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The super-connectivity of the Kneser graph $KG(n, 3)$

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Abstract

A vertex cut S of a connected graph G is a subset of vertices of G whose deletion makes G disconnected. A super vertex cut S of a connected graph G is a subset of vertices of G whose deletion makes G disconnected and there is no isolated vertex in each component of $G - S$. The super-connectivity of graph G is the size of the minimum super vertex cut of G. Let $KG(n, k)$ be the Kneser graph whose vertices are the k-subsets of $\{1, \dots, n\}$, where k is the number of labels of each vertex in G. We have shown in this paper that the conjecture from [G.B. Ekinci and J.B. Gauci, *Discuss. Math. Graph Theory* 39 (2019), 5–11] on the super-connectivity of the Kneser graph $KG(n, k)$ is true when $k = 3$.

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1 Introduction

Let $[n] = \{1, \ldots, n\}$ be n labels. The Kneser graph $G = KG(n, k)$ is the graph whose vertices are the k-subsets of $[n]$, and two vertices are adjacent if these two k-subsets are disjoint, i.e. two vertices do not share labels. Let $V(G)$ be the set of vertices of G. It is clear that $V(KG(n, k)) = \binom{[n]}{k}$ and $KG(n, k)$ is regular with degree $\binom{n-k}{k}$. A vertex cut S of a connected graph G is a subset of vertices of G whose deletion disconnects G. The connectivity κ of G is the size of the minimum vertex cut of G. If the deletion of any vertex cut of size κ in G will isolate a vertex, then G is super-connected. A vertex cut which isolates a single vertex is called a trivial vertex cut of G. When G is super-connected, it makes sense to determine the size of a minimum nontrivial vertex cut of G, that is, the super-connectivity κ_1 of G. And the smallest nontrivial vertex cut is called a super-vertex cut of G. A complete graph K_n is a simple graph with n vertices and an edge between every pair of vertices of K_n.

The concept of the Kneser graph was proposed by Kneser in 1955 [5]. Structural properties of the Kneser graph have been studied extensively: for example, the hamiltonicity, chromatic number and the matchings within the graph. Chen and Lih [2] proved that the Kneser graph is symmetric, vertex-transitive and edge-transitive. Using this property, Ekinci and Gauci [3] showed that the connectivity of the Kneser graph $KG(n, k)$ is $\binom{n-k}{k}$. Harary [4] proposed the concept of super-connectivity in 1983. Subsequently, Balbuena, Marcote and García-Vázquez [1] defined a similar concept, i.e. restricted connectivity of graphs. In this paper we investigate the super-connectivity of the Kneser graph.

It is clear that if $n < 2k$, then $KG(n, k)$ contains no edges, and if $n = 2k$, then $KG(n, k)$ is a set of independent edges. The Kneser graph $KG(n, 1)$ is the complete graph on n vertices. Ekini and Gauci made a conjecture in [3] which states:

Conjecture 1.1 *Let $n \geq 2k + 1$. Then the super-connectivity κ_1 of $KG(n, k)$ is*

$$\kappa_1 = \begin{cases} 2\left(\binom{n-k}{k} - 1\right) & \text{if} \quad 2k + 1 \leq n < 3k, \\ 2\left(\binom{n-k}{k} - 1\right) - \binom{n-2k}{k} & \text{if} \quad n \geq 3k. \end{cases}$$

Ekinci and Gauci [3] proved that this conjecture holds when $k = 2$. In this work we prove the conjecture for the case when $k = 3$.

2 Super-Connectivity of $KG(n, 3)$

In this section we will determine the super-connectivity of $KG(n, 3)$ when $n \geq 7$ and confirm that Conjecture 1.1 is true for $k = 3$.

Theorem 2.1 *The super-connectivity of the Kneser graph $KG(n, 3)$ is*

$$\kappa_1 = \begin{cases} 2\left(\binom{n-3}{3} - 1\right) & \text{if} \quad 7 \leq n \leq 8, \\ 2\left(\binom{n-3}{3} - 1\right) - \binom{n-6}{3} & \text{if} \quad n \geq 9. \end{cases}$$

Proof.
Let $S \subseteq V(G)$ be a super-vertex cut of G. Suppose $n \geq 9$ and $|S| < 2\left(\binom{n-3}{3} - 1\right) - \binom{n-6}{3}$; then we have

$$|G - S| > \binom{n}{3} - 2\left[\binom{n-3}{3} - 1\right] + \binom{n-6}{3} = \frac{54n - 204}{6} = 9n - 34.$$

This means that if κ_1 is less than the bound stated in the conjecture, then there will be more than $9n - 34$ vertices in $G - S$. In the following, we will show that $G - S$ has to be connected if it contains more than $9n - 34$ vertices.

Since S is a super-vertex cut, then $G - S$ has at least two components and each component has at least 2 vertices. If $G - S$ has a component containing exactly two vertices, then it is straightforward that $|S| = \kappa_1$ since S has to contain all the neighbours of these two vertices, and also it is easy to see that there is no isolated vertex in $G - S$.

Now we assume that each component of $G - S$ has at least three vertices. We also assume that $G - S$ has two components C_1, C_2, with $C_2 = G - S - C_1$. Note, in here, C_2 might not be connected. If C_2 is not connected, then C_2 is the union of some connected components with each having at least three vertices. Since C_1 has at least three vertices, let them be v_1, v_2, v_3. These three vertices form possibly two different graphs, either a complete graph K_3 or a path P_3 of length 2. If these three vertices form a path, then there are two possibilities, either the two non-adjacent vertices share only one common label, which we refer to as *Type* 1 *path* or the two non-adjacent vertices share two common labels, which we refer to as *Type* 2 *path*.

We make the following three claims.

Claim 1: If there is a K_3 in C_1, then there are at most 27 vertices in C_2.

Let the three vertices in C_1 be $v_1 = \{1, 2, 3\}$, $v_2 = \{4, 5, 6\}$, $v_3 = \{7, 8, 9\}$. Since C_1 and C_2 are disconnected, every vertex in C_2 has at least one label in common with every vertex in C_1, i.e. any vertex of C_2 has to have a label from $\{1, 2, 3\}$, a label from $\{4, 5, 6\}$ and a label from $\{7, 8, 9\}$. Thus the number of vertices in C_2 is at most $3^3 = 27$.

Claim 2: If there is a Type 1 path in C_1, then there are at most $3n + 3$ vertices in C_2.

Let the three vertices in C_1 be $v_1 = \{1, 2, 3\}$, $v_2 = \{4, 5, 6\}$, $v_3 = \{1, 7, 8\}$; the common label of two end vertices is 1. Then similar to the proof of Claim 1, we have a maximum of $3(n - 2)$ vertices in C_2 contain label 1, since the vertices of C_2 in this case have to use a label in $\{4, 5, 6\}$. In this calculation we have double counted three vertices $\{1, 4, 5\}$, $\{1, 4, 6\}$, $\{1, 5, 6\}$, and therefore there are at most $3(n - 2) - 3$ vertices containing label 1 in C_2. And there are at most $2 \cdot 3 \cdot 2$ vertices in C_2 which do not contain label 1. Hence the number of vertices in C_2 is at most $3(n - 2) - 3 + 12 = 3n + 3$.

Claim 3: If there is a Type 2 path P_3 in C_1, then there are at most $6n - 18$ vertices in C_2.

Let the three vertices in C_1 be $v_1 = \{1, 2, 3\}$, $v_2 = \{4, 5, 6\}$, $v_3 = \{1, 2, 7\}$; the set of common labels of the end vertices are $\{1, 2\}$. Similar to the previous argument, we have a maximum of $3(n-3)$ vertices in C_2 containing label 1, but not label 2. Similarly, we have a maximum of $3(n-3)$ vertices in C_2 containing label 2, but not label 1. And there are at most three vertices in C_2 containing both labels $\{1, 2\}$, and there are at most three vertices in C_2 containing neither label 1 nor label 2. Since we have double counted the 6 vertices $\{1, 4, 5\}$, $\{1, 4, 6\}$, $\{1, 5, 6\}$, $\{2, 4, 5\}$, $\{2, 4, 6\}$, $\{2, 5, 6\}$, it follows that the number of vertices in C_2 is at most $2 \cdot 3(n-3) + 6 - 6 = 6n - 18$.

Next we will show that $|C_1 \cup C_2| \leq 9n - 34$, which implies that $G - S$ has to be connected if it contains more than $9n - 34$ vertices. We consider the following cases.

Case 1: There are K_3s in both C_1 and C_2.

If the three vertices in C_1 form a complete graph K_3, let them be $v_1 = \{1, 2, 3\}$, $v_2 = \{4, 5, 6\}$, $v_3 = \{7, 8, 9\}$. Then by Claim 1, we have the number of vertices in C_2 is at most 27. If C_2 also contains a K_3, for example, $\{1, 4, 7\}$, $\{2, 5, 8\}$, $\{3, 6, 9\}$, then $C_1 \cup C_2$ has at most 54 vertices. In these 54 vertices, we have double counted the six vertices $\{1, 5, 9\}$, $\{1, 6, 8\}$, $\{2, 4, 9\}$, $\{2, 6, 7\}$, $\{3, 4, 8\}$, $\{3, 5, 7\}$. Additionally, the vertex $\{2, 3, 7\}$ can only be in C_1 or S, $\{1, 4, 8\}$ can only be in C_2 or S; howeveri, they are connected, and so one of them must be in S. Similarly for the pairs $\{5, 6, 7\}$ and $\{1, 4, 9\}$, $\{2, 3, 4\}$ and $\{1, 5, 7\}$, and thus $C_1 \cup C_2$ has at most 45 vertices. When $n \geq 9$, $9n - 34$ is larger than 45, and thus $G - S$ is connected, i.e. C_1 and C_2 must be connected in this case, a contradiction. So we know that C_1 and C_2 cannot both contain K_3. See Figure 1 for an illustration.

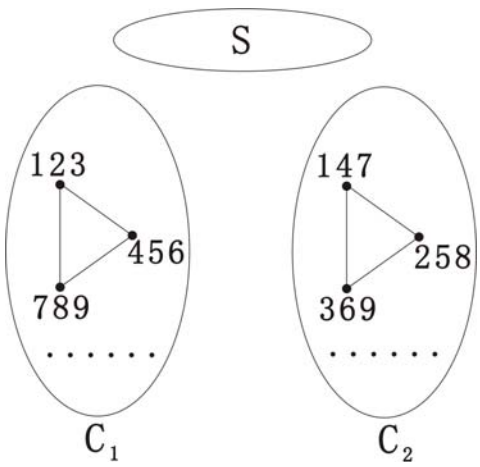

Figure 1: The case in C_1 and C_2

Case 2: There is a K_3 in C_1 or C_2, but not in both.

Suppose C_1 contains a K_3 and C_2 does not have a K_3. Let the three vertices in C_1 be $v_1 = \{1, 2, 3\}$, $v_2 = \{4, 5, 6\}$, $v_3 = \{7, 8, 9\}$. From Claim 1 we know that there are at most 27 vertices in C_2. If all 27 vertices are present in C_2, it is easy to verify that there are 36 K_3s in C_2, with no two K_3s sharing an edge; however, four K_3s will share a vertex, for example, $\{1, 5, 7\}$, $\{2, 4, 8\}$, $\{3, 6, 9\}$, and $\{1, 5, 7\}$,

$\{2,4,9\}$, $\{3,6,8\}$, and $\{1,5,7\}$, $\{2,6,8\}$, $\{3,4,9\}$, and $\{1,5,7\}$, $\{2,6,9\}$, $\{3,4,8\}$ (see Figure 2). To make sure there is no K_3 in C_2, at least nine vertices (such as $\{1,5,7\}$, $\{1,6,8\}$) have to be excluded from these 27 vertices. Thus there are at most $27-9=18$ vertices in C_2. If there are exactly 18 vertices in C_2, it implies that these nine vertices that have been removed all contain a certain label, for example, label 1. Otherwise, more than nine vertices have to be excluded to make sure there is no K_3 in C_2.

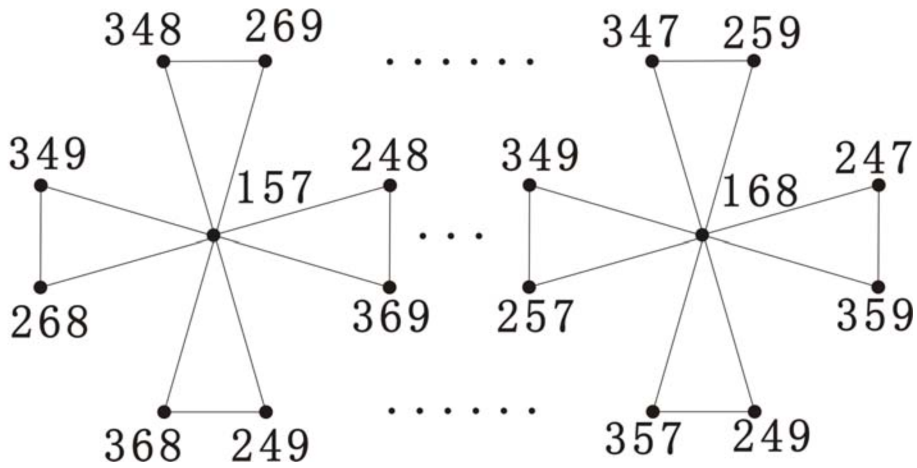

Figure 2: The K_3s in C_2

There must be a path P_3 in C_2, either Type 1 or Type 2, or otherwise there will be an isolated vertex or K_2 in C_2, which contradicts the assumption that the number of vertices in each component of C_2 is at least three.

For the first case, without loss of generality, assume the common label for two end vertices of the path is 1, and the middle vertex in the path contains label 2. We could further assume that the three vertices on the path are $\{1,4,x\}$, $\{2,5,y\}$, $\{1,6,z\} \in C_2$, where $x \neq y \neq z$ and $x,y,z \in \{7,8,9\}$. From the proof of Claim 2, we know that there are at most $3n+3$ vertices in C_1. However, we have double counted seven vertices $\{1,4,y\}$, $\{1,5,7\}$, $\{1,5,8\}$, $\{1,5,9\}$, $\{1,6,y\}$, $\{2,4,z\}$, $\{2,6,x\}$, which should be in either C_1 or C_2 but not in both. Thus, overall, $C_1 \cup C_2$ has no more than $18+3n+3-7=3n+14$ vertices, which is less than $9n-34$ when $n \geq 9$, and then C_1 and C_2 have to be connected, a contradiction. See Figure 3 for an illustration.

For the second case, assume the path consists of three vertices $\{1,4,x\}$, $\{2,5,y\}$, $\{1,4,z\} \in C_2$, where $x \neq y \neq z$ and $x,y,z \in \{7,8,9\}$. From the proof of Claim 3, we know that there are at most $6n-18$ vertices in C_1. Since we have double counted the eight vertices $\{1,4,y\}$, $\{1,5,7\}$, $\{1,5,8\}$, $\{1,5,9\}$, $\{1,6,y\}$, $\{2,4,7\}$, $\{2,4,8\}$, $\{2,4,9\}$, these vertices should be in either C_1 or C_2 but not in both. Thus, overall, $C_1 \cup C_2$ has no more than $18+6n-18-8=6n-8$ vertices, which is less than $9n-34$ when $n \geq 9$, and then C_1 and C_2 have to be connected, a contradiction. See Figure 4 for an illustration.

Case 3: There is no K_3 in either C_1 or C_2, that is, the components C_1 and C_2 contain P_3s.

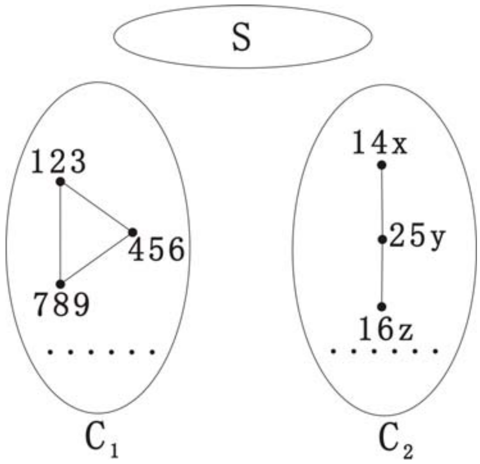

Figure 3: The case in C_1 and C_2

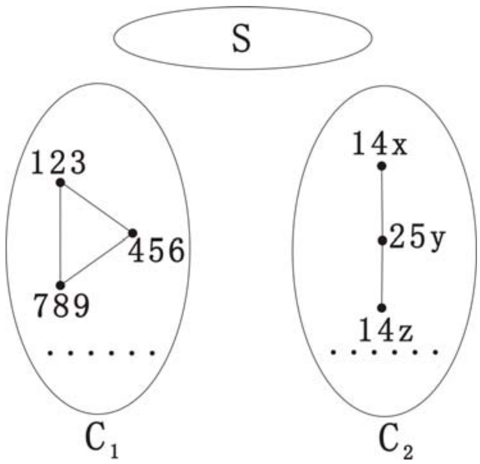

Figure 4: The case in C_1 and C_2

We shall consider the following three sub-cases based on the type of the paths.

Suppose there is a Type 1 path in C_1, and let the three vertices be $v_1 = \{1, 2, 3\}$, $v_2 = \{4, 5, 6\}$, $v_3 = \{1, 7, 8\}$. Then by Claim 2, we know the number of vertices in C_2 is at most $3n + 3$.

Now look at these vertices in C_2; there are at most 12 vertices which do not contain label 1. If all of them are in C_2, i.e. none of them is included in S, then these 12 vertices form two cycles of length 6. Of course, if some of the vertices are in S, then the rest of the vertices in each cycle form a set of paths. The rest of the vertices in C_2 all contain label 1, and thus are not connected to each other, but they are connected to the vertices which do not contain label 1. Next, we claim that either there is a Type 1 path, for example, $\{1, 4, 7\}$, $\{2, 5, 8\}$, $\{1, 6, 9\}$, or there will be no more than $2n + 4$ vertices in C_2. To see this, suppose we have no such desired path, and there are up to $n - 2$ vertices containing both labels $\{1, 4\}$, and there could be up to $n - 2$ vertices in C_2 containing both labels $\{1, 5\}$. Clearly not all 12 vertices with no label 1 are in C_2, since among those 12 vertices, the ones such as $\{2, 6, 7\}$,

$\{2,6,8\}$, $\{3,6,7\}$, $\{3,6,8\}$ will give us a desired path. Thus there are at most eight among these 12 vertices which could be in C_2. Also note that there must be some vertices from these 12 vertices contained in C_2, or otherwise we have a set of singular vertices in C_2. Then the number of vertices in C_2 is at most $2(n-2)+8 = 2n+4$. If there are vertices containing both labels $\{1,6\}$ in C_2, then for sure we see the desired path.

If we have the desired Type 1 path in C_2, let the three vertices be $\{1,4,x\}$, $\{2,5,y\}$, $\{1,6,z\}$, where $x \neq y \neq z$ and $x, z \in \{3,7,\ldots,n\}$, $y \in \{7,8\}$. Then based on the proof of Claim 2, C_1 has maximum $3n+3$ vertices, and thus $C_1 \cup C_2$ has a maximum of $6n+6$ vertices. Also note that we have double counted the vertices of the form $\{1,5,a\}$, where $a \in \{2,3,4,6,\ldots,n\}$, and vertices $\{1,2,4\}$, $\{1,2,6\}$, $\{1,4,y\}$, $\{1,6,y\}$, which both appear in C_1 and C_2 in our calculation. Meanwhile, $\{2,4,6\}$ is only in C_1 or S, and the vertex $\{3,5,7\}$ is either in C_2 or S. Depending on the choice of x, y, z, the vertex $\{3,5,7\}$ could also appear in C_1; for example, in the case $x = 3, y = 8, z = 7$. If $\{3,5,7\}$ is either in C_2 or S, as $\{2,4,6\}$ and $\{3,5,7\}$ are connected, it follows that one of them must be in S. If $\{3,5,7\}$ is in C_1, then $\{3,5,7\}$ is not in C_2, and thus we know the size of C_2 has to be one less than the maximum possible. The same holds for $\{1,2,7\}$ and $\{3,5,8\}$, $\{1,2,8\}$ and $\{3,6,7\}$. Therefore there are no more than $5n-1$ vertices in $C_1 \cup C_2$, and $9n-34$ is larger than $5n-1$ when $n \geq 9$, so then C_1 and C_2 have to be connected, a contradiction. See Figure 5 for an illustration.

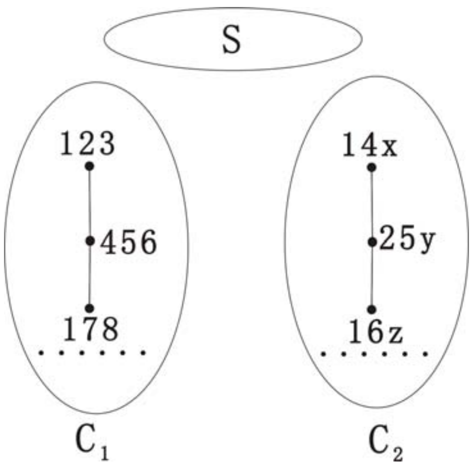

Figure 5: The case in C_1 and C_2

If there is no desired Type 1 path, then C_2 has at most $2n+4$ vertices, and we know there is a Type 2 path in C_2. Let the two shared labels be $\{1,4\}$ and let the three vertices be $\{1,4,x\}$, $\{2,5,y\}$, $\{1,4,z\}$ as shown in Figure 6, where $x \neq y \neq z$ and $x, z \in \{3,6,\ldots,n\}$, $y \in \{7,8\}$. Based on the proof of Claim 3, there is a maximum of $6n-18$ vertices in C_1. Since we have double counted the vertices of the form $\{1,5,a\}$, where $a \in \{2,3,4,6,\ldots,n\}$, and vertices $\{1,2,4\}$, $\{1,2,6\}$, $\{1,4,y\}$, $\{1,6,y\}$, $\{2,4,7\}$, $\{2,4,8\}$, which both appear in C_1 and C_2 in our calculation, therefore, there are no more than $7n-20$ vertices in $C_1 \cup C_2$, and

$9n-34$ is larger than $7n-20$ when $n \geq 9$, so then C_1 and C_2 have to be connected. This is a contradiction.

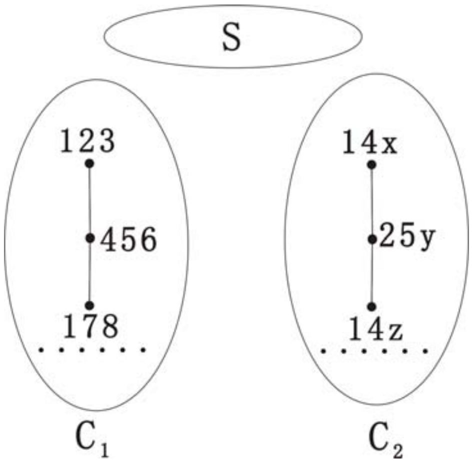

Figure 6: The case in C_1 and C_2

Now assume that there is no Type 1 path in C_1. Then there must be a Type 2 path in C_1. Let the three vertices on the path be $v_1 = \{1,2,3\}$, $v_2 = \{4,5,6\}$, $v_3 = \{1,2,7\}$. Then by Claim 3 we know that the number of vertices in C_2 is at most $6n-18$.

The case where there is a Type 2 path in C_1 and a Type 1 path in C_2 is the same as the case where there is a Type 1 path in C_1 and a Type 2 path in C_2. The latter we have considered before, so here we only consider the case where there is a Type 2 path in C_1 and there is also a Type 2 path in C_2.

Suppose, in C_2, that there are vertices containing both labels $\{1,4\}$ and vertices containing both labels $\{2,5\}$. Then there is no vertex containing both labels $\{1,6\}$ and no vertex containing both labels $\{2,6\}$, and furthermore, there is no vertex containing either label 1 or label 2 in C_2. This implies that there is no vertex containing both labels $\{1,2\}$ in C_2, since the vertices with both $\{1,2\}$ only connect the vertices with no $\{1,2\}$ in C_2, or otherwise a Type 1 path or K_3 will appear in C_2. Then the number of vertices in C_2 is at most $4(n-3)-2$, i.e. at most $n-3$ vertices contain both labels $\{1,4\}$, at most $n-3$ vertices contain both labels $\{1,5\}$, at most $n-3$ vertices contain both labels $\{2,4\}$ and at most $n-3$ vertices contain both labels $\{2,5\}$, and we have double counted the vertices $\{1,4,5\}$ and $\{2,4,5\}$. Then the Type 2 path in C_2 can be $\{1,4,x\}$, $\{2,5,y\}$, $\{1,4,z\}$, where $x \neq y \neq z$ and $x,y,z \in \{3,6,\ldots,n\}$. Based on the proof of Claim 3, there exist a maximum of $6n-18$ vertices in C_1. Since we have double counted the vertices of the form $\{1,5,a\}$ and $\{2,4,b\}$, where $a \in \{2,3,4,6,\ldots,n\}$ and $b \in \{1,3,5,\ldots,n\}$, which both appear in the C_1 and C_2 in our calculation, it follows that there is no more than $8n-28$ vertices in $C_1 \cup C_2$. Now $G-S = 9n-34$ is larger than $8n-28$ when $n \geq 9$, and thus C_1 and C_2 have to be connected, a contradiction. See Figure 7 for an illustration.

So far we have shown that if $n \geq 9$, then $G-S$ has to be connected, and therefore the conjecture is true.

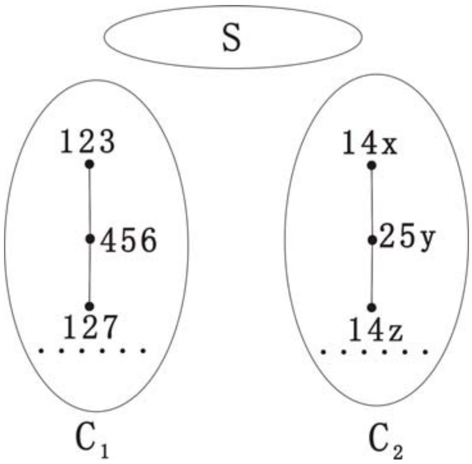

Figure 7: The case in C_1 and C_2

When $n = 7$, only a Type 2 path is possible in the graph. Let the three vertices in C_1 be $v_1 = \{1, 2, 3\}$, $v_2 = \{4, 5, 6\}$, $v_3 = \{1, 2, 7\}$. Then based on the proof of Claim 3, the number of vertices of C_2 is at most $6n - 18 = 24$, that is, nine vertices contain label 1, but not label 2, nine vertices contain label 2, but not label 1, three vertices contain both labels $\{1, 2\}$ and three vertices contain neither label 1 nor label 2.

Because C_2 has at least three vertices, and it only has a Type 2 path, let the three vertices on the path be $\{1, 4, x\}$, $\{2, 5, y\}$, $\{1, 4, z\}$, where $x \neq y \neq z$ and $\{x, y, z\} = \{3, 6, 7\}$; then there are possibly three paths, depending on the choice of y. The three paths are $\{1, 4, 3\}$, $\{2, 5, 6\}$, $\{1, 4, 7\}$ and $\{1, 4, 6\}$, $\{2, 5, 3\}$, $\{1, 4, 7\}$ and $\{1, 4, 3\}$, $\{2, 5, 7\}$, $\{1, 4, 6\}$. If the first path is present in C_2, based on the proof of Claim 3, C_1 has at most $6n - 18 = 24$ vertices, since we have double counted the 16 vertices $\{1, 2, 4\}$, $\{1, 2, 5\}$, $\{1, 2, 6\}$, $\{1, 3, 5\}$, $\{1, 3, 6\}$, $\{1, 4, 5\}$, $\{1, 4, 6\}$, $\{1, 5, 6\}$, $\{1, 5, 7\}$, $\{1, 6, 7\}$, $\{2, 3, 4\}$, $\{2, 4, 5\}$, $\{2, 4, 6\}$, $\{2, 4, 7\}$, $\{3, 5, 7\}$, $\{3, 6, 7\}$. Meanwhile, $\{4, 5, 7\}$ can only be in C_1 or S, $\{2, 3, 6\}$ can only be in C_2 or S, but they are connected, so that one of them must be in S, and the same for $\{3, 4, 5\}$ and $\{2, 6, 7\}$, $\{3, 4, 6\}$ and $\{2, 5, 7\}$, $\{4, 6, 7\}$ and $\{2, 3, 5\}$. Thus, overall, there are no more than $24+24-16-4 = 28$ vertices, which is less than $|G| - \kappa_1 = 29$, so then C_1 and C_2 have to be connected. This is a contradiction. We can discuss the second path and the third path in the same way, to arrive at the same conclusion, i.e. the number of vertices in $C_1 \cup C_2$ is at most $24 + 24 - 16 - 4 = 28$, which is less than $|G| - \kappa_1 = 29$, so then C_1 and C_2 have to be connected, a contradiction.

When $n = 8$, the three vertices in C_1 form a path P_3 of length 2, and it is possible for C_1 and C_2 to contain a Type 1 path or Type 2 path; thus we have to look into each case.

First, let C_1 have a Type 1 path $v_1 = \{1, 2, 3\}$, $v_2 = \{4, 5, 6\}$, $v_3 = \{1, 7, 8\}$. Then based on the proof of Claim 2, the number of vertices in C_2 is at most $3n + 3 = 27$, that is, 15 vertices contain label 1 and 12 vertices do not contain label 1.

If we have a Type 1 path in C_2, for example, $\{1, 4, x\}$, $\{2, 5, y\}$, $\{1, 6, z\}$, where

$x \neq y \neq z$ and $x, z \in \{3, 7, 8\}$, $y \in \{7, 8\}$, then there are possibly four paths; they are $\{1, 4, 3\}$, $\{2, 5, 7\}$, $\{1, 6, 8\}$ and $\{1, 4, 8\}$, $\{2, 5, 7\}$, $\{1, 6, 3\}$ and $\{1, 4, 3\}$, $\{2, 5, 8\}$, $\{1, 6, 7\}$ and $\{1, 4, 7\}$, $\{2, 5, 8\}$, $\{1, 6, 3\}$, respectively. If the first path is present in C_2, based on the proof of Claim 2, C_1 has at most $3n+3 = 27$ vertices, since we have double counted the 13 vertices $\{1, 2, 4\}$, $\{1, 2, 5\}$, $\{1, 2, 6\}$, $\{1, 3, 5\}$, $\{1, 4, 5\}$, $\{1, 4, 7\}$, $\{1, 5, 6\}$, $\{1, 5, 7\}$, $\{1, 5, 8\}$, $\{1, 6, 7\}$, $\{2, 4, 8\}$, $\{3, 5, 8\}$, $\{3, 6, 7\}$. Meanwhile, $\{1, 2, 7\}$ can only be in C_1 or S, $\{3, 4, 8\}$ can only be in C_2 or S, but they are connected, so then one of them must be in S, and the same holds i for the pairs $\{2, 4, 6\}$ and $\{3, 5, 7\}$, $\{4, 5, 8\}$ and $\{2, 6, 7\}$, $\{4, 7, 8\}$ and $\{1, 3, 6\}$. Thus, overall, there are no more than $27+27-13-4 = 37$ vertices, which is less than $|G|-\kappa_1 = 38$, so then C_1 and C_2 have to be connected, a contradiction. We can discuss the second path, the third path and the fourth path in the same way and arrive at the same contradiction.

If we have a Type 2 path in C_2, for example, $\{1, 4, x\}$, $\{2, 5, y\}$, $\{1, 4, z\}$, where $x \neq y \neq z$ and $x, z \in \{3, 6, 7, 8\}$, $y \in \{7, 8\}$, then based on the proof of Claim 3, C_1 has at most $6n - 18 = 30$ vertices. We have double counted 13 vertices: $\{1, 2, 4\}$, $\{1, 2, 5\}$, $\{1, 2, 6\}$, $\{1, 3, 5\}$, $\{1, 4, 5\}$, $\{1, 4, y\}$, $\{1, 5, 6\}$, $\{1, 5, 7\}$, $\{1, 5, 8\}$, $\{1, 6, y\}$, $\{2, 4, 7\}$, $\{2, 4, 8\}$, $\{3, 4, y\}$. Meanwhile, $\{1, 2, 7\}$ can only be in C_1 or S. The vertex $\{3, 5, 8\}$ is either in C_2 or S. Depending on the choice of x, y, z, the vertex $\{3, 5, 8\}$ could also appear in C_1, for example, when $x = 3, y = 7, z = 8$. If $\{3, 5, 8\}$ is either in C_2 or S, as $\{1, 2, 7\}$ and $\{3, 5, 8\}$ are connected, then one of them must be in S. If $\{3, 5, 8\}$ is in C_1, then we know the size of C_2 has to be one less than the maximum possible. The same holds for $\{1, 2, 8\}$ and $\{3, 5, 7\}$, $\{3, 4, 5\}$ and $\{2, 6, 7\}$, $\{4, 5, 7\}$ and $\{2, 6, 8\}$, $\{4, 5, 8\}$ and $\{3, 6, 7\}$, $\{2, 4, 5\}$ and $\{3, 6, 8\}$. Now $\{4, 7, 8\}$ can only be in C_1 or S, $\{1, 3, 6\}$ can only be in C_2 or S, but they are connected, so then one of them must be in S. Thus, overall, there are no more than $27 + 30 - 13 - 7 = 37$ vertices, which is less than $|G| - \kappa_1 = 38$, so then C_1 and C_2 have to be connected, a contradiction.

Second, let C_1 have a Type 2 path $v_1 = \{1, 2, 3\}$, $v_2 = \{4, 5, 6\}$, $v_3 = \{1, 2, 7\}$. Then based on the proof of Claim 3, the number of vertices of C_2 is at most $6n-18 = 30$, that is, 12 vertices contain label 1, but not label 2, 12 vertices contain label 2, but not label 1, three vertices contain both labels $\{1, 2\}$ and three vertices contain neither label 1 nor label 2.

The case where there is a Type 2 path in C_1 and a Type 1 path in C_2 is similar to the case where there is a Type 1 path in C_1 and a Type 2 path in C_2. The latter we have considered already, so here we only need to consider the case where there is a Type 2 path in C_1 and there is also a Type 2 path in C_2.

Suppose, in C_2, that there are vertices containing both labels $\{1, 4\}$ and vertices containing both labels $\{2, 5\}$ in C_2; then there is no vertex containing both labels $\{1, 6\}$, and there is no vertex containing both labels $\{2, 6\}$ in C_2, or otherwise a Type 1 path will appear in C_2. Then the number of vertices in C_2 is at most $4 \cdot 5 + 6 - 2 = 24$, i.e. at most five vertices contain both labels $\{1, 4\}$, at most five vertices contain both labels $\{1, 5\}$, at most five vertices contain both labels $\{2, 4\}$ and at most five vertices contain both labels $\{2, 5\}$; at most three vertices contain both labels $\{1, 2\}$ and at most three vertices contain neither label 1 nor label 2, and we have double counted the vertices $\{1,4,5\}$ and $\{2,4,5\}$. Then the Type 2 path in C_2 can be $\{1,4,x\}$, $\{2,5,y\}$, $\{1,4,z\}$, where $x \neq y \neq z$ and $x,y,z \in \{3,6,7,8\}$; based on the proof of Claim 3, there is a maximum of $6n - 18 = 30$ vertices in C_1. Note we have double counted the 14 vertices $\{1,2,4\}$, $\{1,2,5\}$, $\{1,2,6\}$, $\{1,3,5\}$, $\{1,4,5\}$, $\{1,4,y\}$, $\{1,5,6\}$, $\{1,5,7\}$, $\{1,5,8\}$, $\{2,3,4\}$, $\{2,4,5\}$, $\{2,4,6\}$, $\{2,4,7\}$, $\{2,4,8\}$, which all appear in the C_1 and C_2 in our calculation. Furthermore, $\{3,4,5\}$ can only be in C_1 or S, and the vertex $\{1,6,7\}$ is either in C_2 or S. Depending on the choice of x,y,z, the vertex $\{1,6,7\}$ could also appear in C_1, for example, when $x = 6, y = 7, z = 8$. If $\{1,6,7\}$ is either in C_2 or S, as $\{3,4,5\}$ and $\{1,6,7\}$ are connected, then one of them must be in S. If $\{1,6,7\}$ is in C_1, then we know the size of C_2 has to be one less than the maximum possible. The same holds for pairs $\{4,5,7\}$ and $\{1,6,8\}$, $\{4,5,8\}$ and $\{1,3,6\}$, $\{1,2,8\}$ and $\{3,5,7\}$. Therefore, there are no more than $24 + 30 - 14 - 4 = 36$ vertices in $C_1 \cup C_2$, and $|G| - \kappa_1 = 38$ is larger than 36, so then C_1 and C_2 have to be connected, a contradiction.

In summary, we have proved that when $k = 3$ the conjecture is true, and the bound is achieved only in the case that one of the disconnected components contains just two vertices linked by an edge.

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