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The structure of cube tilings

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Abstract

It is shown that every cube tiling of $\mathbb{R}^2$ is layered and the structure of non-layered cube tilings of $\mathbb{R}^3$ is described. It is also shown that in every cube tiling of $\mathbb{R}^4$ a cylinder contains a column. Simultaneously, this provides another proof of Keller's cube tiling conjecture about columns for dimensions $d \leq 4$.

1 Introduction

Let $T \subseteq \mathbb{R}^d$. A family of cubes $[0,1)^d + T$ is called a *cube tiling* of $\mathbb{R}^d$ if elements of this family are pairwise disjoint and the union of cubes of this family is the whole space $\mathbb{R}^d$. A set T is said to be a set that determines a cube tiling. If T forms a lattice, then we say that it determines a lattice cube tiling of $\mathbb{R}^d$. In 1896 Minkowski [13] conjectured that in every lattice cube tiling of $\mathbb{R}^d$ there is a pair of cubes which have a $(d-1)$-dimensional face in common. In 1930 Keller [5] generalized Minkowski's problem and conjectured that in every, not only lattice, cube tiling there is such a pair of cubes.

A family of cubes F in $\mathbb{R}^d$ is called an *l-column*, $1 \leq l < d$, if there is a set of vectors $S \subseteq \mathbb{R}^d$ such that:

1. $F = [0,1)^d + S$;
2. there is an index $i \in \{1, \dots, d\}$ such that the mapping $x \mapsto x_i$ transforms the set S bijectively into a set that determines a cube tiling of $\mathbb{R}$;
3. there are l indices $i \in \{1, \dots, d\}$ such that the sets $\{x_i \colon x = (x_1, \dots, x_d) \in S\}$ are singletons.

Then the set S is said to be a set that determines an l-column. Every $(d-1)$-column in $\mathbb{R}^d$ is called a *column*.

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For example, in Figure 1 a part of a cube tiling of $\mathbb{R}^2$ is shown. The horizontal coordinate axis is the first axis, and the vertical coordinate axis is the second. The grey cubes form a 1-column, and as it is in 2-dimensional space, we say that it is a column. The lower left corners of grey cubes, which are pointed in Figure 1, are elements of the set S, i.e.,

$$S = \{\dots, (-6; -1.4), (-5; -1.4), \dots, (2; -1.4), (3; -1.4), (4; -1.4), \dots\}.$$

Then there is an index $i \in \{1, 2\}$, i.e., $i = 1$, such that the mapping $x \mapsto x_1$ transforms the set S bijectively into a set that determines a cube tiling of $\mathbb{R}$ and there is one index $i \in \{1, 2\}$, i.e., $i = 2$, such that the set $\{x_2 \colon x = (x_1, x_2) \in S\} = \{-1.4\}$ is a singleton.

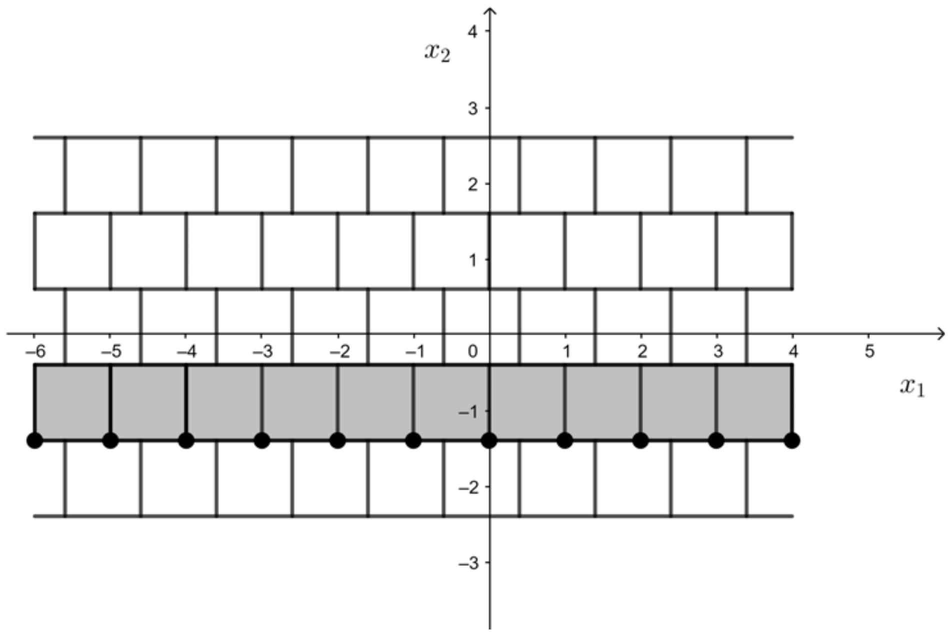

Fig. 1
A part of a cube tiling of $\mathbb{R}^2$.
The grey cubes form a column (1-column).

Minkowski's conjecture can be equivalently formulated in the form: in every lattice cube tiling of $\mathbb{R}^d$ there is a column, whereas Keller's conjecture gives two problems: about a pair of cubes and about a column. In 1940 Perron [14] published the proof of the weaker Keller conjecture for dimensions not exceeding 6, whereas in 1941 Hajós [4] showed that Minkowski's conjecture is true for all $d \geq 1$. Next, in 1992 Lagarias and Shor [9] discovered a counterexample to Keller's conjecture about a pair of cubes in dimension 10 and ten years later Mackey [12] found such a counterexample in dimension 8. This implies that both Keller's conjectures are not valid in any dimension greater than 7. In 2012 Łysakowska and Przesławski [11] showed that Keller's conjecture about a column is true for $d \leq 6$. In the years

2011–2017, Keller's conjecture about a pair of cubes for dimension 7 was proved in some special cases (see [2, 8, 7]), and in 2020 Brakensiek, Heule, Mackey, Narváez [1], using computer calculations, showed that this conjecture is true in dimension 7. As a result, Keller's conjecture about a pair of cubes is completely verified, while Keller's conjecture about a column is still open for dimension 7.

In this paper the structure of cube tilings is investigated. Let $T \subseteq \mathbb{R}^d$ be a set that determines a cube tiling of $\mathbb{R}^d$ and let $W \subseteq T$. The set W determines a *cylinder* (in the direction of the i-th coordinate axis), if there is $i \in \{1,\dots,d\}$ and $\alpha \in \mathbb{R}$ such that

$$W = \{t = (t_1,\dots,t_d) \in T\colon t_i \in \alpha + \mathbb{Z}\}.$$

Then the family $[0,1)^d + W = \{[0,1)^d + w\colon w \in W\}$ is called a cylinder. Let $S \subseteq \mathbb{R}^d$ and $S_i = \{s_i\colon s = (s_1,\dots,s_d) \in S\}$ for $i \in \{1,\dots,d\}$. A family of cubes $[0,1)^d + S$ is said to be *layered*, if there is $i \in \{1,\dots,d\}$ and $\alpha \in \mathbb{R}$ such that $S_i \subseteq \alpha + \mathbb{Z}$. A family of cubes which is not layered is called *non-layered*.

In [10] authors pointed out an example of a cube tiling of $\mathbb{R}^5$ in which there is a cylinder without any column. This implies that for $d \geq 5$ there are cylinders in cube tilings without l-columns, $l \geq 4$. It is worthwhile to see that this is one of the reasons why the methods used by authors in [11] in the proof of Keller's conjecture about columns for $d \in \{1,\dots,6\}$ do not work in dimension 7.

In contrast to 3-dimensional space, the fact that each cylinder in a cube tiling of $\mathbb{R}^4$ contains a column is not sufficient to describe the structure of all non-layered cube tilings in 4-dimensional space. Moreover, Sikirić and Łysakowska [3] showed that there are 183 non-isomorphic two-periodic non-layered cube tilings of $\mathbb{R}^4$, where a cube tiling is said to be *two-periodic* if the set T that determines it has the property such that $T = T + 2\mathbf{e}_i$ for every vector $\mathbf{e}_i$ of the standard basis. By comparison, in $\mathbb{R}^3$ there is only one such cube tiling. The higher the dimension, the more complicated the structure of cube tilings. In $\mathbb{R}^4$ there are cube tilings that cannot be described in the same way as in $\mathbb{R}^3$. So it seems to be impossible to describe the structure of all non-layered cube tilings of $\mathbb{R}^4$. However, as in Lemma 4.1, we can analyse the structure of cylinders, especially the existence of l-columns in them, in higher dimensions.

In this article it is shown that every cube tiling of $\mathbb{R}^2$ is layered and the structure of non-layered cube tilings in three-dimensional space is described. It is also proved that in a cube tiling of $\mathbb{R}^4$ every cylinder contains a column. A part of these results is known (see [10, 6]); however in this paper the new important thing is Lemma 4.1, telling us that in every 4-dimensional cube tiling each cylinder contains a 2-column. Moreover, the other methods are used, which can be exploited to analyse the structure of cube tilings in higher dimensions, especially to solve Keller's conjecture in dimension 7. Additionally, the results at the same time provide a new proof of Keller's conjecture about columns for dimensions $d \leq 4$.

2 Preliminaries

The terminology and notation are taken from the paper [11]. As usual, the set of all integers is denoted by $\mathbb{Z}$, and the set of positive integers is denoted by $\mathbb{N}$. The mapping $\varepsilon\colon \mathbb{R}^d \to \mathbb{N}^d$ given by

$$\varepsilon(x) = \varepsilon(x_1, \dots, x_d) = (\varepsilon_1(x_1), \dots, \varepsilon_d(x_d)),$$

where for each $i \in \{1, \dots, d\}$ a mapping $\varepsilon_i\colon \mathbb{R} \to \mathbb{N}$ is defined such that for every $x \in \mathbb{R}$ the restriction $\varepsilon_i | x + \mathbb{Z}$ is a bijection between the sets $x + \mathbb{Z} = \{x + k\colon k \in \mathbb{Z}\}$ and $\mathbb{N}$, is called a *code*.

In 1930 Keller [5] showed that *if* $[0,1)^d + T$ *is a cube tiling of* $\mathbb{R}^d$*, then for each pair of distinct elements* $s = (s_1, \dots, s_d), t = (t_1, \dots, t_d) \in T$ *there is an index* $i \in \{1, \dots, d\}$ *such that* $|s_i - t_i| \in \mathbb{N}$. Basing on Keller's result, in [11] the authors proved the following theorem.

Theorem 2.1 *Let* $\varepsilon\colon \mathbb{R}^d \to \mathbb{N}^d$ *be a code. Then a set* $T \subseteq \mathbb{R}^d$ *determines a cube tiling of* $\mathbb{R}^d$ *if and only if* $\varepsilon(T) = \mathbb{N}^d$ *and for every pair of distinct elements* $s, t \in T$ *there is* $i \in \{1, \dots, d\}$ *such that* $|s_i - t_i| \in \mathbb{N}$.

Notice that if $T \subseteq \mathbb{R}^m$ is a set that determines a cube tiling of $\mathbb{R}^m$, $m > d$, $I = \{i_1, \dots, i_{m-d}\} \subseteq \{1, \dots, m\}$, and $T_I^k = \{t = (t_1, \dots, t_m) \in T\colon (t_{i_1}, \dots, t_{i_{m-d}}) = k\}$ for $k \in \mathbb{R}^{m-d}$, then by Theorem 2.1 we obtain that a set $T^k(\{1, \dots, m\} \setminus I) = \{(t_{i_{m-d+1}}, \dots, t_{i_m})\colon t = (t_1, \dots, t_m) \in T_I^k\}$ determines a cube tiling of $\mathbb{R}^d$. This implies the following corollary.

Corollary 2.1 *If every cube tiling of* $\mathbb{R}^d$ *contains a column, then for each* $m > d$ *every cube tiling of* $\mathbb{R}^m$ *contains a* $(d-1)$*-column.*

Two vectors $x = (x_1, \dots, x_d), y = (y_1, \dots, y_d) \in \mathbb{R}$ are called *distinguishable* if there is an index $i \in \{1, \dots, d\}$ such that $x_i \neq y_i$ and $x_i \in y_i + \mathbb{Z}$. A system of vectors is said to be distinguishable if any two vectors of it are distinguishable.

Two sets of vectors $F, G \subseteq \mathbb{R}^d$ are said to be *isomorphic* if there are a bijection $f\colon F \to G$ and a permutation σ of the set $\{1, \dots, d\}$ such that $x_i = y_i$ if and only if $f(x)_{\sigma(i)} = f(y)_{\sigma(i)}$, and $|x_i - y_i| \in \mathbb{N}$ if and only if $|f(x)_{\sigma(i)} - f(y)_{\sigma(i)}| \in \mathbb{N}$ for every pair of vectors $x = (x_1, \dots, x_d), y = (y_1, \dots, y_d) \in F$ and for each $i \in \{1, \dots, d\}$.

We will write $x\colon x_1 \dots x_d$ instead of $x = (x_1, \dots, x_d)$. The coordinates of vectors will be denoted by Roman or Greek lower case letters. If we talk about the set determining a cube tiling of $\mathbb{R}^d$, then we will tacitly assume that a code of $\mathbb{R}^d$ is defined and we will use Theorem 2.1 without explicitly referring to it. Roman letters will occur only with lower indices, while Greek letters will occur with lower and upper indices. A coordinate of a vector will be denoted by a Greek letter when it is not explicit. Lower indices of coordinates of a vector will correspond to the code of this vector. When i-th coordinates of two vectors are denoted by the same lower case letter with the same upper index (if there is any), then they differ by an integer, and when they are denoted by different Roman letters, then their difference is not an integer. For example, $w_1\colon a_1\alpha_4^2\beta_2^1a_5$, $w_2\colon b_1\alpha_1^2\beta_2^3a_3$ means that vectors $w_1 = (a_1, \alpha_4^2, \beta_2^1, a_5)$ and $w_2 = (b_1, \alpha_1^2, \beta_2^3, a_3)$ have codes $\varepsilon(w_1) = (1,4,2,5)$ and $\varepsilon(w_2) = (1,1,2,3)$. Differences $(w_1)_2 - (w_2)_2 = \alpha_4^2 - \alpha_1^2$ and $(w_1)_4 - (w_2)_4 = a_5 - a_3$ are non-zero integers, and the difference $(w_1)_1 - w(2)_1 = a_1 - b_1$ is not an integer. Moreover, the third coordinates of the vectors w_1 and w_2 have the same code but it is not decided whether they are equal or different. As another example, let us see that the vectors

$$\begin{array}{llllll} w_1: & a_1 & a_1 & \alpha_2^1 & a_1 & \alpha_1^2, \\ w_2: & a_1 & a_2 & \alpha_2^3 & a_1 & \alpha_1^4, \\ w_3: & a_1 & a_3 & \alpha_2^5 & a_1 & \alpha_1^6, \\ & & \vdots & & & \end{array}$$

determine a 2-column in 5-dimensional space and they can be written in the short form

$$w_l: \; a_1 \;\; a_l \;\; \alpha_2^{2l-1} \;\; a_1 \;\; \alpha_1^{2l}, \qquad l \geq 1.$$

If $A \subseteq X$, then the set $X \setminus A$ will be denoted by A'. Let $x \in \mathbb{R}$, $A \subseteq \mathbb{N}$ and let $\varepsilon\colon \mathbb{R} \to \mathbb{N}$ be a code. Then the family $\{x_i\colon i \in A\}$ will be denoted by x_A. We will denote by the symbol $*$ an unspecified member of the family $x_{\mathbb{N}}$. These symbols will be used in the following way. Let A_1, $A_2, \ldots$, $A_n \subseteq \mathbb{N}$ and $T \subseteq \mathbb{R}^n$. We will denote by $a_{A_1}a_{A_2}\ldots a_{A_n}$ the following family of vectors from T:

$$a_{A_1}a_{A_2}\ldots a_{A_n} = \{t \in T\colon \; t : a_{i_1}a_{i_2}\ldots a_{i_n}, \; i_1 \in A_1, i_2 \in A_2, \ldots, i_n \in A_n\}.$$

The inscription a_Aa_B* will denote an unspecified member of the family of sets

$$\bigcup_{(i,j)\in\; A\times B} a_ia_jx(i,j)_{\mathbb{N}},$$

where x runs over all functions from $A \times B$ to $\mathbb{R}$. The symbol $\star + \mathbb{Z}$ will denote an unspecified member of the family $\alpha + \mathbb{Z}$, $\alpha \in \mathbb{R}$, and the inscription $X \times (\star + \mathbb{Z})$ will denote an unspecified member of the family

$$\bigcup_{x\in X} \{x\} \times (\alpha(x) + \mathbb{Z}),$$

where α runs over all functions from X to $\mathbb{R}$.

Using geometric arguments it is easy to see that in $\mathbb{R}^2$ every cube tiling is layered. We prove this obvious fact by using our notation to demonstrate how it works.

Theorem 2.2 *Every cube tiling of* $\mathbb{R}^2$ *is layered.*

Proof. Suppose that $T \subseteq \mathbb{R}^2$ determines a non-layered cube tiling of $\mathbb{R}^2$. According, with our notation we can assume that the vector of T with the code (1, 1) has the form

$$w_1: \; a_1 \;\; a_1.$$

As the tiling determined by T is non-layered, the set T contains the following vectors

$$\begin{array}{llll} w_2: & \alpha_k & b_1, \\ w_3: & b_1 & \beta_l \end{array}$$

for some $k, l \in \mathbb{N}$, up to an isomorphism. By distinguishability of vectors w_1, w_2 and w_1, w_3, we have $\alpha_k = a_k$, $k \neq 1$ and $\beta_l = a_l$, $l \neq 1$. As a result, vectors w_2 and w_3 are not distinguishable, which is impossible. □

3 Cube tilings of 3-dimensional space

Theorem 3.1 *Let $T \subseteq \mathbb{R}^3$ be a set determining a cube tiling of $\mathbb{R}^3$ and let $W \subseteq T$ be a set determining a cylinder. Then in W there is a set that determines a column.*

Proof. Let W be a set that determines a cylinder in $[0, 1)^3 + T$, i.e.

$$W = \{t = (t_1, t_2, t_3) \in T : t_3 \in a_{\mathbb{N}}\},$$

up to an isomorphism. Passing to an isomorphic system if necessary, we can assume that the vector with the code $(1, 1, 1)$ belongs to W and it has the form

$$w_{1,\,1}: \quad a_1 \quad a_1 \quad a_1.$$

Consider the vectors of T with codes $(l, 1, 1)$, $l \geq 2$. By their distinguishability from $w_{1,\,1}$, they have the form

$$w_{1,\,l}: \quad a_l \quad \alpha_1^{2l-3} \quad \alpha_1^{2l-2}, \qquad l \geq 2.$$

If $\alpha_1^i = a_1$ for all $i \geq 1$, then the vectors $w_{1,\,l}$, $l \geq 2$, belong to W and together with $w_{1,\,1}$ determine a column. Suppose that $\alpha_1^i \neq a_1$ for some $i \geq 1$. We have to consider two cases:

1. $\alpha_1^i \neq a_1$ for some $i \in \{1, 3, \ldots\}$.

Assume that $\alpha_1^1 \neq a_1$. (If $\alpha_1^1 = a_1$ and for example $\alpha_1^3 \neq a_1$, then we replace the code ε_1 by ε_1', where $\varepsilon_1'|(a_1 + \mathbb{Z}) = (2\ 3) \circ \varepsilon_1$ and $\varepsilon_1' = \varepsilon_1$ on the complement of the set $a_1 + \mathbb{Z}$.) Take into account the vectors with codes $(1, l, 1)$, $l \geq 2$. By distinguishability, they can be written as follows

$$w_{2,\,l-1}': \quad a_1 \quad a_l \quad \beta_1^{l-1}, \qquad l \geq 2.$$

If $\beta_1^i = a_1$ for all $i \geq 1$, then the vectors $w_{2,\,l}'$, $l \geq 1$, lie in W and together with $w_{1,\,1}$ determine a column. Suppose that $\beta_1^i \neq a_1$ for some $i \geq 1$. We can assume that $\beta_1^1 \neq a_1$. (If $\beta_1^1 = a_1$ and $\beta_1^i \neq a_1$ for some $i \geq 2$, then we can change the code ε_2 in an appropriate way.) Consider the vectors with codes $(1, 1, l)$, $l \geq 2$. They have the form

$$w_{3,\,l-1}': \quad \gamma_1^{l-1} \quad a_1 \quad a_l, \qquad l \geq 2.$$

Notice that the vectors $w'_{3,\,l}$, $l \geq 1$, belong to W. If $\alpha_1^i \neq a_1$ for some $i \in \{2, 4, \ldots\}$, then by distinguishability of $w_{1,\,l}$, $l \geq 2$, and $w'_{3,\,l}$, $l \geq 1$, we obtain $\gamma_1^i = a_1$ for all $i \geq 1$ and the vectors $w_{1,\,1}$, $w'_{3,\,l}$, $l \geq 1$, determine a column. Suppose that $\alpha_1^i = a_1$ for $i = 2, 4, \ldots$ and $\gamma_1^i \neq a_1$ for some $i \geq 1$. We can assume that $\gamma_1^1 \neq a_1$. (If $\gamma_1^1 = a_1$ and $\gamma_1^i \neq a_1$ for some $i \geq 2$, then we can change the code ε_3 respectively.) Take the vectors with codes $(l, 1, 2)$, $l \geq 2$. By distinguishability, they are as follows

$$w'_{4,\,l-1}: \quad \gamma_l^1 \quad a_1 \quad a_2, \qquad l \geq 2.$$

The vectors $w'_{4,\,l}$, $l \geq 1$, belong to W and together with $w'_{3,\,1}$ determine a column.

2. $\alpha_1^i \neq a_1$ for some $i \in \{2, 4, \ldots\}$. We can assume that $\alpha_1^2 \neq a_1$. (If $\alpha_1^2 = a_1$ and for example $\alpha_1^4 \neq a_1$, then we replace the code ε_1 by ε'_1 so that $\varepsilon'_1|(a_1 + \mathbb{Z}) = (2\ 3) \circ \varepsilon_1$ and $\varepsilon'_1 = \varepsilon_1$ on the complement of the set $a_1 + \mathbb{Z}$.) Consider the vectors with codes $(1, 1, l)$, $l \geq 2$. By distinguishability from $w_{1,\,l}$, $l \geq 1$, they have the form

$$w_{2,\,l-1}: \quad a_1 \quad \beta_1^{l-1} \quad a_l, \qquad l \geq 2.$$

The vectors $w_{2,\,l}$, $l \geq 1$, belong to W. If $\beta_1^i = a_1$ for all $i \geq 1$, then the vectors $w_{1,\,1}$ and $w_{2,\,l}$, $l \geq 1$, determine a column. Suppose that $\beta_1^i \neq a_1$ for some $i \geq 1$. We can assume that $\beta_1^1 \neq a_1$. (If not, then we can change the code ε_3 in an appropriate way.) Now take the vectors with codes $(1, l, 1)$, $l \geq 2$. By distinguishability from $w_{1,\,1}$ and $w_{2,\,l}$, $l \geq 1$, they can be written in the form

$$w_{3,\,l-1}: \quad \gamma_1^{l-1} \quad a_l \quad a_1, \qquad l \geq 2.$$

Notice that the vectors $w_{3,\,l}$, $l \geq 1$ belong to W. If $\alpha_1^i \neq a_1$ for some $i \in \{1, 3, \ldots\}$, then by distinguishability of $w_{1,\,l}$, $l \geq 2$, and $w_{3,\,l}$, $l \geq 1$, it follows that $\gamma_1^i = a_1$ for all $i \geq 1$ and the vectors $w_{1,\,1}$, $w_{3,\,l}$, $l \geq 1$, determine a column. So suppose that $\alpha_1^i = a_1$ for $i = 1, 3, \ldots$ and $\gamma_1^i \neq a_1$ for some $i \geq 1$. We can assume that $\gamma_1^1 \neq a_1$. (If not, then we can change the code ε_2 respectively.) Consider the vectors with codes $(l, 2, 1)$, $l \geq 2$. They are as follows

$$w_{4,\,l-1}: \quad \gamma_l^1 \quad a_2 \quad a_1, \qquad l \geq 2.$$

The vectors $w_{4,\,l}$, $l \geq 1$, belong to W and together with $w_{3,\,1}$ determine a column. □

Lemma 3.1 *Let $T \subseteq \mathbb{R}^3$ be a set determining a non-layered cube tiling of $\mathbb{R}^3$, which contains a column in the direction of the third coordinate axis. Then there are proper subsets A and B of $\mathbb{Z}$ and real numbers α and β such that a subset of T that contains the vectors determining all columns in the direction of the third coordinate axis has the form*

$$(\alpha + A) \times (\beta + B) \times (\star + \mathbb{Z}).$$

Proof. First we will show that if a subset of T that contains the vectors determining all columns in the direction of the third coordinate axis has the form $(\alpha + A) \times (\beta +$

$B) \times (\star + \mathbb{Z})$, then sets A and B have to be proper subsets of $\mathbb{Z}$. Suppose that $A = \mathbb{Z}$ or $B = \mathbb{Z}$. We can assume that $A = \mathbb{Z}$. Then for every vector $t = (t_1, t_2, t_3) \in T \setminus ((\alpha + A) \times (\beta + B) \times (\star + \mathbb{Z}))$ we have $t_2 \in \beta + \mathbb{Z}$. It means that T determines a layered tiling. This implies that sets A and B are proper subsets of $\mathbb{Z}$.

According with our notation, it remains to show that there are proper subsets C and D of $\mathbb{N}$ such that a subset of T that contains the vectors determining all columns in the direction of the third coordinate axis has the form $a_C\ a_D\ *$.

It is sufficient to show that if the set T contains vectors $\{\alpha^1\} \times \{\alpha^2\} \times (\alpha^3 + \mathbb{Z})$ and $\{\beta^1\} \times \{\beta^2\} \times (\beta^3 + \mathbb{Z})$ determining two columns such that $\alpha^1 \neq \beta^1$ and $\alpha^2 \neq \beta^2$, then the vectors $\{\alpha^1\} \times \{\beta^2\} \times (\gamma + \mathbb{Z})$ for some $\gamma \in \mathbb{R}$ belong to T.

Passing to an isomorphic system if necessary, we can assume that the vectors of T which determine a column in the direction of the third coordinate axis have the form

$$w_{1,\,l}: \quad a_1 \quad a_1 \quad a_l, \qquad l \geq 1.$$

Suppose that the set T contains the vectors determining another column in the third direction, i.e. the following vectors

$$w_{2,\,l}: \quad \alpha_i^1 \quad \alpha_j^2 \quad \alpha_l^3, \qquad l \geq 1,$$

for some $i, j \in \mathbb{N}$. If $\alpha_i^1 \neq a_i$, then by distinguishability we have $\alpha_j^2 = a_j$ and $j \neq 1$. Now, the vectors with codes (r, k, s), $k \in \mathbb{N} \setminus \{1, j\}$, for arbitrary $r, s \in \mathbb{N}$ imply that T determines a layered tiling. Similarly, if $\alpha_j^2 \neq a_j$, then $\alpha_i^1 = a_i$ and $i \neq 1$. Next, the vectors with codes (k, r, s), $k \in \mathbb{N} \setminus \{1, i\}$, for arbitrary $r, s \in \mathbb{N}$ imply that T determines a layered tiling. As a result we have

$$w_{2,\,l}: \quad a_i \quad a_j \quad \alpha_l^3, \qquad l \geq 1.$$

We can assume that $i \neq 1$ and $j \neq 1$. Consider the vectors with codes $(1, j, l)$ and $(i, 1, l)$, $l \geq 1$. By distinguishability, they have the form

$$\begin{aligned} w_{3,\,l}&: \quad a_1 \quad a_j \quad \beta_l^1, \\ w_{4,\,l}&: \quad a_i \quad a_1 \quad \beta_l^2, \qquad l \geq 1. \end{aligned}$$

These vectors also determine columns in the direction of the third coordinate axis. It implies that the set of vectors determining all columns in the third direction has the desired form. □

Lemma 3.1 is obviously also true for columns in the direction of the first or second instead of third coordinate axis.

Theorem 3.2 *Let $T \subseteq \mathbb{R}^3$ be a set that determines a non-layered cube tiling of $\mathbb{R}^3$. Then there are proper subsets A, B, C of $\mathbb{Z}$ and real numbers α, β, γ such that the set T can be presented as the union of the following sets of vectors:*

$$\begin{gathered} (\alpha + A) \times (\beta + B) \times (\star + \mathbb{Z}), \\ (\alpha + A') \times (\star + \mathbb{Z}) \times (\gamma + C), \\ (\star + \mathbb{Z}) \times (\beta + B') \times (\gamma + C'), \\ (\alpha + A') \times (\beta + B) \times (\gamma + C'), \\ (\alpha + A) \times (\beta + B') \times (\gamma + C). \end{gathered}$$

Proof. In [11] the authors showed that the set T contains vectors that determine a column. Passing to an isomorphic system if necessary, we can assume that it is a column in the direction of the third coordinate axis. By Lemma 3.1, a set of vectors determining all columns in the third direction can be written in the form

$$K_1 = a_A \quad a_B \quad *,$$

for some proper subsets A, B of $\mathbb{N}$. As the tiling determined by T is non-layered, T contains the vector

$$w_{1,\,j}: \quad \alpha_i^1 \quad b_j \quad \alpha_k^2,$$

for some $i, j, k \in \mathbb{N}$. By distinguishability of $w_{1,\,j}$ with vectors of the set K_1, we have $\alpha_i^1 = a_i$ and $i \notin A$. Consider vectors of the set T with codes (i, l, k), $l \in \mathbb{N} \setminus \{j\}$. By distinguishability, they have the form

$$w_{1,\,l}: \quad a_i \quad b_l \quad \beta_k^l, \qquad l \in \mathbb{N} \setminus \{j\}.$$

If $\beta_k^l \neq \alpha_k^2$ for some $l \in \mathbb{N}\setminus\{j\}$, then taking the vectors with codes (i, j, r), $r \in \mathbb{N}\setminus\{k\}$, which have the form

$$a_i \quad b_j \quad \alpha_r^2, \qquad r \in \mathbb{N} \setminus \{k\},$$

we obtain the vectors that determine a column in the third direction, but they do not belong to K_1, which is impossible. Thus $\beta_k^l = \alpha_k^2$ for all $l \in \mathbb{N} \setminus \{j\}$ and the vectors $w_{1,\,l}$, $l \geq 1$, determine a column in the direction of the second coordinate axis. Moreover, this column is disjoint with all columns determined by the set K_1. By an appropriate version of Lemma 3.1 and definition of distinguishability, the vectors determining all columns in the second direction disjoint with columns determined by vectors of K_1 are contained in the set

$$K_2 = a_{A'} \quad * \quad a_C,$$

for some proper subset C of $\mathbb{N}$. Again, as the tiling determined by T is non-layered, T contains the vector

$$w_{2,\,r}: \quad b_r \quad \gamma_s^1 \quad \gamma_t^2,$$

for some $r, s, t \in \mathbb{N}$. By distinguishability of $w_{2,\,r}$ from vectors of sets K_1 and K_2, we have $\gamma_s^1 = a_s$, $s \notin B$ and $\gamma_t^2 = a_t$, $t \notin C$. Now, take the vectors of T with codes (l, s, t), $l \in \mathbb{N} \setminus \{r\}$. They are as follows

$$w_{2,\,l}: \quad b_l \quad a_s \quad a_t, \qquad l \in \mathbb{N} \setminus \{r\}.$$

The vectors $w_{2,\,l}$, $l \geq 1$, determine a column in the direction of the first coordinate axis disjoint with all columns determined by the vectors of sets K_1 and K_2. By an appropriate version of Lemma 3.1 and definition of distinguishability, vectors determining all columns in the first direction disjoint with columns determined by the vectors of sets K_1 and K_2 are contained in the set

$$K_3 = * \quad a_{B'} \quad a_{C'}.$$

By distinguishability, the vectors of the set $T \setminus \left(\bigcup_{i=1}^{3} K_i\right)$ are contained in the union of the following sets:

$$\begin{array}{lll} a_{A'} & a_B & a_{C'}, \\ a_A & a_{B'} & a_C. \end{array}$$

As vectors of the set

$$S = a_{A'} a_B a_{C'} \cup a_A a_{B'} a_C \cup \bigcup_{i=1}^{3} K_i$$

are pairwise distinguishable and $\varepsilon(S) = \mathbb{N}^3$, where $\varepsilon \colon \mathbb{R}^3 \to \mathbb{N}^3$ is a code, by Theorem 2.1 we obtain $T = S$. □

Theorem 3.2 is illustrated in Figure 2.

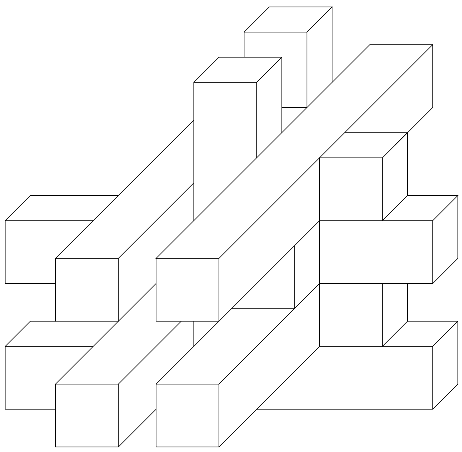

Fig. 2
A part of a non-layered cube tiling of $\mathbb{R}^3$.
Each column can be partitioned into cubes in an arbitrary way.

4 Cube tilings of 4-dimensional space

Lemma 4.1 *Let $T \subseteq \mathbb{R}^4$ be a set that determines a cube tiling of $\mathbb{R}^4$ and let $W \subseteq T$ be a set that determines a cylinder. Then the set W contains vectors that determine a 2-column.*

Proof. If $W \subseteq T$ is a set that determines a cylinder of the tiling $[0,1)^4 + T$, then it has the form

$$W = \{t = (t_1, t_2, t_3, t_4) \in T \colon t_4 \in a_{\mathbb{N}}\},$$

up to an isomorphism. Suppose that W does not contain vectors determining a 2-column. Passing to an isomorphic system if necessary, we can assume that the vector

$$w_{1,1} \colon \quad a_1 \quad a_1 \quad a_1 \quad a_1$$

belongs to W. Consider the vectors of T with codes $(1,1,1,l)$, $l \geq 2$. By distinguishability, they have the form

$$w_{1,\,l}:\quad \alpha_1^{3l-5}\quad \alpha_1^{3l-4}\quad \alpha_1^{3l-3}\quad a_l,\qquad l\geq 2.$$

The vectors $w_{1,\,l}$, $l \geq 2$, belong to the set W. As W does not contain vectors determining a 2-column, we can assume that $\alpha_1^i \neq a_1$ for some $i \in \{1,4,\ldots\}$ and $\alpha_1^i \neq a_1$ for some $i \in \{2,5,\ldots\}$. We can suppose that $\alpha_1^1 \neq a_1$. (If $\alpha_1^1 = a_1$ and for example $\alpha_1^4 \neq a_1$, then we replace the code ε_4 by ε_4', where $\varepsilon_4'|(a_1+\mathbb{Z}) = (2\ 3)\circ\varepsilon_4$ and $\varepsilon_4' = \varepsilon_4$ on the complement of the set $a_1+\mathbb{Z}$.) Take into account the vectors with codes $(l,1,1,1)$, $l \geq 2$. They are as follows

$$w_{2,\,l-1}:\quad a_l\quad \beta_1^{2l-3}\quad \beta_1^{2l-2}\quad a_1,\qquad l\geq 2.$$

The vectors $w_{2,\,l}$, $l \geq 1$, belong to W. Again, as W does not contain vectors determining a 2-column, we can suppose that $\beta_1^i \neq a_1$ for some $i \in \{1,3,\ldots\}$ and $\beta_1^i \neq a_1$ for some $i \in \{2,4,\ldots\}$. We can assume that $\beta_1^1 \neq a_1$. (If $\beta_1^1 = a_1$ and for example $\beta_1^3 \neq a_1$, then we replace the code ε_1 by ε_1', where $\varepsilon_1'|(a_1+\mathbb{Z}) = (2\ 3)\circ\varepsilon_1$ and $\varepsilon_1' = \varepsilon_1$ on the complement of the set $a_1+\mathbb{Z}$.) Now, consider the vectors with codes $(1,l,1,1)$, $l \geq 2$. By distinguishability, they have the form

$$w_{3,\,l-1}:\quad a_1\quad a_l\quad \gamma_1^{l-1}\quad a_1,\qquad l\geq 2.$$

The vectors $w_{3,\,l}$, $l \geq 1$, belong to W and together with $w_{1,\,1}$ determine a 2-column. □

Theorem 4.1 *Let $T \subseteq \mathbb{R}^4$ be a set that determines a cube tiling of $\mathbb{R}^4$ and let $W \subseteq T$ be a set that determines a cylinder. Then W contains vectors determining a column.*

Proof. From the definition it follows that W has the form

$$W = \{t = (t_1,t_2,t_3,t_4) \in T\colon t_4 \in a_{\mathbb{N}}\},$$

up to an isomorphism. By Lemma 4.1, W contains vectors determining a 2-column. We have to consider two cases:

I. Vectors determining a 2-column have the form (up to permutations of the first, second, and third coordinate axes):

$$\begin{array}{lllll} w_{1,\,1}: & a_1 & a_1 & a_1 & a_1, \\ w_{1,\,l}: & \alpha_1^{l-1} & a_1 & a_1 & a_l, \qquad l\geq 2. \end{array}$$

If $\alpha_1^i = a_1$ for all $i \geq 1$, then the vectors $w_{1,\,k}$, $k \geq 1$, determine a column. Suppose that $\alpha_1^i \neq a_1$ for some $i \geq 1$. We can assume that $\alpha_1^1 \neq a_1$. (If $\alpha_1^1 = a_1$ and for example $\alpha_1^2 \neq a_1$, then we replace the code ε_4 by ε_4', where $\varepsilon_4'|(a_1+\mathbb{Z}) = (2\ 3)\circ\varepsilon_4$

and $\varepsilon_4' = \varepsilon_4$ on the complement of the set $a_1 + \mathbb{Z}$.) Consider the vectors with codes $(l, 1, 1, 1)$, $l \geq 2$. By distinguishability, they have the form

$$w_{2,\,l-1}: \ \ a_l \quad \beta_1^{2l-3} \quad \beta_1^{2l-2} \quad a_1, \qquad l \geq 2.$$

The vectors $w_{2,\,l}$, $l \geq 1$, belong to W. If $\beta_1^i = a_1$ for all $i \geq 1$, then the vectors $w_{1,\,1}$ and $w_{2,\,l}$, $l \geq 1$, determine a column. Suppose that $\beta_1^i \neq a_1$ for some $i \geq 1$. We can assume that $\beta_1^1 \neq a_1$. (If $\beta_1^1 = a_1$ and for example $\beta_1^4 \neq a_1$, then we change the order of the second and third coordinates and replace the code ε_1 by ε_1', where $\varepsilon_1'|(a_1 + \mathbb{Z}) = (2\ 3) \circ \varepsilon_1$ and $\varepsilon_1' = \varepsilon_1$ on the complement of the set $a_1 + \mathbb{Z}$.) Take the vectors with codes $(2, l, 1, 1)$, $l \geq 2$. They are as follows

$$w_{3,\,l-1}: \ \ a_2 \quad \beta_l^1 \quad \gamma_1^{l-1} \quad a_1, \qquad l \geq 2.$$

The vectors $w_{3,\,l}$, $l \geq 1$, belong to W. If $\gamma_1^i = \beta_1^2$ for all $i \geq 1$, then the vectors $w_{2,\,1}$ and $w_{3,\,l}$, $l \geq 1$, determine a column. Suppose that $\gamma_1^i \neq \beta_1^2$ for some $i \geq 1$. We can assume that $\gamma_1^1 \neq \beta_1^2$. (If not, then we change the code ε_2 in an appropriate way.) Then we can also assume that $\gamma_1^1 \neq a_1$. (If $\gamma_1^1 = a_1$ and $\beta_1^2 \neq a_1$, then we replace the code ε_2 by ε_2', where $\varepsilon_2'|(\beta_1^1 + \mathbb{Z}) = (1\ 2) \circ \varepsilon_2$ and $\varepsilon_2' = \varepsilon_2$ on the complement of the set $\beta_1^1 + \mathbb{Z}$.) Consider the vectors with codes $(2, 2, l, 1)$, $l \geq 2$. By distinguishability, they have the form

$$w_{4,\,l-1}: \ \ a_2 \quad \beta_2^1 \quad \gamma_l^1 \quad a_1, \qquad l \geq 2.$$

The vectors $w_{4,\,l}$, $l \geq 1$, belong to W and together with $w_{3,\,1}$ determine a column.

II. Vectors determining a 2-column have the form (up to permutations of the first, second, and third coordinate axes):

$$\begin{array}{rlllll} w_{1,1}: & a_1 & a_1 & a_1 & a_1, & \\ w_{1,l}: & \alpha_1^{l-1} & a_l & a_1 & a_1, & \quad l \geq 2. \end{array}$$

If $\alpha_1^i = a_1$ for all $i \geq 1$, then the vectors $w_{1,\,k}$, $k \geq 1$, determine a column. Suppose that $\alpha_1^i \neq a_1$ for some $i \geq 1$. We can assume that $\alpha_1^1 \neq a_1$. (If $\alpha_1^1 = a_1$ and for example $\alpha_1^2 \neq a_1$, then we replace the code ε_2 by ε_2', where $\varepsilon_2'|(a_1 + \mathbb{Z}) = (2\ 3) \circ \varepsilon_2$ and $\varepsilon_2' = \varepsilon_2$ on the complement of the set $a_1 + \mathbb{Z}$.) Consider the vectors with codes $(l, 2, 1, 1)$, $l \geq 2$. By distinguishability, they have the form

$$w_{2,\,l-1}: \ \ \alpha_l^1 \quad a_2 \quad \beta_1^{2l-3} \quad \beta_1^{2l-2}, \qquad l \geq 2.$$

If $\beta_1^i = a_1$ for all $i \geq 1$, then the vectors $w_{2,\,l}$, $l \geq 1$, belong to W and together with $w_{1,\,2}$ determine a column. We have to consider the following three cases:

1. $\beta_1^i \neq a_1$ for some $i \in \{1, 3, \ldots\}$ and $\beta_1^i = a_1$ for $i = 2, 4, \ldots$.

Then the vectors $w_{1,\,l}$, $l \geq 1$, belong to W. We can assume that $\beta_1^1 \neq a_1$. (If $\beta_1^1 = a_1$ and for example $\beta_1^3 \neq a_1$, then we replace the code ε_1 by ε_1', where $\varepsilon_1'|(\alpha_1^1 + \mathbb{Z}) =$

$(2\ 3) \circ \varepsilon_1$ and $\varepsilon_1' = \varepsilon_1$ on the complement of the set $\alpha_1^1 + \mathbb{Z}$.) Consider the vectors with codes $(2, 2, l, 1)$, $l \geq 2$. They are as follows

$$u_{3,\,l-1}: \quad \alpha_2^1 \quad a_2 \quad \beta_l^1 \quad \gamma_1^{l-1}, \qquad l \geq 2.$$

If $\gamma_1^i = a_1$ for all $i \geq 1$, then the vectors $u_{3,\,l}$, $l \geq 1$, belong to W and together with $w_{2,\,1}$ determine a column. Suppose that $\gamma_1^i \neq a_1$ for some $i \geq 1$. We can assume that $\gamma_1^1 \neq a_1$. (If not, then we change the code ε_3 in an appropriate way.) Take the vectors with codes $(2, 2, 1, l)$, $l \geq 2$. They have the form

$$u_{4,\,l-1}: \quad \delta_2^{2l-3} \quad \delta_2^{2l-2} \quad \beta_1^1 \quad a_l, \qquad l \geq 2.$$

The vectors $u_{4,\,l}$, $l \geq 1$, belong to W. If $\delta_2^i = \alpha_2^1$ for $i = 1, 3, \ldots$ and $\delta_2^i = a_2$ for $i = 2, 4, \ldots$, then the vectors $w_{2,\,1}$ and $u_{4,\,l}$, $l \geq 1$, determine a column. Suppose that $\delta_2^i \neq \alpha_2^1$ for some $i \in \{1, 3, \ldots\}$. We can assume that $\delta_2^1 \neq \alpha_2^1$. (If $\delta_2^1 = \alpha_2^1$ and for example $\delta_2^3 \neq \alpha_2^1$, then we replace the code ε_4 by ε_4', where $\varepsilon_4'|(a_1 + \mathbb{Z}) = (2\ 3) \circ \varepsilon_4$ and $\varepsilon_4' = \varepsilon_4$ on the complement of the set $a_1 + \mathbb{Z}$.) Consider the vectors with codes $(l, 2, 1, 2)$, $l \in \mathbb{N} \setminus \{2\}$. By distinguishability, they can be written in the form

$$\begin{aligned} u'_{5,\,1}: &\quad \delta_1^1 \quad \eta_2^1 \quad\ \ \beta_1^1 \quad a_2, \\ u'_{5,\,l-1}: &\quad \delta_l^1 \quad \eta_2^{l-1} \quad \beta_1^1 \quad a_2, \qquad l \geq 3. \end{aligned}$$

The vectors $u'_{5,\,l}$, $l \geq 1$, belong to W. If $\eta_2^i = \delta_2^2$ for all $i \geq 1$, then the vectors $u_{4,\,1}$, $u'_{5,\,l}$, $l \geq 1$, determine a column. Suppose that $\eta_2^i \neq \delta_2^2$ for some $i \geq 1$. We can assume that $\eta_2^1 \neq \delta_2^2$. (If not, then we change the code ε_1 appropriately.) Then we can also assume that $\eta_2^1 \neq a_2$. (If $\eta_2^1 = a_2$ and $\delta_2^2 \neq a_2$, then we replace the code ε_1 by ε_1', where $\varepsilon_1'|(\delta_1^1 + \mathbb{Z}) = (1\ 2) \circ \varepsilon_1$ and $\varepsilon_1' = \varepsilon_1$ on the complement of the set $\delta_1^1 + \mathbb{Z}$.) Take the vectors with codes $(1, l, 1, 2)$, $l \in \mathbb{N} \setminus \{2\}$. They have the form

$$\begin{aligned} u'_{6,\,1}: &\quad \delta_1^1 \quad \eta_1^1 \quad \beta_1^1 \quad a_2, \\ u'_{6,\,l-1}: &\quad \delta_1^1 \quad \eta_l^1 \quad \beta_1^1 \quad a_2, \qquad l \geq 3. \end{aligned}$$

The vectors $u'_{6,\,l}$, $l \geq 1$, belong to W and together with $u'_{5,\,1}$ determine a column. Suppose now that $\delta_2^i \neq a_2$ for some $i \in \{2, 4, \ldots\}$. We can assume that $\delta_2^2 \neq a_2$. (If $\delta_2^2 = a_2$ and for example $\delta_2^4 \neq a_2$, then we replace the code ε_4 by ε_4', where $\varepsilon_4'|(a_1 + \mathbb{Z}) = (2\ 3) \circ \varepsilon_4$ and $\varepsilon_4' = \varepsilon_4$ on the complement of the set $a_1 + \mathbb{Z}$.) Consider the vectors with codes $(2, l, 1, 2)$, $l \in \mathbb{N} \setminus \{2\}$. By distinguishability, they are as follows

$$\begin{aligned} u_{5,\,1}: &\quad \eta_2^1 \quad\ \ \delta_1^2 \quad \beta_1^1 \quad a_2, \\ u_{5,\,l-1}: &\quad \eta_2^{l-1} \quad \delta_l^2 \quad \beta_1^1 \quad a_2, \qquad l \geq 3. \end{aligned}$$

The vectors $u_{5,\,l}$, $l \geq 1$, belong to W. If $\eta_2^i = \delta_2^1$ for all $i \geq 1$, then the vectors $u_{4,\,1}$, $u_{5,\,l}$, $l \geq 1$, determine a column. Suppose that $\eta_2^i \neq \delta_2^1$ for some $i \geq 1$. We can assume that $\eta_2^1 \neq \delta_2^1$. (If not, then we change the code ε_2 in an appropriate way.) Then we can also assume that $\eta_2^1 \neq \alpha_2^1$. (If $\eta_2^1 = \alpha_2^1$ and $\delta_2^1 \neq \alpha_2^1$, then we replace the code ε_2 by ε_2', where $\varepsilon_2'|(\delta_1^2 + \mathbb{Z}) = (1\ 2) \circ \varepsilon_2$ and $\varepsilon_2' = \varepsilon_2$ on the complement of the set $\delta_1^2+\mathbb{Z}$.) Now take the vectors with codes $(l,1,1,2)$, $l\in\mathbb{N}\setminus\{2\}$. They have the form

$$\begin{aligned} u_{6,\,1}:&\quad \eta_1^1 \quad \delta_1^2 \quad \beta_1^1 \quad a_2,\\ u_{6,\,l-1}:&\quad \eta_l^1 \quad \delta_1^2 \quad \beta_1^1 \quad a_2, \qquad l\geq 3.\end{aligned}$$

The vectors $u_{6,\,l}$, $l\geq 1$, belong to W and together with $u_{5,\,1}$ determine a column.

2. $\beta_1^i=a_1$ for $i=1,3,\ldots$ and $\beta_1^i\neq a_1$ for some $i\in\{2,4,\ldots\}$.

We can assume that $\beta_1^2\neq a_1$. (If $\beta_1^2=a_1$ and for example $\beta_1^4\neq a_1$, then we replace the code ε_1 by ε_1', where $\varepsilon_1'|(\alpha_1^1+\mathbb{Z})=(2\ 3)\circ\varepsilon_1$ and $\varepsilon_1'=\varepsilon_1$ on the complement of the set $\alpha_1^1+\mathbb{Z}$.) Consider the vectors with codes $(1,2,1,l)$, $l\geq 2$. By distinguishability from $w_{1,\,l}$, $l\geq 1$, and $w_{2,\,l}$, $l\geq 1$, they are as follows

$$v_{3,\,l-1}:\quad \alpha_1^1 \quad \gamma_2^{2l-3} \quad \gamma_1^{2l-2} \quad a_l, \qquad l\geq 2.$$

The vectors $v_{3,\,l}$, $l\geq 1$, belong to W. If $\gamma_2^i=a_2$ for $i=1,3,\ldots$ and $\gamma_1^i=a_1$ for $i=2,4,\ldots$, then vectors $w_{1,\,2}$, $v_{3,\,l}$, $l\geq 1$, determine a column. Suppose that $\gamma_2^i\neq a_2$ for some $i\in\{1,3,\ldots\}$. We can assume that $\gamma_2^1\neq a_2$. (If not, then we change the code ε_4 appropriately.) Consider the vectors with codes $(1,l,1,2)$, $l\in\mathbb{N}\setminus\{2\}$. They have the form

$$\begin{aligned} v_{4,\,1}':&\quad \alpha_1^1 \quad \gamma_1^1 \quad \delta_1^1 \quad a_2,\\ v_{4,\,l-1}':&\quad \alpha_1^1 \quad \gamma_l^1 \quad \delta_1^{l-1} \quad a_2, \qquad l\geq 3.\end{aligned}$$

The vectors $v_{4,\,l}'$, $l\geq 1$, belong to W. If $\delta_1^i=\gamma_1^2$ for all $i\geq 1$, then vectors $v_{3,\,1}$, $v_{4,\,l}'$, $l\geq 1$, determine a column. Suppose that $\delta_1^i\neq\gamma_1^2$ for some $i\geq 1$. We can assume that $\delta_1^1\neq\gamma_1^2$. (If not, then we change the code ε_2 in an appropriate way.) Then we can also assume that $\delta_1^1\neq a_1$. (If $\delta_1^1=a_1$ and $\gamma_1^2\neq a_1$, then we replace the code ε_2 by ε_2', where $\varepsilon_2'|(\gamma_1^1+\mathbb{Z})=(1\ 2)\circ\varepsilon_2$ and $\varepsilon_2'=\varepsilon_2$ on the complement of the set $\gamma_1^1+\mathbb{Z}$.) Take the vectors with codes $(1,1,l,2)$, $l\geq 2$. By distinguishability, they can be written in the form

$$v_{5,\,l-1}':\quad \alpha_1^1 \quad \gamma_1^1 \quad \delta_l^1 \quad a_2, \qquad l\geq 2.$$

The vectors $v_{5,\,l}'$, $l\geq 1$, belong to W and together with $v_{4,\,1}'$ determine a column. Now, suppose that $\gamma_1^i\neq a_1$ for some $i\in\{2,4,\ldots\}$. We can assume that $\gamma_1^2\neq a_1$. (If $\gamma_1^2=a_1$ and for example $\gamma_1^4\neq a_1$, then we replace the code ε_4 by ε_4', where $\varepsilon_4'|(a_1+\mathbb{Z})=(2\ 3)\circ\varepsilon_4$ and $\varepsilon_4'=\varepsilon_4$ on the complement of the set $a_1+\mathbb{Z}$.) Consider the vectors with codes $(1,2,l,2)$, $l\geq 2$. They have the form

$$v_{4,\,l-1}:\quad \alpha_1^1 \quad \delta_2^{l-1} \quad \gamma_l^2 \quad a_2, \qquad l\geq 2.$$

The vectors $v_{4,\,l}$, $l\geq 1$, belong to W. If $\delta_2^i=\gamma_2^1$ for all $i\geq 1$, then vectors $v_{3,\,1}$ and $v_{4,\,l}$, $l\geq 1$, determine a column. Suppose that $\delta_2^i\neq\gamma_2^1$ for some $i\geq 1$. We can assume that $\delta_2^1\neq\gamma_2^1$. (If not, then we change the code ε_3 in an appropriate way.) Then we can also assume that $\delta_2^1\neq a_2$. (If $\delta_2^1=a_2$ and $\gamma_2^1\neq a_2$, then we replace the code ε_3 by ε_3', where $\varepsilon_3'|(\gamma_1^2+\mathbb{Z})=(1\ 2)\circ\varepsilon_3$ and $\varepsilon_3'=\varepsilon_3$ on the complement of the set $\gamma_1^2 + \mathbb{Z}$.) Now, take the vectors with codes $(1, l, 2, 2)$, $l \in \mathbb{N} \setminus \{2\}$. By distinguishability, they are as follows

$$\begin{array}{llllll} v_{5,\,1}: & \alpha_1^1 & \delta_1^1 & \gamma_2^2 & a_2, & \\ v_{5,\,l-1}: & \alpha_1^1 & \delta_l^1 & \gamma_2^2 & a_2, & l \geq 3. \end{array}$$

The vectors $v_{5,\,l}$, $l \geq 1$, belong to W and together with $v_{4,\,1}$ determine a column.

3. $\beta_1^i \neq a_1$ for some $i \in \{1, 3, \ldots\}$ and $\beta_1^i \neq a_1$ for some $i \in \{2, 4, \ldots\}$.

We can assume that $\beta_1^1 \neq a_1$. (If $\beta_1^1 = a_1$ and for example $\beta_1^3 \neq a_1$, then we replace the code ε_1 by ε_1', where $\varepsilon_1'|(\alpha_1^1 + \mathbb{Z}) = (2\ 3) \circ \varepsilon_1$ and $\varepsilon_1' = \varepsilon_1$ on the complement of the set $\alpha_1^1 + \mathbb{Z}$.) Set $r \in \{2, 4 \ldots\}$ such that $\beta_1^r \neq a_1$. Consider the vectors with codes $(1, 2, 1, l)$, $l \geq 2$. By distinguishability from $w_{1,\,l}$, $l \geq 1$, and $w_{2,\,l}$, $l \geq 1$, they have the form

$$w_{3,\,l-1}: \quad \alpha_1^1 \quad \gamma_2^{2l-3} \quad \gamma_1^{2l-2} \quad a_l, \qquad l \geq 2.$$

The vectors $w_{3,\,l}$, $l \geq 1$, belong to W. If $\gamma_2^i = a_2$ for $i = 1, 3, \ldots$ and $\gamma_1^i = a_1$ for $i = 2, 4, \ldots$, then vectors $w_{1,\,2}$ and $w_{3,\,l}$, $l \geq 2$, determine a column. Suppose that $\gamma_2^i \neq a_2$ for some $i \in \{1, 3, \ldots\}$. We can assume that $\gamma_2^1 \neq a_2$. (If not, then we change the code ε_4 appropriately.) Consider the vectors with codes $(1, l, 1, 2)$, $l \in \mathbb{N} \setminus \{2\}$. They have the form

$$\begin{array}{llllll} w''_{4,\,1}: & \alpha_1^1 & \gamma_1^1 & \delta_1^1 & a_2, & \\ w''_{4,\,l-1}: & \alpha_1^1 & \gamma_l^1 & \delta_1^{l-1} & a_2, & l \geq 3. \end{array}$$

The vectors $w''_{4,\,l}$, $l \geq 1$, belong to W. If $\delta_1^i = \gamma_1^2$ for all $i \geq 1$, then vectors $w_{3,\,1}$ and $w''_{4,\,l}$, $l \geq 1$, determine a column. Suppose that $\delta_1^i \neq \gamma_1^2$ for some $i \geq 1$. We can assume that $\delta_1^1 \neq \gamma_1^2$. (If not, then we change the code ε_2 in an appropriate way.) Then we can also assume that $\delta_1^1 \neq a_1$. (If $\delta_1^1 = a_1$ and $\gamma_1^2 \neq a_1$, then we replace the code ε_2 by ε_2', where $\varepsilon_2'|(\gamma_1^1 + \mathbb{Z}) = (1\ 2) \circ \varepsilon_2$ and $\varepsilon_2' = \varepsilon_2$ on the complement of the set $\gamma_1^1 + \mathbb{Z}$.) Take the vectors with codes $(1, 1, l, 2)$, $l \geq 2$. They are as follows

$$w''_{5,\,l-1}: \quad \eta_1^{l-1} \quad \gamma_1^1 \quad \delta_l^1 \quad a_2, \qquad l \geq 2.$$

The vectors $w''_{5,\,l}$, $l \geq 1$, belong to W. If $\beta_1^{r-1} \neq \delta_1^1$, then, by distinguishability of $w_{2,\,\frac{r}{2}}$ and $w''_{5,\,l}$, $l \geq 1$, we have $\eta_1^i = \alpha_1^1$ for all $i \geq 1$ and the vectors $w''_{4,\,1}$, $w''_{5,\,l}$, $l \geq 1$, determine a column. Suppose that $\beta_1^{r-1} = \delta_1^1$ and $\eta_1^i \neq \alpha_1^1$ for some $i \geq 1$. We can assume that $\eta_1^1 \neq \alpha_1^1$. (If not, then we change the code ε_3 appropriately.) Consider the vectors with codes $(l, 1, 2, 2)$, $l \geq 2$. By distinguishability, they have the form

$$w''_{6,\,l-1}: \quad \eta_l^1 \quad \gamma_1^1 \quad \delta_2^1 \quad a_2, \qquad l \geq 2.$$

The vectors $w''_{6,\,l}$, $l \geq 1$, belong to W and together with $w''_{5,\,1}$ determine a column. Now, suppose that $\gamma_1^i \neq a_1$ for some $i \in \{2, 4, \ldots\}$. We can assume that $\gamma_1^2 \neq a_1$. (If $\gamma_1^2 = a_1$ and for example $\gamma_1^4 \neq a_1$, then we replace the code ε_4 by ε_4', where

$\varepsilon_4'|(a_1+\mathbb{Z}) = (2\ 3)\circ\varepsilon_4$ and $\varepsilon_4' = \varepsilon_4$ on the complement of the set $a_1+\mathbb{Z}$.) Consider the vectors with codes $(1,2,l,2)$, $l \geq 2$. They have the following form

$$w_{4,\,l-1}:\ \ \delta_1^{2l-3}\ \ \ \delta_2^{2l-2}\ \ \ \gamma_l^2\ \ \ a_2, \qquad l \geq 2.$$

The vectors $w_{4,\,l}$, $l \geq 1$, belong to W. If $\beta_1^{r-1} \neq \gamma_1^2$, then by distinguishability of $w_{2,\,\frac{r}{2}}$ and $w_{4,\,l}$, $l \geq 1$, we have $\delta_1^i = \alpha_1^1$ for $i = 1,3,\ldots$. If now $\delta_2^i = \gamma_2^1$ for $i = 2,4,\ldots$, then vectors $w_{3,\,1}$ and $w_{4,\,l}$, $l \geq 1$, determine a column. Suppose that $\delta_2^i \neq \gamma_2^1$ for some $i \in \{2,4,\ldots\}$. We can assume that $\delta_2^2 \neq \gamma_2^1$. (If not, then we change the code ε_3 in an appropriate way.) Then we can also suppose that $\delta_2^2 \neq a_2$. (If $\delta_2^2 = a_2$ and $\gamma_2^1 \neq a_2$, then we replace the code ε_3 by ε_3', where $\varepsilon_3'|(\gamma_1^2+\mathbb{Z}) = (1\ 2)\circ\varepsilon_3$ and $\varepsilon_3' = \varepsilon_3$ on the complement of the set $\gamma_1^2+\mathbb{Z}$.) Take the vectors with codes $(1,l,2,2)$, $l \in \mathbb{N}\setminus\{2\}$. They have the form

$$\begin{array}{llllll} \alpha_1^1 & \delta_1^2 & \gamma_2^2 & a_2, & & \\ \alpha_1^1 & \delta_l^2 & \gamma_2^2 & a_2, & & l \geq 3. \end{array}$$

These vectors belong to W and together with $w_{4,\,1}$ determine a column. Thus $\beta_1^{r-1} = \gamma_1^2$. If $\delta_1^i = \alpha_1^1$ for $i = 1,3,\ldots$ and $\delta_2^i = \gamma_2^1$ for $i = 2,4,\ldots$, then vectors $w_{3,\,1}$, $w_{4,\,l}$, $l \geq 1$, determine a column. Suppose that $\delta_2^i \neq \gamma_2^1$ for some $i \in \{2,4,\ldots\}$. We can assume that $\delta_2^2 \neq \gamma_2^1$. (If not, then we change the code ε_3 appropriately.) Consider the vectors with codes $(1,l,2,2)$, $l \in \mathbb{N}\setminus\{2\}$. By distinguishability, they are as follows

$$\begin{array}{llllll} w_{5,\,1}': & \eta_1^1 & \delta_1^2 & \gamma_2^2 & a_2, & \\ w_{5,\,l-1}': & \eta_1^{l-1} & \delta_l^2 & \gamma_2^2 & a_2, & l \geq 3. \end{array}$$

The vectors $w_{5,\,l}'$, $l \geq 1$, belong to W. If $\eta_1^i = \delta_1^1$ for all $i \geq 1$, then vectors $w_{4,\,1}$ and $w_{5,\,l}'$, $l \geq 1$, determine a column. Suppose that $\eta_1^i \neq \delta_1^1$ for some $i \geq 1$. We can assume that $\eta_1^1 \neq \delta_1^1$. (If not, then we change the code ε_2 in an appropriate way.) Then we can also assume that $\eta_1^1 \neq \alpha_1^1$. (If $\eta_1^1 = \alpha_1^1$ and $\delta_1^1 \neq \alpha_1^1$, then we replace the code ε_2 by ε_2', where $\varepsilon_2'|(\delta_1^2+\mathbb{Z}) = (1\ 2)\circ\varepsilon_2$ and $\varepsilon_2' = \varepsilon_2$ on the complement of the set $\delta_1^2+\mathbb{Z}$.) Take the vectors with codes $(l,1,2,2)$, $l \geq 2$. They have the form

$$w_{6,\,l-1}':\ \ \eta_l^1\ \ \delta_1^2\ \ \gamma_2^2\ \ \nu_2^{l-1}, \qquad l \geq 2.$$

If $\nu_2^i = a_2$ for all $i \geq 1$, then the vectors $w_{6,\,l}'$, $l \geq 1$, belong to W and together with $w_{5,\,1}'$ determine a column. Suppose that $\nu_2^i \neq a_2$ for some $i \geq 1$. We can assume that $\nu_2^1 \neq a_2$. (If not, then we change the code ε_1 in an appropriate way.) Then, by distinguishability of $w_{6,\,1}'$, $w_{1,\,1}$, and $w_{1,\,2}$, we obtain $\eta_2^1 = a_2$ and $\delta_1^2 = a_1$. Take the vectors with codes $(1,2,l,1)$, $l \geq 2$. By distinguishability, they are as follows

$$w_{7,\,l-1}':\ \ \alpha_1^1\ \ a_2\ \ a_l\ \ a_1, \qquad l \geq 2.$$

The vectors $w_{7,\,l}'$, $l \geq 1$, belong to W and together with $w_{1,\,2}$ determine a column. Now, suppose that $\delta_1^i \neq \alpha_1^1$ for some $i \in \{1,3,\ldots\}$. We can assume that $\delta_1^1 \neq \alpha_1^1$.

(If not, then we change the code ε_3 appropriately.) Consider the vectors with codes $(l, 2, 2, 2)$, $l \geq 2$. They have the form

$$w_{5,\, l-1}: \quad \delta_l^1 \quad \eta_2^{l-1} \quad \gamma_2^2 \quad a_2, \qquad l \geq 2.$$

The vectors $w_{5,\, l}$, $l \geq 1$, belong to W. If $\eta_2^i = \delta_2^2$ for all $i \geq 1$, then vectors $w_{4,\, 1}$, $w_{5,\, l}$, $l \geq 1$, determine a column. Suppose that $\eta_2^i \neq \delta_2^2$ for some $i \geq 1$. We can assume that $\eta_2^1 \neq \delta_2^2$. (If not, then we change the code ε_1 in an appropriate way.) Then we can also assume that $\eta_2^1 \neq \gamma_2^1$. (If $\eta_2^1 = \gamma_2^1$ and $\delta_2^2 \neq \gamma_2^1$, then we replace the code ε_1 by ε_1', where $\varepsilon_1' | (\delta_1^1 + \mathbb{Z}) = (1\ 2) \circ \varepsilon_1$ and $\varepsilon_1' = \varepsilon_1$ on the complement of the set $\delta_1^1 + \mathbb{Z}$.) Consider the vectors with codes $(2, l, 2, 2)$, $l \in \mathbb{N} \setminus \{2\}$. They are as follows

$$\begin{aligned} w_{6,\, 1}: &\quad \delta_2^1 \quad \eta_1^1 \quad \gamma_2^2 \quad \nu_2^1, \\ w_{6,\, l-1}: &\quad \delta_2^1 \quad \eta_l^1 \quad \gamma_2^2 \quad \nu_2^{l-1}, \qquad l \geq 3. \end{aligned}$$

If $\nu_2^i = a_2$ for all $i \geq 1$, then the vectors $w_{6,\, l}$, $l \geq 1$, belong to W and together with $w_{5,\, 1}$ determine a column. Suppose that $\nu_2^i \neq a_2$ for some $i \geq 1$. We can assume that $\nu_2^1 \neq a_2$. (If not, then we change the code ε_2 appropriately.) Then, by distinguishability of $w_{6,\, 1}$, $w_{1,\, 1}$, and $w_{1,\, 2}$, we obtain $\eta_1^1 = a_1$ and $\delta_2^1 = a_2$. Take the vectors with codes $(1, 2, l, 1)$, $l \geq 2$. By distinguishability, they have the form

$$w_{7,\, l-1}: \quad \alpha_1^1 \quad a_2 \quad a_l \quad a_1, \qquad l \geq 2.$$

The vectors $w_{7,\, l}$, $l \geq 1$, belong to W and together with $w_{1,\, 2}$ determine a column. □

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References

[1] J. Brakensiek, M. Heule, J. Mackey and D. Narváez, The Resolution of Keller's Conjecture, *arXiv*:1910.03740v3 [math.CO] 7 Apr 2020.

[2] J. Debroni, Aj. D. Eblen, M. A. Langston, W. Myrvold, P. W. Shor and D. Weerapurage, A complete resolution of the Keller maximum clique problem, *SODA* (2011), 129–135.

[3] M. Dutour Sikirić and M. Łysakowska, On the structure of two-periodic cube tilings of the 4-dimensional Euclidean space, *Utilitas Math.* (to appear).

[4] G. Hajós, Über einfache und mehrfache Bedeckung des n-dimensionalen Raumes mit einem Würfelgitter, *Math. Z.* **47** (1941), 427–467.

[5] O. H. Keller, Über die lückenlose Erfülung des Raumes Würfeln, *J. Reine Angew. Math.* **163** (1930), 231–248.

[6] A. P. Kisielewicz, On the structure of cube tilings of $\mathbb{R}^3$ and $\mathbb{R}^4$, *J. Combin. Theory Ser. A* **120** (2013), 1–10.

[7] A. P. Kisielewicz, Rigid polyboxes and Keller's conjecture, *Adv. Geom.* **17** (2017), 203–230.

[8] A. P. Kisielewicz and M. Łysakowska, On Keller's Conjecture in Dimension Seven, *Electron. J. Comb.* **22** (1) (2015), #P1.16.

[9] J. C. Lagarias and P. W. Shor, Keller's cube-tiling conjecture is false in high dimensions, *Bull. Amer. Math. Soc.* **27** (1992), 279–287.

[10] M. Łysakowska and K. Przesławski, On the structure of cube tilings and unextendible systems of cubes in low dimension, *European J. Combin.* **32** (2011), 1417–1427.

[11] M. Łysakowska and K. Przesławski, Keller's conjecture on the existence of columns in cube tilings of $\mathbb{R}^n$, *Adv. Geom.* **12** (2012), 329–352.

[12] J. Mackey, A cube tiling of dimension eight with no facesharing, *Discr. Comput. Geom.* **28** (2002), 275–279.

[13] H. Minkowski, *Geometrie der Zahlen*, Taubner, Leipzig, 1896.

[14] O. Perron, Über lückenlose Ausfüllung des n-dimensionalen Raumes durch kongruente Würfel I, II, *Math. Z.* **46** (1940), 1–26, 161–180.

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