattice (of rank at least 3); being infinite, Lemma 6(b) applies, to tell us that the cardinality of the set of *its* hyperplanes is $|\downarrow h_0|$, which, as we just observed, equals $|L|$—but the set of its hyperplanes is $C$. Thus $|C| = |L|$. As $C$ is the disjoint union of $C_2$ and $C_3$, but $|C_2| < |L|$ and $L$ is infinite, we conclude that $|C_3| = |L|$.

We now define our matching as follows: Since $|\mathcal{A}(L) \setminus \underline{h_0}| \leq |C_3|$, take any injection $b : \mathcal{A}(L) \setminus \underline{h_0} \to C_3$ and let $f(p) = p \vee b(p)$ for $p \in \mathcal{A}(L) \setminus \underline{h_0}$. For $p \in \underline{h_0}$, let $f(p)$ be any cover of $g(p)$ except $h_0$ or, in case $g(p) = b(p')$ for some $p' \in \mathcal{A}(L) \setminus \underline{h_0}$, except $f(p')$.

(We can do this since $b(p') \in C_3$.) If $x', x'' \in C$ and $x' \neq x''$, then $\uparrow x' \cap \uparrow x'' = \uparrow h_0$; hence if $p, p' \in \mathcal{A}(L) \setminus \underline{h_0}$ and $p \neq p'$ but $f(p) = f(p')$ (so $p \vee b(p) = p' \vee b(p')$), then $f(p) \geq h_0$; but $r\big(f(p)\big) = r(h_0)$, so $f(p) = h_0$ and $p \leq h_0$, a contradiction.

If for some $p \in \underline{h_0}$ and $p' \in \mathcal{A}(L) \setminus \underline{h_0}$ we have $f(p) = f(p')$, then $g(p) \lessdot p' \vee b(p')$. Since $g(p), b(p') \lessdot h_0$, then $\{g(p), b(p'), h_0, p' \vee b(p')\}$ would be a 4-element crown (also called a "cycle") of elements in consecutive ranks—impossible in a lattice—unless $g(p) = b(p')$, which we have ruled out.

If $p, p' \in \underline{h_0}$ and $f(p) = f(p')$ but $p \neq p'$, then $g(p) \neq g(p')$ and $g(p), g(p') \lessdot h_0$ and $f(p)$ is a cover of $g(p), g(p')$ distinct from $h_0$, so we get another impossible 4-crown.

Hence $f$ is one-to-one. □

This answers the question of Björner from 1976 that was the first question he stated at the 1981 Banff Conference on Ordered Sets.

The Pólya Prize-winner went on to ask at the Banff Conference if there exists a family $M$ of pairwise disjoint maximal chains in $L \setminus \{0, 1\}$ whose union contains the set of atoms, saying, "I showed this is true for modular $L$, and J. Mason showed it to be true for finite $L$." Björner conjectured this in 1977 ([5, p. 18], [4, p. 10]), writing in 1976, "Another challenging question, related to the existence of matchings, is whether maximal families of pairwise disjoint maximal proper chains do exist in infinite geometric lattices. . . ." He refers the reader to [11].

A good approach to the second question would be to use [1] and [10]; Theorem 13 of the latter, when this writer first read it, made this writer feel that it could hold its own alongside many classic results in combinatorics.

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