AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **81(1)** (2021), Pages 58–88

The second neighborhood conjecture for oriented graphs missing a $\{C_4, \overline{C_4}, S_3, \text{chair and } \overline{\text{chair}}\}$-free graph

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Abstract

Seymour's Second Neighborhood Conjecture (SNC) asserts that every oriented graph has a vertex whose first out-neighborhood is at most as large as its second out-neighborhood. In this paper, we prove that if G is a graph containing no induced C_4, $\overline{C_4}$, S_3, chair and $\overline{\text{chair}}$, then every oriented graph missing G satisfies this conjecture. As a consequence, we deduce that the conjecture holds for every oriented graph missing a threshold graph, a generalized comb or a star.

1 Introduction

Throughout this paper, all graphs are considered to be simple, that is, there are no loops and no multiple edges. Given a graph G, the vertex-set and the edge-set of G are denoted by $V(G)$ and $E(G)$ respectively. Given an edge xy of G, the vertices x and y are called the endpoints of xy and they are said to be adjacent. Two edges of G are said to be adjacent if they have a common endpoint. The neighborhood of

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ISSN: 2202-3518

a vertex v in G, denoted by $N_G(v)$, is the set of all the vertices adjacent to v. The degree $d_G(v)$ of v in G is defined to be $d_G(v) := |N_G(v)|$. Note that we may omit the subscript if the graph is clear from the context. Given two sets of vertices U and W of G, we denote by $E[U, W]$ the set of all edges in G that join a vertex in U to a vertex in W. For $A \subseteq V(G)$, $G[A]$ denotes the subgraph of G induced by A. If $G[A]$ is an empty graph, then A is called a stable set, that is, there is no edge that joins any two distinct vertices of A. However, if $G[A]$ is a complete graph, then A is called a clique set, that is, any two distinct vertices of A are adjacent. The complement graph $\overline{G}$ of G is defined as follows: $V(\overline{G}) = V(G)$ and $xy \in E(\overline{G})$ if and only if $xy \notin E(G)$. A graph H is called a forbidden subgraph of G if H is not (isomorphic to) an induced subgraph of G. In this case, we say that G is an H-free graph.

A digraph D is an ordered pair $D = (V(D), E(D))$ where $V(D)$ is a non-empty set of elements called the vertices of D, and $E(D) \subseteq \{(x, y); x, y \in V(D)$ and $x \neq y\}$ and it is called the arc-set of D. Thus, a digraph contains neither loops nor multiple arcs. An oriented graph D is a digraph that contains no digons, that is, if $(x, y) \in E(D)$, then $(y, x) \notin E(D)$ for all $x, y \in V(D)$. In other words, an oriented graph is an orientation of a simple graph. Given a digraph D, for $(x, y) \in E(D)$ with $x, y \in V(D)$, we say that y is an out-neighbor of x, x is an in-neighbor of y and x and y are adjacent. The (first) out-neighborhood (respectively in-neighborhood) $N^+_D(v)$ (respectively $N^-_D(v)$) of a vertex v in D is the set of all out-neighbors (respectively in-neighbors) of v. Moreover, the second out-neighborhood $N^{++}_D(v)$ of v in D is the set of vertices that are at distance 2 from v, that is, $N^{++}_D(v) := \{x \in V(D) - N^+_D(v);$ $\exists\ y \in N^+_D(v)|\ (y, x) \in E(D)\}$. The out-degree, the in-degree and the second out-degree of v in D are defined as follows: $d^+_D(v) := |N^+_D(v)|$, $d^-_D(v) := |N^-_D(v)|$ and $d^{++}_D(v) := |N^{++}_D(v)|$, respectively. Note that we omit the subscript if the digraph is clear from the context. For short, we write $x \to y$ if the arc $(x, y) \in E(D)$. Also, we write $x_1 \to x_2 \to \cdots \to x_n$, if $x_i \to x_{i+1}$ for every $1 \leqslant i \leqslant n-1$.

Let D be an oriented graph and let $v \in V(D)$, we say that v has the second neighborhood property SNP if $d^+(v) \leq d^{++}(v)$. In 1990, Seymour (see [2]) conjectured the following:

Conjecture 1. *Every oriented graph has a vertex satisfying the SNP.*

The above conjecture is called "The Second Neighborhood Conjecture", and is abbreviated as "SNC". The SNC on tournaments is called Dean's conjecture, where tournaments are orientations of complete graphs. In 1996, Fisher [4] proved Dean's Conjecture. In 2000, a shorter proof of Dean's conjecture was given by Havet and Thomassé [10] using a tool called the median order. In 2007, Fidler and Yuster [3] proved the SNC for tournaments missing a matching, using local median orders and dependency digraphs. In 2012, Ghazal [7] proved the weighted version of SNC for tournaments missing a threshold graph. Then, in 2013, Ghazal [5] proved the SNC for tournaments missing a comb, a cycle of length 4 or 5. In 2015, Ghazal [6] refined the result of [3] and he showed in particular that every tournament missing a matching has a certain "feed vertex" satisfying the SNP.

In this paper, we prove the SNC for any oriented graph missing a graph G, where

G contains no C_4, $\overline{C_4}$, S_3, chair and $\overline{\text{chair}}$ as induced subgraphs. This generalizes the results of [7] and [5] that confirm the SNC for oriented graphs missing either a threshold graph or a comb, respectively. For this purpose, we introduce in Section 2 some necessary definitions and preliminary results established by Ghazal [7, 8] on the structure of threshold graphs, generalized combs and $\{C_4, \overline{C_4}, S_3$, chair and $\overline{\text{chair}}\}$-free graphs. Then in Section 3, as a key step, we characterize the graphs of our interest using dependency digraphs.

2 Definitions and Preliminaries

A chair is a graph G whose vertex-set is $V(G) = \{x, y, z, t, v\}$ and whose edge-set is $E(G) = \{xy, yz, zt, zv\}$. The complement of a chair is defined as $\overline{\text{chair}}$. We denote by $C_n = v_1v_2 \dots v_nv_1$ the cycle on n vertices, by $P_n = v_1v_2 \dots v_n$ the path on n vertices and by S_3 the graph on 6 vertices indicated in Figure 1. A graph G is a called a split graph if its vertex-set is the disjoint union of a stable set S and a clique set K. In this case, we write G is an $\{S,\ K\}$-split graph. For an $\{S,\ K\}$-split graph G, if $sx \in E(G)$ for all $s \in S$ and for all $x \in K$, then G is called a complete split graph. Otherwise if $E[S, K]$ forms a perfect matching of G, then G is called a perfect split graph.

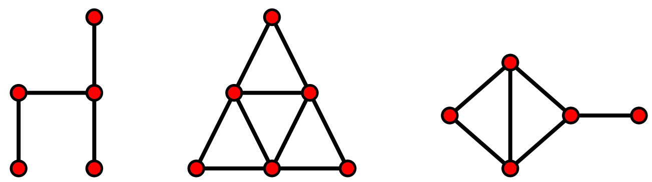

Figure 1: A chair, an S_3 and a $\overline{\text{chair}}$

In [9] and [1], the notion of a threshold graph (also known as a generalized star) is introduced as follows:

Definition 2.1. A graph G is called a threshold graph if:

1. $V(G) := \bigcup_{i=1}^{n+1}(X_i \cup A_{i-1})$, where the A_i and the X_i are pairwisely disjoint sets.

2. $K := \bigcup_{i=1}^{n+1} X_i$ is a clique and the X_i are nonempty, except possibly X_{n+1}.

3. $S := \bigcup_{i=0}^{n} A_i$ is a stable set and the A_i are nonempty, except possibly A_0.

4. For all i and j with $1 \leq j \leq i \leq n$, $G[A_i \cup X_j]$ is a complete split graph.

5. The only edges of G are the edges of the subgraphs mentioned above.

In this case, G is called an $\{S, K\}$-threshold graph.

On the structure of threshold graphs, Hammer and Chvàtal noticed the following:

Theorem 2.2. *(Hammer and Chvàtal [9], [1]) G is a threshold graph if and only if C_4, $\overline{C}_4$ and P_4 are forbidden subgraphs of G.*

As a generalization of threshold graphs, Ghazal introduced the notion of generalized combs as follows:

Definition 2.3. *(Ghazal [8])* A graph G is called a generalized comb if:

1. $V(G)$ is the disjoint union of the sets $A_0, \dots, A_n, M_1, \dots, M_l, X_1, \dots, X_{n+1}, Y_2, \dots, Y_{l+2}$ with $Y_1 = X_1$.
2. $S := A_0 \cup A \cup M$ is a stable set, where $M = \bigcup_{i=1}^{l} M_i$ and $A = \bigcup_{i=1}^{n} A_i$.
3. $K := X \cup Y$ is a clique, where $X = \bigcup_{i=1}^{n+1} X_i$ and $Y = \bigcup_{i=1}^{l+2} Y_i$.
4. For all i and j with $1 \leq j \leq i \leq n$, $G[A_i \cup X_j]$ is a complete split graph.
5. $G[A \cup Y]$ is a complete split graph.
6. For all i with $1 \leq i \leq l$, $G[Y_i \cup M_i]$ is a perfect split graph or $M_i = \emptyset$.
7. For all i and j with $1 \leq i < j \leq l + 1$, $G[Y_j \cup M_i]$ is a complete split graph.
8. $X_{n+1}, Y_{l+2}, Y_{l+1}$ and A_0 are the only possibly empty sets among the X_i, Y_i and A_i.
9. The only edges of G are the edges of the subgraphs mentioned above.

In this case, we say that G is an $\{S, K\}$-generalized comb.

On the structural properties of generalized combs, Ghazal proved in the same paper the following characterization:

Theorem 2.4. *(Ghazal [8]) G is a generalized comb if and only if C_4, $\overline{C}_4$, C_5, S_3, chair and $\overline{\text{chair}}$ are forbidden subgraphs of G.*

The next corollary shows that generalized combs can be viewed as generalizations of threshold graphs:

Corollary 2.5. *(Ghazal [8]) Every threshold graph is a generalized comb.*

Even though the converse of Corollary 2.5 is not necessarily true, the next proposition shows that every generalized comb contains a threshold graph as a subgraph:

Proposition 2.6. *Let G be a generalized comb defined as in Definition 2.3. Then $G' = G - \bigcup_{1\leq i\leq l} E[Y_i, M_i]$ is a threshold graph.*

Proof. According to Theorem 2.2, it is enough to prove that G' contains no induced C_4, $\overline{C}_4$ and P_4. Assume to the contrary that G' contains an induced C_4, say $C_4 = v_1v_2v_3v_4v_1$. Assume first that $v_1v_3 \notin E(G)$, then $v_2v_4 \in E(G)$, since otherwise C_4 would be an induced subgraph of G. Thus, according to Theorem 2.4, G is not a generalized comb, a contradiction. This implies that $v_1v_3 \in E(G)$ and so $v_1v_3 \in E[Y_i, M_i]$ for some $1 \leq i \leq l$, say $v_1 \in M_i$ and $v_3 \in Y_i$. But $v_1v_2 \in E(G')$ and so $v_1v_2 \in E(G)$ as well. Recall that the neighbors of each vertex in M are only in Y, due to Definition 2.3. Thus, it follows that $v_2 \in Y$. But $v_3 \in Y_i$, then $v_2v_3 \notin E(G)$ and so $v_2v_3 \notin E(G')$, a contradiction. This verifies that G' contains no induced C_4. Proceeding in a similar way, we can prove that G' contains no $\overline{C}_4$ and P_4 as induced subgraphs. This ends the proof. □

On the structure of graphs containing no C_4, $\overline{C_4}$, S_3, chair and $\overline{\text{chair}}$ as induced subgraphs, Ghazal exhibited a partition of their vertex-sets as follows:

Theorem 2.7. *(Ghazal [8]) C_4, $\overline{C_4}$, S_3, chair and $\overline{\text{chair}}$ are forbidden subgraphs of a graph G if and only if $V(G)$ is the disjoint union of three sets S, K and C such that:*

1. *$G[S \cup K]$ is an $\{S, K\}$-generalized comb.*

2. *$G[C]$ is empty or isomorphic to the cycle C_5.*

3. *Every vertex in C is adjacent to every vertex in K but to no vertex in S.*

From now on, if $G = C_5$, we set $G = xyzuvx$. If G is an $\{S, K\}$-generalized comb, we follow the same notation as in Definition 2.3. Moreover, if G is a $\{C_4$, $\overline{C_4}$, S_3, chair and $\overline{\text{chair}}\}$-free graph, we use the notation in Theorem 2.7. Note that if G is defined as in Theorem 2.7 and $G[C]$ is empty, then G is a generalized comb.

3 Characterization Using Dependency Digraphs

Let D be an oriented graph. For two vertices x and y of D, we say that xy is a missing edge of D if $(x, y) \notin E(D)$ and $(y, x) \notin E(D)$. A vertex v of D is called a whole vertex if it is not incident to any missing edge, i.e., $N^+(v) \cup N^-(v) = V(D) - \{v\}$. Otherwise, we say that v is a non-whole vertex. The missing graph G of D is defined to be the graph formed by the missing edges of D, formally, G is the graph whose edge-set is the set of all the missing edges of D and whose vertex-set is the set of the non-whole vertices. In this case, we say that D is missing G. Given two missing edges x_1y_1 and x_2y_2 of D, we say that x_1y_1 loses to x_2y_2 if: $x_1 \to x_2$ and $y_2 \notin N^+(x_1) \cup N^{++}(x_1)$, $y_1 \to y_2$ and $x_2 \notin N^+(y_1) \cup N^{++}(y_1)$. In this case, we say that there is a losing relation between the missing edges x_1y_1 and x_2y_2.

The dependency digraph Δ_D (or simply Δ) of D is defined to be the digraph whose vertex-set consists of all the missing edges of D, and whose arc-set contains the arc (ab, cd) if and only if the missing edge ab loses to the missing edge cd. Note that Δ may contain digons. These digraphs were used in [3, 5] to prove the SNC for some oriented graphs.

In [7], Ghazal distinguished between the missing edges as follows:

Definition 3.1. *(Ghazal [7]) A missing edge ab is called good if one of the following holds:*

(i) *For all $v \in V\backslash\{a, b\}$, if $v \to a$ then $b \in N^+(v) \cup N^{++}(v)$.*

(ii) *For all $v \in V\backslash\{a, b\}$, if $v \to b$ then $a \in N^+(v) \cup N^{++}(v)$.*

If ab satisfies (i), then (a, b) is said to be a convenient orientation of ab. Else, (b, a) is called a convenient orientation of ab.

The next lemma is an immediate consequence of Definition 3.1 and the definition of the losing relation between two missing edges:

Lemma 3.2. *(Ghazal [5]) Let D be an oriented graph and let Δ denote its dependency digraph. A missing edge ab is good if and only if its in-degree in Δ is zero.*

In [7], threshold graphs are characterized using dependency digraphs as follows:

Theorem 3.3. *(Ghazal [7]) Let G be a graph. The following statements are equivalent:*

(i) *G is a threshold graph.*

(ii) *Every missing edge of every oriented graph missing G is good.*

(iii) *The dependency digraph of every oriented graph missing G is empty.*

Given that $\mathcal{H}$ is a family of digraphs, a graph G is said to be *$\mathcal{H}$-forcing* if the dependency digraph of every oriented graph missing G is a member of $\mathcal{H}$. The set of all $\mathcal{H}$-forcing graphs is denoted by $\mathcal{F}(\mathcal{H})$. Formally, $\mathcal{F}(\mathcal{H}) = \{G$ *is a graph; for all oriented graph D missing G*, $\Delta_D \in \mathcal{H}\}$.

Denoting by $\mathcal{E}$ the class of all empty digraphs, the characterization of threshold graphs presented in Theorem 3.3 can be restated in terms of $\mathcal{E}$ as follows: *The only $\mathcal{E}$-forcing graphs are the generalized stars.* Recall that Ghazal [7] proved the SNC for every oriented graph missing a generalized star. This motivates us to go further and ask about the characterization of $\overrightarrow{\mathcal{P}}$-forcing graphs for the family of vertex disjoint directed paths $\overrightarrow{\mathcal{P}}$:

Problem 1. *Let $\overrightarrow{\mathcal{P}}$ be the family of all digraphs consisting of vertex disjoint directed paths and let $\mathcal{F}(\overrightarrow{\mathcal{P}}) = \{G$ is a graph; for all D missing G, $\Delta_D \in \overrightarrow{\mathcal{P}}\}$. Characterize $\mathcal{F}(\overrightarrow{\mathcal{P}})$.*

The next proposition establishes a relation between a $\overrightarrow{\mathcal{P}}$-forcing graph and its induced subgraphs:

Proposition 3.4. *$G \in \mathcal{F}(\overrightarrow{\mathcal{P}})$ if and only if $G' \in \mathcal{F}(\overrightarrow{\mathcal{P}})$, for every induced subgraph G' of G.*

Proof. To prove the sufficient condition, simply take $G' = G$. For the necessary condition, we assume first that $G' = G - v$ for some $v \in V(G)$, and we consider an oriented graph D' missing G'. We construct from D' an oriented graph D whose missing graph is G in the following way: Add to the vertex-set of D' the vertices v, α and β, where α and β are two distinct extra vertices that are neither in D' nor in G. In other words, the vertex-set of D is $V(D) = V(D') \cup \{v, \alpha, \beta\}$. Add to the arc-set of D' the arcs (α, v), (v, β) and (α, β), the arcs (x, α) and (β, x) for every $x \in V(D')$, and the arcs (x, v) for every $x \in V(D')$ in case that $xv \notin E(G)$. The addition of v, α and β to D' in this way neither affects the losing relations between the missing edges of D' nor creates new ones. This means that Δ_D is equal to $\Delta_{D'}$ plus a set of isolated vertices that correspond to the edges of G incident to v. According to the facts that G is a $\overrightarrow{\mathcal{P}}$-forcing graph and D is missing G, it follows that $\Delta_D \in \overrightarrow{\mathcal{P}}$. Whence, $\Delta_{D'} \in \overrightarrow{\mathcal{P}}$ and so $G' \in \mathcal{F}(\overrightarrow{\mathcal{P}})$. This proves the case where $G' = G - v$. The rest of the proof follows by induction on the number of vertices removed from G to obtain the induced subgraph G'. □

A quick verification leads to this easy observation:

Proposition 3.5. *$\overline{C}_4$, chair and $\overline{\text{chair}}$ are not in $\mathcal{F}(\overrightarrow{\mathcal{P}})$.*

Proof. Let D be the oriented graph whose vertex-set is $V(D) = \{a, b, c, d\}$ and whose arc-set is $E(D) = \{(a, c), (b, d), (d, a), (c, b)\}$. Observe that D is missing $\overline{C}_4$, ab loses to cd and cd loses to ba. Thus $\Delta_D \notin \overrightarrow{\mathcal{P}}$ and so $\overline{C}_4 \notin \mathcal{F}(\overrightarrow{\mathcal{P}})$. To prove that chairs are not $\overrightarrow{\mathcal{P}}$-forcing graphs, consider the oriented graph D' whose vertex-set is $V(D') = \{a, b, c, d, x\}$ and whose arc-set is $E(D') = \{(a, d), (b, c), (c, a), (b, x), (x, a), (x, c)\}$. Observe that D' is missing a chair, ab loses to both dc and dx. Thus $\Delta_{D'} \notin \overrightarrow{\mathcal{P}}$. It remains to prove that $\overline{\text{chair}} \notin \mathcal{F}(\overrightarrow{\mathcal{P}})$. For this sake, let D'' be the oriented graph with vertex-set $V(D'') = \{a, b, c, d, x\}$ and arc-set $E(D'') = \{(a, c), (b, d), (d, a), (a, x)\}$. It is easy to see that D'' is missing a $\overline{\text{chair}}$, ab loses to both dc and dx. Thus $\Delta_{D''} \notin \overrightarrow{\mathcal{P}}$. □

Based on the two propositions above and Theorem 2.4, we are able to prove the following characterization on generalized combs:

Theorem 3.6. *Let G be a $\{C_4, C_5, S_3\}$-free graph. Then $G \in \mathcal{F}(\overrightarrow{\mathcal{P}})$ if and only if G is a generalized comb.*

Proof. To prove the first implication, observe that Proposition 3.4, Proposition 3.5 and the fact that $G \in \mathcal{F}(\overrightarrow{\mathcal{P}})$ imply that G contains no $\overline{C}_4$, chair and $\overline{\text{chair}}$ as induced subgraphs of G. But C_4, C_5 and S_3 are also not induced subgraphs of G, then Theorem 2.4 forces G to be a generalized comb.

To verify the second implication, consider an oriented graph D missing a generalized comb G and denote by Δ its dependency digraph. Using the definition of G, each possible losing relation can occur only between two edges in $E[Y_t, M_t]$ for some $1 \leq t \leq l$. Suppose now that $a_ix_i \in E[Y_t, M_t]$ with $a_i \in M_t$ and $x_i \in Y_t$ for $i \in \{1, 2, 3\}$. Assume to the contrary that a_1x_1 loses to the two other edges. This gives that $a_1 \to x_3$, $x_1 \to a_2$, $a_2 \notin N^+(a_1) \cup N^{++}(a_1)$ and $x_3 \notin N^+(x_1) \cup N^{++}(x_1)$. By the definition of G, the only edge of $G[Y_t \cup M_t]$ incident to a_2 is a_2x_2. Thus, a_2x_3 is not a missing edge and so either $a_2 \to x_3$ or $x_3 \to a_2$. Whence, either $x_3 \in N^+(x_1) \cup N^{++}(x_1)$ or $a_2 \in N^+(a_1) \cup N^{++}(a_1)$, a contradiction to the initial assumption. This implies that the maximum out-degree of Δ is 1. Similarly, it can be proved that the maximum in-degree of Δ is 1. This gives that Δ is composed of directed cycles and paths only. It remains to prove that Δ contains no directed cycles. Assume that the contrary is true, and let $a_1b_1 \to \cdots \to a_nb_n \to a_1b_1$ be a directed cycle in Δ with $a_i \in M_t$ and $b_i \in Y_t$ for some t. Due to the losing relations, it is easy to see that $a_{i+1} \to a_i$ for all $i < n$ and $a_1 \to a_n$ in D. We will show now by induction on i that $a_i \to a_n$ for all $1 \leq i < n$. The case $i = 1$ is true. Assume that $a_{i-1} \to a_n$. Because a_ia_n is not a missing edge of D, we must have $a_i \to a_n$, since otherwise $a_{i-1} \to a_n \to a_i$ in D, which contradicts the losing relation between $a_{i-1}b_{i-1}$ and a_ib_i. This proves that for all $1 \leq i < n$, $a_i \to a_n$. In particular, $a_{n-1} \to a_n$, a contradiction. Therefore, Δ has no directed cycles and thus $G \in \mathcal{F}(\overrightarrow{\mathcal{P}})$. $\square$

The next proposition shows the dependency digraphs structure of oriented graphs missing the cycle C_5 of length 5:

Proposition 3.7. *C_5 is a $\overrightarrow{\mathcal{P}} \cup \overrightarrow{\mathcal{C}_5}$-forcing graph, where $\overrightarrow{\mathcal{C}_5} = \{\overrightarrow{C_5}\}$ and $\overrightarrow{C_5}$ is the directed cycle of length* 5*. That is, $C_5 \in \mathcal{F}(\overrightarrow{\mathcal{P}} \cup \overrightarrow{\mathcal{C}_5})$.*

Proof. Let D be an oriented graph missing C_5 and let Δ denote its dependency digraph. Assume that a vertex in Δ, say (without loss of generality) xy, loses to two others. Since xu, xz, yu and yv are the only non-missing edges of $D[V(C_5)]$ incident with x and y, then we may assume that xy loses to uv and zu. Due to the losing relation $xy \to zu$, we have $y \to u$ in D and so $u \in N^+(y) \cup N^{++}(y)$, a contradiction to the losing relation $xy \to uv$. Thus the maximum out-degree in Δ is 1. Similarly, we can prove that the maximum in-degree in Δ is 1. This implies that Δ is composed of directed cycles and paths only.
We claim that the only possible directed cycle in Δ is of length 5. Assume to the contrary that Δ contains a directed cycle of length 2, say $xy \to uv \to xy$, then D contains a directed cycle of length 2 $x \to u \to x$, a contradiction. Assume now that Δ contains a directed cycle of length 3, say $xy \to uv \to yz \to xy$. Thus $y \to x$ in D, a contradiction to the fact that yx is a missing edge of D. Finally, assume that Δ contains a directed cycle of length 4, say $xy \to uv \to yz \to vx \to xy$, then $v \to x$ in

D, a contradiction. This proves our claim. This implies that Δ is either a directed cycle of length 5 or the union of vertex disjoint directed paths.
For the case where Δ is formed of the union of vertex disjoint directed paths, Δ has at most 4 arcs because there are only 5 missing edges. We may check by cases that one of the following occurs up to isomorphism:

(i) Δ has no arcs.

(ii) Δ has exactly one arc, say $uv \to xy$.

(iii) Δ has exactly two arcs, say $uv \to xy$ and $xv \to yz$.

(iv) Δ has exactly two arcs, say $uv \to xy \to zu$.

(v) Δ has exactly three arcs, say $uv \to xy \to zu \to vx$.

(vi) Δ has exactly three arcs, say $uv \to xy \to zu$ and $xv \to zy$. □

In view of the preceding, we get the following about $\{C_4, \overline{C_4}, S_3$, chair and $\overline{\text{chair}}\}$-free graphs:

Corollary 3.8. *Let G be a graph such that $V(G)$ is the disjoint union of three sets S, K and C such that:*

1. *$G[S \cup K]$ is an $\{S, K\}$-generalized comb;*

2. *$G[C]$ is empty or isomorphic to the cycle C_5;*

3. *every vertex in C is adjacent to every vertex in K but to no vertex in S.*

Then $G \in \mathcal{F}(\overrightarrow{\mathcal{P}} \cup \overrightarrow{\mathcal{C}_5})$. That is, G is a $\overrightarrow{\mathcal{P}} \cup \overrightarrow{\mathcal{C}_5}$- forcing graph.

Proof. Observe that every edge in $E(G) - E(C)$ is incident to a vertex in K. This implies that there is no losing relation between an edge in $E(C)$ and an edge in $E(G) - E(C)$, since otherwise there is an edge ab with $a \in C$, $b \in K$ and $ab \notin E(G)$. This contradicts the fact that every vertex in C is adjacent in G to every vertex in K. In the same way, we can prove that there is no losing relation between an edge in $E(G[S \cup K])$ and an edge in $E[K, C]$, or between two edges in $E[K, C]$. Thus the only possible losing relations hold either between two edges in $G[S \cup K]$ or between two edges in $G[C]$. However, $G[S \cup K]$ is a generalized comb, then by Theorem 3.6 it is in $\mathcal{F}(\overrightarrow{\mathcal{P}})$. Moreover, $G[C]$ is empty or isomorphic to the cycle C_5, whence by Proposition 3.7 it is in $\mathcal{F}(\overrightarrow{\mathcal{P}} \cup \overrightarrow{\mathcal{C}_5})$. Therefore, $G \in \mathcal{F}(\overrightarrow{\mathcal{P}} \cup \overrightarrow{\mathcal{C}_5})$. □

4 Main theorem

Let $L = v_1 v_2 \dots v_n$ be an ordering of the vertices of a digraph D. An arc $(v_i, v_j) \in E(D)$ is called forward with respect to L if $i < j$. Otherwise, it is called backward with respect to L. L is called a median order of D if it maximizes the set of forward arcs of D with respect to L, that is, the set $\{(v_i, v_j) \in E(D); \ i < j\}$. An interval $[x_i; x_j]$ of L is the part of L that contains all the vertices $x_i, x_{i+1}, \dots, x_j$. The last vertex of a median order is called the *feed* vertex. The following well-known proposition gives a fundamental property of median orders, called the feedback property:

Proposition 4.1. *Let $L = x_1 x_2 \dots x_n$ be a median order of a digraph D. For each interval $I = [x_i; x_j]$ of L, we have:*

$$d_I^+(x_i) \geq d_I^-(x_i); \tag{1}$$

$$d_I^-(x_j) \geq d_I^+(x_j). \tag{2}$$

We will use frequently what follows, in the proof of our main theorem.

Lemma 4.2. *Suppose that rs loses to ab with $s \to b$ in an oriented graph D. If $f \to a$ in D and fs is not a missing edge of D, then $f \to s \to b$ in D and thus $b \in N^+(f) \cup N^{++}(f)$.*

Proof. Since fs is not a missing edge, then either $f \to s$ or $s \to f$ in D. If $s \to f$ in D, then $s \to f \to a$ in D and thus $a \in N^{++}(s)$, which contradicts the fact that rs loses to ab. Thus $f \to s$; whence the result follows. □

Proposition 4.3. *Suppose that $L = v_1 v_2 \dots v_n$ is a median order of a digraph D and $e = (v_j, v_i) \in E(D)$ with $i < j$. Then L is a median order of the digraph D' obtained from D by reversing the orientation of e.*

We will heavily use the following theorem:

Theorem 4.4. *(Havet et al. [10]) Every feed vertex of a tournament has the SNP.*

Now we are ready to prove our main theorem:

Theorem 4.5. *Let D be an oriented graph missing a $\{C_4$, $\overline{C_4}$, S_3, chair and $\overline{\text{chair}}\}$-free graph G. Then D satisfies the SNC.*

Proof. Let D be an oriented graph whose missing graph G is a $\{C_4, \overline{C_4}, S_3$, chair and $\overline{\text{chair}}\}$-free graph. For the vertex-set of G, we assume that $V(G)$ is the disjoint union of three sets K, S and C as mentioned in Theorem 2.7. Let Δ denote the dependency digraph of D and let $\Delta[E(C)]$ denote the subdigraph of Δ induced by the subset of the vertices of Δ that correspond to the edges of $G[C]$. Due to the proof of Corollary 3.8, it follows that either Δ consists of the union of vertex disjoint directed paths whose arcs occur between two edges in the same set $E[Y_j, M_j]$ for some $1 \leq j \leq l$ or between two edges of C only, or Δ consists of the disjoint union of vertex disjoint directed paths whose arcs occur only between two edges in the same set $E[Y_j, M_j]$ for some $1 \leq j \leq l$ and a directed cycle C_5 of length 5 whose arcs occur between two edges of C. Let $P = m_0y_0 \to \cdots \to m_iy_i \to \cdots \to m_ky_k$ be a maximal directed path in Δ, with $m_i \in M_j$ $y_i \in Y_j$ and $k \geq 0$. Due to the maximality of P and due to Lemma 3.2, we get that m_0y_0 is a good missing edge of D. If (m_0, y_0) is a convenient orientation, we add the arcs (m_{2i}, y_{2i}) and (y_{2i+1}, m_{2i+1}) to D. Else, we add to D the arcs (y_{2i}, m_{2i}) and (m_{2i+1}, y_{2i+1}). In each main case below, we do this procedure for every maximal directed path in Δ whose vertices are edges in $E[Y_j, M_j]$. To complete the proof, we distinguish between the possible cases of $\Delta[E(C)]$.

Case I. *$G[C]$ is empty, that is, $\Delta[E(C)]$ is empty.*

In this case, the obtained oriented graph D' is missing $G' = G - \cup E[Y_j, M_j]$ which is a threshold graph by Proposition 2.6. We assign to every missing edge of D' (which is good by Theorem 3.3) a convenient orientation and we add it to D'. The obtained oriented graph T is a tournament. Let L be a median order of T and let f denote its feed vertex. Then, by Theorem 4.4, f has the SNP in T. Reorienting all the missing edges incident to f towards f except those whose out-degree in Δ is not zero results in a tournament T'. It can be easily seen, by Proposition 4.3, that L is also a median order of T'. Whence, f has the SNP in T'. We will prove that f has the SNP in D also. For this aim, we consider many cases.

Case I.1. *f is a whole vertex.*

Clearly, f gains no new out-neighbors in T' by comparison to D. We will prove that f gains no new second out-neighbors in T' by comparison to D. To this end, we assume that $f \to a \to b \to f$ in T'. Since f is a whole vertex, then $f \to a$ and $b \to f$ in D. If $a \to b$ in D or (a, b) is a convenient orientation with respect to D, then $b \in N^{++}_D(f)$. If $a \to b$ in $D' - D$ and (a, b) is not a convenient orientation with respect to D, then there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N^{++}_D(s)$. But $f \to a$ in D and fs is a non-missing edge of D, then Lemma 4.2 gives that $b \in N^{++}_D(f)$. If $a \to b$ in $T' - D'$, then (a, b) is a convenient orientation w.r.t D'. Hence, $b \in N^{++}_{D'}(f)$. This implies the existence of a vertex a' such that $f \to a' \to b$ in D'. Since $f \to a'$ in D' and f is a whole vertex, then $f \to a'$ in D. But this is already treated above, then $b \in N^{++}_D(f)$.

Case I.2. *$f \in M_t$ for some $1 \leq t \leq l$, that is, there is a maximal directed path $P = m_0y_0 \to \cdots \to m_iy_i \to \cdots \to m_ky_k$ in Δ such that $f = m_i$.*

Case I.2.1. $(y_i, m_i) \in E(D')$. Clearly, f gains no new first out-neighbors in T' by comparison to D. To see the previous fact, recall that all the missing edges of D incident to f are directed towards f in T' except those whose out-degree in Δ is not zero. However, due to the structure of G, the only missing edges of D with possible out-degree greater than zero in Δ are the edges in $E[Y_t, M_t]$ for some $1 \leqslant t \leqslant l$ because the possible losing relations occur only between two edges in $E[Y_t, M_t]$. Thus, the only missing edge of D incident to m_i with possible out-degree greater than zero in Δ is the edge y_im_i. But y_i is an in-neighbor of m_i in D' and so in T' as well. We claim that f gains no new second out-neighbors in T' by comparison to D. Assume that $m_i \to a \to b \to m_i$ in T'. Then $(m_i, a) \in E(D)$ and $(a, b) \in E(T)$. We consider the following three possibilities:

Subcase I.2.1.a. If $(a, b) \in E(D)$, then clearly $b \in N_D^{++}(f)$.

Subcase I.2.1.b. If $(a, b) \in E(D') - E(D)$, then either (a, b) is a convenient orientation with respect to D and hence $b \in N_D^{++}(f)$, or there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N_D^{++}(s)$. Then there is an integer $j \in \{1, 2, \ldots, l\}$ such that $rs, ab \in E[Y_j, M_j]$. Clearly, m_i is distinct from a and b. We will prove that m_i is also different from r and s. Assume first that $m_i = r$. It follows that $y_i = s$, $a = y_{i+1}$ and $b = m_{i+1}$. Since $(y_i, m_i) \in E(D')$, then $(m_{i+1}, y_{i+1}) \in E(D')$, that is, $(b, a) \in E(D')$, a contradiction. This means that m_i cannot be equal to r. Assume now that $m_i = s$. This gives that $r = y_i$, $a = m_{i+1}$ and $b = y_{i+1}$. Since M is stable in G, then $m_i m_{i+1}$ is not a missing edge of D. Consequently, due to the losing relation $y_i m_i \to m_{i+1} y_{i+1}$, we get that $(m_{i+1}, m_i) \in E(D)$ and so $(a, m_i) \in E(D)$, a contradiction. Thus m_i is not s also. Now we prove that $m_i s$ is not a missing edge of D. If $b \in Y_j$, then $s \in M_j$ and thus $m_i s$ is not a missing edge of D. Else if $b \in M_j$, then $a \in Y_j$ and $s \in Y_j$. But $m_i a$ is not a missing edge of D, then the definition of G gives that $m_i s$ is also not a missing edge of D. But $f \to a$ in D and $a \notin N_D^{++}(s)$, then Lemma 4.2 implies that $b \in N_D^{++}(f)$.

Subcase I.2.1.c. If $(a, b) \in E(T) - E(D')$, then (a, b) is a convenient orientation w.r.t D'. Since $f \to a$ in D, then $f \to a$ in D' and so $b \in N_{D'}^{++}(f)$. But this is already treated above in Subcase I.2.1.a and Subcase I.2.1.b.

Case I.2.2. $(m_i, y_i) \in E(D')$. Here there are two cases to be consider.

Case I.2.2.1. *$i = k$, that is, $f = m_k$.* Note that possibly $k = 0$. In this case, the missing edge $m_k y_k$ has out-degree zero in Δ. Thus $(y_k, m_k) \in E(T')$. This implies that f gains no new out-neighbors in T' by comparison to D. We will prove that f gains no new second out-neighbors in T' by comparison to D. Suppose $f \to a \to b \to f$ in T'. Then $(f, a) \in E(D)$ and $(a, b) \in E(T)$.

Subcase I.2.2.1.a. If $(a, b) \in E(D)$, then clearly $b \in N_D^{++}(f)$.

Subcase I.2.2.1.b. If $(a, b) \in E(D') - E(D)$, then either (a, b) is a convenient orientation with respect to D and hence $b \in N_D^{++}(f)$, or there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N_D^{++}(s)$. Then there is $1 \leq j \leq l$ such that $rs, ab \in E[Y_j, M_j]$. Since $f = m_k$, we have $r \neq m_k$ and $s \neq m_k$. If $b \in Y_j$, then $s \in M_j$. Then $m_k s$ is not a missing edge of D. Else if $b \in M_j$, then $a \in Y_j$ and $s \in Y_j$. Since $m_k a$ is not a missing edge of D, then the definition of G gives that $fs = m_k s$ is also not a missing edge of D. But $f \to a$ in D and $a \notin N_D^{++}(s)$, then by Lemma 4.2 we get $b \in N_D^{++}(f)$.

Subcase I.2.2.1.c. If $(a, b) \in E(T) - E(D')$, then (a, b) is a convenient orientation with respect to D' and so $b \in N_{D'}^{++}(f)$. Then there is a vertex a' such that $m_k \to a' \to b$ in D'. Since $(m_k, a) \in E(D)$, then $a \neq y_k$ and for all $j > t, a \notin Y_j$. If we prove that $(m_k, a') \in E(D)$, this case will be reduced to Subcase I.2.2.1.a and Subcase I.2.2.1.b. Thus it is sufficient now to show that $(m_k, a') \in E(D)$, that is, $m_k a'$ is not a missing edge of D. To this end, we need to prove that $a' \neq y_k$. Assume to the contrary that $a' = y_k$. Then $(y_k, b) \in E(D')$ and $b \neq m_k$. This implies that $(y_k, b) \notin E[Y_j, M_j]$ for all $1 \leqslant j \leqslant l$. Thus the edge $y_k b$ is not a missing edge of D and so $(y_k, b) \in E(D)$. This means that $b \notin A \cup X \cup Y \cup C$. Then either b is a whole vertex or $b \in M$. If b is a whole vertex, then ab is not a missing edge of D, a contradiction. Thus, $b \in M$ and so there exists an integer α such that $b \in M_\alpha$. If $\alpha < t$, then the definition of G forces $y_k b$ to be an edge in G, that is, $y_k b$ is a missing edge of D, a contradiction. Thus $\alpha \geq t$. Since $b \in M_\alpha$ with $\alpha \geq t$ and ab is a missing edge of D', then by the definition of G, $a \in Y_j$ for some $j > \alpha$. Thus $a \in Y_j$ for some $j > t$, a contradiction. This proves that $a' \neq y_k$ and so $m_k a'$ is not a missing edge of D.

Case I.2.2.2. $i < k$. In this case, f gains only y_i as a first out-neighbor in T' by comparison to D. We will prove that f gains only m_{i+1} as a second out-neighbor in T' by comparison to D.

Subcase I.2.2.2.a. Suppose that $m_i \to y_i \to b$ in T' such that $b \neq m_{i+1}$. Then $(y_i, b) \notin E(D') - E(D)$ and $(y_i, b) \in E(T)$.

Subcase I.2.2.2.a.1. If $(y_i, b) \in E(D)$, then $y_i b$ is not a missing edge of D. But $y_i \in Y_t$, then $y_{i+1} \in Y_t$. Then, by the definition of G, $y_{i+1} b$ is not a missing edge of D. Since $y_i \to b$ in D and $y_{i+1} \notin N_D^{++}(y_i)$, then we must have $y_{i+1} \to b$ in D. Thus, $m_i \to y_{i+1} \to b$ in D.

Subcase I.2.2.2.a.2. If $(y_i, b) \in E(T) - E(D')$, then (y_i, b) is a convenient orientation with respect to D'. Then $b \in N_{D'}^{++}(m_i)$, that is, there is vertex a such that $m_i \to a \to b$ in D'.

Subcase I.2.2.2.a.2.1. If $a = y_i$, then by the definition of G, $(y_i, b) \in E(D)$, a contradiction to the fact that $y_i b$ is a missing edge of D.

Subcase I.2.2.2.a.2.2. If $a \neq y_i$, then $(m_i, a) \in E(D)$. If $(a, b) \in E(D)$, then clearly $b \in N_D^{++}(f)$. Else $(a, b) \in E(D') - E(D)$, then either (a, b) is a convenient orientation with respect to D and hence $b \in N_D^{++}(f)$, or there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N_D^{++}(s)$. Thus $\exists\ j$ such that $rs, ab \in E[Y_j, M_j]$. Assume that $r = m_i$, then $y_i = s$, $a = y_{i+1}$ and $b = m_{i+1}$, a contradiction to the fact that $b \neq m_{i+1}$. Thus, $r \neq m_i$. Assume now that $s = m_i$, then $a = m_{i+1}$. However, $(m_{i+1}, m_i) \in E(D)$, then $(a, m_i) \in E(D)$, a contradiction. So $s \neq m_i$. Now we will prove that $m_i s$ is not a missing edge. If $b \in Y_j$, then $s \in M_j$ and thus $m_i s$ is not a missing edge of D. Else if $b \in M_j$, then $a \in Y_j$ and $s \in Y_j$. Since $m_i a$ is not a missing edge of D, then by the definition of G, $fs = m_i s$ is also not a missing edge of D. But $f \to a$ in D and $a \notin N_D^{++}(s)$, then by Lemma 4.2 we get that $b \in N_D^{++}(f)$.

Subcase I.2.2.2.b. Suppose that $m_i \to a \to b$ in T', with $a \neq y_i$ and $b \neq m_{i+1}$. Then $(m_i, a) \in E(D)$ and $(a, b) \in E(T)$.

Subcase I.2.2.2.b.1. If $(a, b) \in E(D')$, we proceed in a similar way to Subcase I.2.2.2.a.2.2.

Subcase I.2.2.2.b.2. If $(a, b) \in E(T) - E(D')$, then (a, b) is a convenient orientation with respect to D' and thus $b \in N^{++}_{D'}(f)$. This means that there is a vertex a' such that $m_i \to a' \to b$ in D'. If $a' = y_i$, then $(y_i, b) \in E(D)$. This case is already treated in Subcase I.2.2.2.a.1. Else, we proceed in the same method as in Subcase I.2.2.2.a.2.2.

Case I.3. *$f \in Y_t$ for some $1 \leq t \leq l$ such that $M_t \neq \emptyset$, that is, there is a maximal directed path $P = m_0y_0 \to \cdots \to m_iy_i \to \cdots \to m_ky_k$ in Δ such that $f = y_i$.*

Case I.3.1. $(m_i, y_i) \in E(D')$. Clearly, f gains no new first out-neighbors in T' by comparison to D. We will prove that f gains no new second out-neighbors in T' by comparison to D. Assume that $y_i \to a \to b \to y_i$ in T'. Then $(y_i, a) \in E(D)$ and $(a, b) \in E(T)$.

Subcase I.3.1.a. If $(a, b) \in E(D)$, then clearly $b \in N^{++}_D(f)$.

Subcase I.3.1.b. If $(a, b) \in E(D') - E(D)$, then either (a, b) is a convenient orientation with respect to D and hence $b \in N^{++}_D(f)$, or there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N^{++}_D(s)$. Thus $\exists\ j$ such that $rs, ab \in E[Y_j, M_j]$. Assume first that $r = y_i$. Then $s = m_i$, $a = m_{i+1}$ and $b = y_{i+1}$. Since $(m_i, y_i) \in E(D')$, then $(y_{i+1}, m_{i+1}) \in E(D')$, that is, $(b, a) \in E(D')$, a contradiction. Then $r \neq y_i$. Assume now that $s = y_i$. It follows that $a = y_{i+1}$ and $y_ia = y_iy_{i+1}$ is a missing edge of D, a contradiction. So $s \neq y_i$. Now we prove that y_is is not a missing edge of D. If $a \in Y_j$, then y_ia is a missing edge of D, a contradiction. So $a \in M_j$ and hence $s \in M_j$. Since y_ia is not a missing edge of D, then by the definition of G, y_is is also not a missing edge of D. Since $y_i \to a$ in D and $a \notin N^{++}(s)$, then by Lemma 4.2 we get that $b \in N^{++}_D(f)$.

Subcase I.3.1.c. If $(a, b) \in E(T) - E(D')$, then (a, b) is a convenient orientation with respect to D'. Whence, $b \in N^{++}_{D'}(f)$. Then there is a vertex a' such that $y_i \to a' \to b$ in D'. Since $(m_i, y_i) \in E(D')$, then $(y_i, a') \in E(D)$. This case is already treated in Subcase I.3.1.a and Subcase I.3.1.b.

Case I.3.2. $(y_i, m_i) \in E(D')$.

Case I.3.2.1. *$i = k$, that is, $f = y_k$.* Note that possibly $k = 0$. Clearly, f gains no new out-neighbors in T' by comparison to D. We prove that f gains no new second out-neighbors in T' by comparison to D. Suppose that $f \to a \to b \to f$ in T'. Then $(f, a) \in E(D)$ and $(a, b) \in E(T)$.

Subcase I.3.2.1.a. If $(a, b) \in E(D)$, then clearly $b \in N^{++}_D(f)$.

Subcase I.3.2.1.b. If $(a, b) \in E(D') - E(D)$, then either (a, b) is a convenient orientation with respect to D and hence $b \in N^{++}_D(f)$, or there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N^{++}_D(s)$. Then $\exists\ j$ such that $rs, ab \in E[Y_j, M_j]$. Since $f = y_k$, then $r \neq y_k$ and $s \neq y_k$. Since $(y_k, a) \in E(D)$, then $a \notin Y$. Whence, $a \in M$ and $s \in M$. Since y_ka is not a missing edge of D, then by the definition of G, y_ks is also not a missing edge of D. Since $f \to a$ in D and $a \notin N^{++}_D(s)$, then by Lemma 4.2 we get $b \in N^{++}_D(f)$.

Subcase I.3.2.1.c. If $(a, b) \in E(T) - E(D')$, then it is a convenient orientation with respect to D'. But $f \to a$ in D and thus in D', then $b \in N^{++}_{D'}(f)$. So there is a vertex a' such that $f \to a' \to b$ in D'.

Subcase I.3.2.1.c.1. If $a' = m_k$, then $a'b$ is not a missing edge of D. Since $y_k a$ is not a missing edge of D, then either a is a whole vertex of D or $a \in M - \{m_k\}$. Since $(a, b) \in E(T) - E(D)$, then a is not a whole vertex. Thus $\exists\ j$ such that $a \in M_j - \{m_k\}$. The definition of G together with the facts that $f = y_k \in Y_t$, $a \in M_j - \{m_k\}$ and $y_k a$ is not a missing edge imply that $j \geq t$. Since $(a, b) \notin E(D')$ and $a \in M_j$, then $\exists\ \alpha > j$ such that $b \in Y_\alpha$. Thus $ba' \in E[Y_\alpha, M_t]$ with $\alpha > t$. So, by using the definition of G, $a'b$ is a missing edge of D and D', a contradiction since $(a', b) \in E(D')$.

Subcase I.3.2.1.c.2. If $a' \neq m_k$, then $(y_k, a') \in E(D)$. But $(a', b) \in E(D')$, this is already discussed in Subcase I.3.2.1.a and Subcase I.3.2.1.b.

Case I.3.2.2. *$i < k$, that is, $f = y_i$ for some $i < k$.* Clearly, f gains only m_i as an out-neighbor in T' by comparison to D. We will prove that f gains only y_{i+1} as a second out-neighbor in T' by comparison to D.

Subcase I.3.2.2.a. Suppose that $f \to m_i \to b \to f$ in T' with $b \neq y_{i+1}$. Then $(y_i, m_i) \in E(D')$ and $(m_i, b) \in E(T)$.

Subcase I.3.2.2.a.1. If $(m_i, b) \in E(D)$, then $m_i b$ is not a missing edge of D. But $b \neq y_{i+1}$, then by the definition of G, $m_{i+1} b$ is also not a missing edge of D. Since $m_i \to b$ in D and $m_{i+1} \notin N^{++}(m_i)$, then we must have $m_{i+1} \to b$ in D. Thus $y_i \to m_{i+1} \to b$ in D.

Subcase I.3.2.2.a.2. If $(m_i, b) \in E(D') - E(D)$, then $m_i b = m_i y_i$ and hence $b = y_i$, a contradiction. So this case does not hold.

Subcase I.3.2.2.a.3. If $(m_i, b) \in E(T) - E(D')$, then (m_i, b) is a convenient orientation with respect to D'. Since $y_i \to m_i$ in D', then $b \in N^{++}_{D'}(y_i)$. Then there is a vertex a' such that $y_i \to a' \to b$ in D'. If $a' = m_i$, then $(m_i, b) \in E(D')$, a contradiction. Thus $a' \neq m_i$ and so $(y_i, a') \in E(D)$ and $(a', b) \in E(D')$.

Subcase I.3.2.2.a.3.1. If $(a', b) \in E(D)$, then $y_i \to a' \to b$ in D.

Subcase I.3.2.2.a.3.2. If $(a', b) \in E(D') - E(D)$, then either (a', b) is a convenient orientation with respect to D and hence $b \in N^{++}_D(f)$, or there is $rs \to a'b$ in Δ, namely $s \to b$ and $a' \notin N^{++}_D(s)$. If $rs = m_i y_i$, then $a'b = m_{i+1} y_{i+1}$. But $b \neq y_{i+1}$, then $b = m_{i+1}$. Since $(y_i, m_i) \in E(D')$, then $(m_{i+1}, y_{i+1}) \in E(D')$, that is, $(b, a') \in E(D')$, a contradiction. Thus, m_i is neither r nor s. Now we claim that $y_i s$ is not a missing edge of D. Since $y_i a'$ is not a missing edge, then $a' \notin Y$. Whence, $a' \in M$ and $s \in M$. Therefore, using the definition of G and the fact that $y_i a'$ is not a missing edge of D, we reach our claim. Since $y_i \to a'$ in D and $a' \notin N^{++}_D(s)$, then by Lemma 4.2 we get that $b \in N^{++}_D(f)$.

Subcase I.3.2.2.b. Assume that $y_i \to a \to b \to y_i$ in T', with $a \neq m_i$ and $b \neq y_{i+1}$. Then $(y_i, a) \in E(D)$ and $(a, b) \in E(T)$. If $(a, b) \in E(D')$, this is already treated in

Subcase I.3.2.2.a.3.1 and Subcase I.3.2.2.a.3.2. Else if $(a, b) \in E(T) - E(D')$, then it has a convenient orientation with respect to D' and hence $b \in N^{++}_{D'}(y_i)$. Then there is a vertex a' such that $y_i \to a' \to b$ in D'. If $a' = m_i$, this is already treated in Subcase I.3.2.2.a.1 and Subcase I.3.2.2.a.2. Else if $a' \neq m_i$, this is already treated in Subcase I.3.2.2.a.3.1 and Subcase I.3.2.2.a.3.2.

Case I.4. *$f \in Y_t$ for some $1 \leq t \leq l + 1$ such that $M_t = \emptyset$.* Clearly, f gains no new out-neighbor in T' by comparison to D. We will prove that it gains no new second out-neighbor in T' by comparison to D. To this end, suppose that $f \to a \to b \to f$ in T'. Then $(f, a) \in E(D)$ and $(a, b) \in E(T)$. We consider the following cases.

Subcase I.4.a. If $(a, b) \in E(D)$, then clearly $b \in N^{++}_D(f)$.

Subcase I.4.b. If $(a, b) \in E(D') - E(D)$, then either (a, b) is a convenient orientation with respect to D and hence $b \in N^{++}_D(f)$, or there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N^{++}_D(s)$. Then $\exists\ j$ such that $rs, ab \in E[Y_j, M_j]$. Since fa is not a missing edge of D, then $a \notin Y$. Whence, $a \in M_j$ and $s \in M_j$. Since $a, s \in M_j$ and fa is not a missing edge of D, then fs is not also a missing edge. But $f \to a$ in D and $a \notin N^{++}_D(s)$, then by Lemma 4.2 we get that $b \in N^{++}_D(f)$.

Subcase I.4.c. If $(a, b) \in E(T) - E(D')$, then (a, b) is a convenient orientation with respect to D'. But $f \to a$ in D and thus in D', then $b \in N^{++}_{D'}(f)$. This induces the existence of a vertex a' such that $f \to a' \to b$ in D'. Then $(f, a') \in E(D)$. But this is already treated in Subcase I.4.a and Subcase I.4.b.

Case I.5. $f \in Y_{l+2}$. This case is exactly the same as Case I.4, with only one difference: In Subcase I.4.b, we have proved that fs is not a missing edge of D. However, in Subcase I.5.b, there is no need to verify it, because $E[Y_{l+2}, M_j] = \emptyset$ by the definition of G.

Case I.6. $f \in V(G) - (Y \cup M)$. Clearly, f gains no new out-neighbors in T' by comparison to D. We will prove that it gains no new second out-neighbors in T' by comparison to D. For this aim, we suppose that $f \to a \to b \to f$ in T'. Then $(f, a) \in E(D)$ and $(a, b) \in E(T)$. We consider the following cases.

Subcase I.6.a. If $(a, b) \in E(D)$, then clearly $b \in N^{++}_D(f)$.

Subcase I.6.b. If $(a, b) \in E(D') - E(D)$, then either (a, b) is a convenient orientation with respect to D and hence $b \in N^{++}_D(f)$, or there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N^{++}_D(s)$. Thus $\exists\ j$ such that $rs, ab \in E[Y_j, M_j]$. If $a \in Y_j$, then fa is a missing edge of D, a contradiction. So $a \in M_j$ and hence $s \in M_j$. Then fs is not a missing edge of D. But $f \to a$ in D and $a \notin N^{++}_D(s)$, then by Lemma 4.2 we get that $b \in N^{++}_D(f)$.

Subcase I.6.c. If $(a, b) \in E(T) - E(D')$, then (a, b) is a convenient orientation with respect to D'. But $f \to a$ in D and so in D', then $b \in N^{++}_{D'}(f)$. Thus there is a vertex a' such that $f \to a' \to b$ in D'. Then $(f, a') \in E(D)$ and $(a', b) \in E(D')$. But this is already discussed in Subcase I.6.a and Subcase I.6.b.

On the grounds of the preceding, f has the SNP in D if the missing graph of D is a generalized comb. This completes the proof of Case I. $\Diamond$

Case II. $\Delta[E(C)]$ *contains exactly one arc, say* $uv \to xy$.

Assume without loss of generality that (u, v) is a convenient orientation of the good missing edge uv. We add to D the arcs (u, v) and (x, y), we assign to the good missing edges xv, yz and zu a convenient orientation and then we add them to D. The obtained oriented graph D' is missing $G' = G - (\cup E[Y_j, M_j] \cup E(C))$ which is a threshold graph. We assign to the missing edges of D' convenient orientations and we add them to D' to get a tournament T. Let L be a median order of T and let f denote its feed vertex. Reorient all the missing edges incident to f towards f, except those whose out-degree in Δ is not zero. The same order L is a median order of the obtained tournament T', f is also a feed vertex of L and thus f has the SNP in T'. We will prove that f has the SNP in D as well. For this purpose, we consider the following cases.

Case II.1. *f is a whole vertex.* This is the same as Case I.1.

Case II.2. *There exists* $1 \le t \le l$ *such that* $f \in M_t$. Exactly same as Case I.2, with only one difference in the subcases where $f \to a \to b$, $(f, a) \in E(D)$ and $(a, b) \in E(D') - E(D)$ such that it is not convenient with respect to D. Such subcase is called unsteady. As usual, since (a, b) is not convenient with respect to D, then there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N^{++}_D(s)$. The difference is that in the unsteady subcases of Case II.2 either $rs, ab \in E[Y_j, M_j]$ for some j or $rs = uv, ab = xy$. If $rs, ab \in E[Y_j, M_j]$ for some j, we proceed exactly in the same way as in the unsteady subcases of Case I.2. Else if $(r, s) = (u, v)$ and $(a, b) = (x, y)$, then fs is not a missing edge of D because $E[M, C] = \emptyset$. Since $f \to a$ in D and $a \notin N^{++}_D(s)$, then by Lemma 4.2 we get that $b \in N^{++}_D(f)$.

Case II.3. *There is* $1 \le t \le l$ *such that* $f \in Y_t$ *and* $M_t \ne \emptyset$. Exactly same as Case I.3, with only one difference in the unsteady subcases, that is, in the cases where $f \to a \to b$, $(f, a) \in E(D)$ and $(a, b) \in E(D') - E(D)$ such that it is not convenient with respect to D. As usual, since (a, b) is not convenient with respect to D, then there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N^{++}_D(s)$. The difference is that in the unsteady subcases of Case II.3 there are two cases to be consider: Either $rs, ab \in E[Y_j, M_j]$ for some j, or $rs = uv$ and $ab = xy$. If $rs, ab \in E[Y_j, M_j]$ for some j, we proceed exactly in the same way as in the unsteady subcases of Case I.3. Else if $(r, s) = (u, v)$ and $(a, b) = (x, y)$, then $fa = fx$ is a missing edge of D because $G[Y \cup C]$ is a complete split graph, which contradicts the definition of G.

Case II.4. *There is* $1 \le t \le l + 1$ *such that* $f \in Y_t$ *and* $M_t = \emptyset$. Exactly same as Case I.4, with only one difference. In Subcase II.4.b, there are two possibilities for the edges rs, ab: Either $rs, ab \in E[Y_j, M_j]$ for some j, or $(r, s) = (u, v)$ and $(a, b) = (x, y)$. The first case is already treated in Subcase I.4.b. However, the second case does not exist, since otherwise $fa = fx$ would be a missing edge of D because $G[Y \cup C]$ is a complete split graph, which contradicts the fact that $(f, a) \in E(D)$.

Case II.5. $f \in Y_{l+2}$. Exactly same as Case I.4, with exactly two differences. The first difference is that in Subcase II.5.b there are two possibilities for the edges rs, ab: Either $rs, ab \in E[Y_j, M_j]$ for some j, or $(r, s) = (u, v)$ and $(a, b) = (x, y)$. The first case is already treated in Subcase I.4.b. However, the second case does not exist, since otherwise $fa = fx$ would be a missing edge of D because $G[Y \cup C]$ is a complete split graph, which contradicts the fact that $(f, a) \in E(D)$. The second difference is that in Subcase II.5.b there is no need to prove that fs is not a missing edge of D because $E[Y_{l+2}, M_j] = \emptyset$ by the definition of G, while in Subcase I.4.b we have proved it.

Case II.6. $f = u$. Clearly, u gains only v as a new first out-neighbor in T' by comparison to D. We will prove that it gains only y as a new second out-neighbor in T' by comparison to D.

Case II.6.1. $u \to v \to b \to u$ *in* T'*, with* $b \neq y$. Then $(v, b) \in E(T) - E(T')$. Note that $b \neq x$ because $u \to x$ in D while $b \to u$ in T'.

Subcase II.6.1.a. If $(v, b) \in E(D)$, then either $b = z$, $b \in S$ or b is a whole vertex. Then by the losing relation $uv \to xy$, we get that $b \in N_D^{++}(u)$.

Subcase II.6.1.b. If $(v, b) \in E(D') - E(D)$, then $b = x$ or $b = u$, a contradiction. Thus this case does not exist.

Subcase II.6.1.c. If $(v, b) \in E(T) - E(D')$, then $b \in K$ and (v, b) is a convenient orientation with respect to D'. Then there exists a vertex v' such that $u \to v' \to b \to u$ in D'. Since $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C$. Since $v' \notin C$ and $(f, v') \in E(D')$, then $(f, v') \in E(D)$.

Subcase II.6.1.c.1. If $(v', b) \in E(D)$, then $b \in N_D^{++}(f)$.

Subcase II.6.1.c.2. If $(v', b) \in E(D') - E(D)$, then (v', b) is a convenient orientation w.r.t D and hence $b \in N_D^{++}(f)$, or there is $rs \to v'b$ in Δ, namely $s \to b$ and $v' \notin N_D^{++}(s)$. Since $v' \notin C$, then $\exists\ j$ such that $rs, v'b \in E[Y_j, M_j]$. Since $b \in K$, then $b \in Y_j$. Whence, $v' \in M_j$ and $s \in M_j$. Thus, by the definition of G, fs is not a missing edge of D. But $(f, v') \in E(D)$ and $v' \notin N_D^{++}(s)$, then by Lemma 4.2 we get that $b \in N_D^{++}(f)$.

Case II.6.2. $u \to a \to b \to u$ *in* T'*, with* $a \neq v$ *and* $b \neq y$. Then $(u, a) \in E(D)$ and $(a, b) \in E(T)$. Note that $a \notin K \cup \{u, v, y, z\}$ and $b \notin \{u, v, x, y\}$.

Subcase II.6.2.a. If $(a, b) \in E(D)$, then $b \in N_D^{++}(u)$.

Subcase II.6.2.b. If $(a, b) \in E(D') - E(D)$, then either $ab \in E(C)$ or $ab \in E[Y_j, M_j]$ for some j. If $ab \in E(C)$, then $(a, b) = (x, z)$ because $a \notin \{u, v, y, z\}$ and $b \notin \{u, v, y, x\}$. Thus xz is a missing edge of D, a contradiction. It follows that $ab \in E[Y_j, M_j]$ for some j. Then either (a, b) is a convenient orientation with respect to D and hence $b \in N_D^{++}(u)$, or there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N_D^{++}(s)$. Since $a \notin K$, then $a \in M_j$ and thus $s \in M_j$. Hence, by the definition of G, us is not a missing edge of D. But $(u, a) \in E(D)$ and $a \notin N_D^{++}(s)$, then by Lemma 4.2 we get that $b \in N_D^{++}(u)$.

Subcase II.6.2.c. If $(a, b) \in E(T) - E(D')$, then $b \in K$ and (a, b) is a convenient orientation with respect to D'. Since $u \to a$ in D and so in D', there exists a vertex v' such that $u \to v' \to b$ in D'. Since $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C$. Since $v' \notin C$ and $(u, v') \in E(D')$, then $(u, v') \in E(D)$. But this is already treated in Subcase II.6.1.c.1 and Subcase II.6.1.c.2.

Case II.7. $f \in C - \{u\}$. It is clear that f gains no new first out-neighbors in T' by comparison to D. We will prove that it gains no new second out-neighbors in T' by comparison to D. Suppose that $f \to a \to b \to f$ in T'. Then $(f, a) \in E(D)$, $(a, b) \in E(T)$ and $a \notin K$.

Subcase II.7.a. If $(a, b) \in E(D)$, then $b \in N_D^{++}(u)$.

Subcase II.7.b. If $(a, b) \in E(D') - E(D)$, then either (a, b) is a convenient orientation with respect to D and hence $b \in N_D^{++}(f)$, or there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N_D^{++}(s)$. If $(r, s) = (u, v)$ and $(a, b) = (x, y)$, then $f \notin \{x, y\}$. Note that $f \neq v$, since otherwise $fa = vx$ is a missing edge of D, a contradiction. Since $f \in C - \{u, v, x, y\}$, then $f = z$ and hence $fs = zv$ is not a missing edge of D. Else if $rs, ab \in E[Y_j, M_j]$ for some j, then $r \neq f$ and $s \neq f$. Since $a \notin K$, then $a \notin Y_j$ and so $a \in M_j$. Whence, $s \in M_j$. Thus fs is not missing edge of D by the definition of G. Therefore, by the losing relation $rs \to ab$ in Δ, we get that $b \in N_D^{++}(f)$.

Subcase II.7.c. If $(a, b) \in E(T) - E(D')$, then $b \in K$ and (a, b) is a convenient orientation with respect to D'. So there is a vertex v' such that $f \to v' \to b$ in D'. Since $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C$. Since $v' \notin C$ and $(f, v') \in E(D')$, then $(f, v') \in E(D)$. But this is already treated in Subcase II.6.1.c.1 and Subcase II.6.1.c.2.

Case II.8. $f \in V(G) - (Y \cup M \cup C) = A \cup (X - X_1) = A \cup (X - Y_1)$. Exactly same as Case I.6, with only one difference. The difference is that in Subcase II.8.b there are two possibilities for the edges rs, ab: Either $rs, ab \in E[Y_j, M_j]$ for some j, or $(r, s) = (u, v)$ and $(a, b) = (x, y)$. The first case is already treated in Subcase I.6.b. However, for the case where $(r, s) = (u, v)$ and $(a, b) = (x, y)$, f must belong to A, since otherwise f would be a vertex of X and so $fx = fa$ must be a missing edge of D, because $G[X \cup C]$ is a complete split graph, which contradicts the fact that $(f, a) \in E(D)$. Thus $fs = fv$ is not a missing edge of D, because $E[A, C]$ is empty by the definition of G. Since $f \to a$ in D and $a \notin N_D^{++}(s)$, then by Lemma 4.2 we get that $b \in N_D^{++}(f)$.

Therefore, f has the SNP in D when Δ has exactly one arc between two edges of C. $\Diamond$

Case III. $\Delta[E(C)]$ *has exactly two arcs, say* $uv \to xy$ *and* $vx \to yz$.

In this case, $u \to x \to z \to v \to y$ in D and uv, vx, uz are good missing edges of D. Assume without loss of generality that (u, v) is a convenient orientation with respect to D. Add the arcs (u,v) and (x,y) to D. If (v,x) is a convenient orientation of vx, then add the arcs (v,x) and (y,z). Otherwise, add the arcs (x,v) and (z,y). Then assign to uz a convenient orientation and add it to D. The obtained oriented graph D' is missing $G' = G - (\cup E[Y_j, M_j] \cup E(C))$ which is a threshold graph. Thus, due to Theorem 3.3, all the missing edges of D' are good. We assign to them a convenient orientation and we add them to D'. The resultant digraph is a tournament, say T. Let L be a median order of T and let f denote its feed vertex. Reorient all the missing edges incident to f towards f except those whose out-degree in Δ is not zero. The same order L is again a median order of the obtained tournament T'. Due to Theorem 4.4, f has the SNP in T'. We will prove that f has the SNP in D as well. We consider the following cases.

Case III.1. *f is a whole vertex.* This is the same as Case I.1.

Case III.2. *There is $1 \leq t \leq l$ such that $f \in M_t$.* Exactly same as Case I.2, with only one difference in the unsteady subcases, that is, in the subcases where $f \to a \to b$, $(f,a) \in E(D)$ and $(a,b) \in E(D') - E(D)$ and it is not convenient with respect to D. As usual, since (a,b) is not convenient with respect to D, then there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N_D^{++}(s)$. The difference is that in the unsteady subcases of Case III.2 either $rs, ab \in E[Y_j, M_j]$ for some j or $rs, ab \in E(C)$. If $rs, ab \in E[Y_j, M_j]$ for some j, we proceed exactly in the same way as in the unsteady subcases of Case I.2. Else if $rs, ab \in E(C)$, there are many cases: $(r,s) = (u,v)$ and $(a,b) = (x,y)$, $(r,s) = (v,x)$ and $(a,b) = (y,z)$ if (v,x) is a convenient orientation of vx, or $(r,s) = (x,v)$ and $(a,b) = (z,y)$ if (x,v) is a convenient orientation of vx. For all these cases, fs is not a missing edge of D because $E[M,C] = \emptyset$. Since $f \to a$ in D and $a \notin N_D^{++}(s)$, then by Lemma 4.2 we get that $b \in N_D^{++}(f)$.

Case III.3. *There is $1 \leq t \leq l$ such that $f \in Y_t$ and $M_t \neq \emptyset$.* Exactly same as Case I.3, with only one difference in the subcases where $f \to a \to b$, $(f,a) \in E(D)$ and $(a,b) \in E(D') - E(D)$ such that (a,b) is not convenient with respect to D. As usual, since (a,b) is not convenient with respect to D, then there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N_D^{++}(s)$. The difference is that in the unsteady subcases of Case III.3, there are two cases to consider: Either rs, $ab \in E[Y_j, M_j]$ for some j, or rs, $ab \in E(C)$. If rs, $ab \in E[Y_j, M_j]$ for some j, we proceed in the same way as that of the unsteady subcases of Case I.3. Else if rs, $ab \in E(C)$, then fa is a missing edge of D because $G[Y \cup C]$ is a complete split graph, a contradiction. Thus this case does not exist.

Case III.4. *There exists $1 \leq t \leq l+1$ such that $f \in Y_t$ and $M_t = \emptyset$.* Exactly same as Case I.4, with only one difference. In Subcase III.4.b, there are two possibilities for the edges rs and ab instead of the one of Subcase I.4.b: Either rs, $ab \in E[Y_j, M_j]$ for some j, or rs, $ab \in E(C)$. The first case is already treated in Subcase I.4.b. However, the second case does not exist, since otherwise fa would be a missing edge of D because $G[Y \cup C]$ is a complete split graph, which contradicts the fact that $(f,a) \in E(D)$.

Case III.5. *$f \in Y_{l+2}$.* This case is similar to Case I.4, except for two differences.

The first difference is that in Subcase III.5.b there are two possibilities for the edges rs and ab: Either rs, $ab \in E[Y_j, M_j]$ for some j, or rs, $ab \in E(C)$. The first case is already treated in Subcase I.4.b. However, the second case does not exist, since otherwise fa would be a missing edge of D because $G[Y \cup C]$ is a complete split graph, which is a contradiction to the fact that $(f, a) \in E(D)$. The second difference is that in Subcase III.5.b fs is not a missing edge of D because $E[Y_{l+2}, M_j] = \emptyset$, while in Subcase I.4.b we have proved it.

Case III.6. $f = u$. Clearly, f gains only v as a new first out-neighbor and y as a new second out-neighbor in T' by comparison to D. We prove that it gains only y as a second out-neighbor in T' by comparison to D.

Case III.6.1. $u \to v \to b \to u$ *in* T' *with* $b \neq y$. Then $(v, b) \in E(T) - E(T')$. Since x and z are first and second out-neighbors of u in D respectively, we may assume that $b \notin C$ and hence $(v, b) \notin E(D') - E(D)$.

Subcase III.6.1.a. If $(v, b) \in E(D)$, then either $b \in S$ or b is a whole vertex. Thus, due to the losing relation $uv \to xy$, we get that $b \in N_D^{++}(u)$.

Subcase III.6.1.b. If $(v, b) \in E(T) - E(D')$, then $b \in K$ and (v, b) is a convenient orientation with respect to D'. But this is exactly the same as Subcase II.6.1.c.

Case III.6.2. $u \to a \to b \to u$ *in* T' *with* $a \neq v$ *and* $b \neq y$. Thus $(a, b) \in E(T) - E(T')$. Since $a \neq v$, then $(u, a) \in E(D)$ and hence $a \notin K$. Since x and z are first and second out-neighbors of u in D and $u \to v$ in T', then we may assume $b \notin C$.

Subcase III.6.2.a. If $(a, b) \in E(D)$, then $b \in N_D^{++}(u)$.

Subcase III.6.2.b. If $(a, b) \in E(D') - E(D)$, then either $ab \in E(C)$ or $ab \in E[Y_j, M_j]$ for some j. If $ab \in E(C)$, then $b \in C$, a contradiction. Thus $ab \in E[Y_j, M_j]$ for some j. It follows that either (a, b) is a convenient orientation with respect to D and hence $b \in N_D^{++}(u)$, or there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N_D^{++}(s)$. Since $a \notin K$, then $a \in M_j$ and thus $s \in M_j$. By the definition of G, us is not a missing edge of D. But $(u, a) \in E(D)$ and $a \notin N_D^{++}(s)$, then by Lemma 4.2 we get that $b \in N_D^{++}(u)$.

Subcase III.6.2.c. If $(a, b) \in E(T) - E(D')$, then $b \in K$ and (a, b) is a convenient orientation with respect to D'. Since $u \to a$ in D and so in D', there exists a vertex v' such that $u \to v' \to b$ in D'. But $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C$. Since $v' \notin C$ and $(u, v') \in E(D')$, then $(u, v') \in E(D)$. But this is already treated in Subcase III.6.2.a and Subcase III.6.2.b.

Case III.7. $f = v$. Here there are two cases to be consider.

Case III.7.1. $(v, x) \in E(D')$ *and* $(y, z) \in E(D')$. Then v gains x as a first out-neighbor and z as a second out-neighbor in T' by comparison to D. This case is similar to Case III.6.

Case III.7.2. *$(x, v) \in E(D')$ and $(z, y) \in E(D')$.* Clearly, v gains no new first out-neighbor in T' by comparison to D. We prove that it gains no new second out-neighbor in T' by comparison to D. Suppose $f \to a \to b \to f$ in T'. Then $(f, a) \in E(D)$ and hence $a \notin K$.

Subcase III.7.2.a. If $(a, b) \in E(D)$, then $b \in N_D^{++}(f)$.

Subcase III.7.2.b. If $(a, b) \in E(D') - E(D)$, then either (a, b) is a convenient orientation with respect to D and hence $b \in N_D^{++}(v)$, or there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N_D^{++}(s)$. So either $rs, ab \in E(C)$ or $rs, ab \in E[Y_j, M_j]$ for some j. If $ab \in E(C)$, then $(a, b) = (x, y)$ or $(a, b) = (z, y)$ and hence $b = y$, which is impossible because $b \to f$ in T' while $f \to y$ in D. Thus $rs, ab \in E[Y_j, M_j]$ for some j. Since $a \notin K$, then $a \in M_j$ and thus $s \in M_j$. So by the definition of G, fs is not a missing edge of D. But $(f, a) \in E(D)$ and $a \notin N_D^{++}(s)$, then by Lemma 4.2 we get that $b \in N_D^{++}(f)$.

Subcase III.7.2.c. If $(a, b) \in E(T) - E(D')$, then $b \in K$ and (a, b) is a convenient orientation with respect to D'. Since $f \to a$ in D and so in D', there exists a vertex v' such that $f \to v' \to b$ in D'. But $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C$. Since $v' \notin C$ and $(f, v') \in E(D')$, then $(f, v') \in E(D)$. But this is already treated in Subcase III.7.2.a and Subcase III.7.2.b.

Case III.8. $f = x$. Here there are two cases to be consider.

Case III.8.1. *$(v, x) \in E(D')$ and $(y, z) \in E(D')$.* Clearly, x gains no new first out-neighbor in T' by comparison to D. We prove it gains no new second out-neighbor in T' by comparison to D. Suppose $x \to a \to b \to x$ in T'. Then $(x, a) \in E(D)$, $(a, b) \in E(T)$ and $a \notin K$. Note that $a \notin \{x, y, u, v\}$ and we may assume $b \notin \{x, z, v\}$.

Subcase III.8.1.a. If $(a, b) \in E(D)$, then $b \in N_D^{++}(f)$.

Subcase III.8.1.b. If $(a, b) \in E(D') - E(D)$, then either (a, b) is a convenient orientation with respect to D and hence $b \in N_D^{++}(x)$, or there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N_D^{++}(s)$. So either $rs, ab \in E(C)$ or $rs, ab \in E[Y_j, M_j]$ for some j. If $ab \in E(C)$, then $(a, b) = (x, y)$ or $(a, b) = (y, z)$ and hence $a \in \{x, y\}$, a contradiction. Thus $rs, ab \in E[Y_j, M_j]$ for some j. Since $a \notin K$, then $a \in M_j$ and thus $s \in M_j$. By the definition of G, fs is not a missing edge of D. But $(x, a) \in E(D)$ and $a \notin N_D^{++}(s)$, then by Lemma 4.2 we get that $b \in N_D^{++}(f)$.

Subcase III.8.1.c. If $(a, b) \in E(T) - E(D')$, then $b \in K$ and (a, b) is a convenient orientation with respect to D'. Since $x \to a$ in D and so in D', there exists a vertex v' such that $x \to v' \to b$ in D'. Since $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C$. Since $v' \notin C$ and $(x, v') \in E(D')$, then $(x, v') \in E(D)$. But this is already treated in Subcase III.8.1.a and Subcase III.8.1.b.

Case III.8.2. *$(x, v) \in E(D')$ and $(z, y) \in E(D')$.* Clearly, x gains only v as a first out-neighbor and y as a second out-neighbor in T' by comparison to D. We prove it gains only y as a second out-neighbor in T' by comparison to D. Note that z

and v are respectively first and second out-neighbors of x in D, $(v,u) \notin E(T)$ and $(x,u) \notin E(T)$.

Subcase III.8.2.a. Suppose that $x \to v \to b \to x$ in T' with $b \neq y$. Then $(v,b) \in E(T)$. By the previous note, we may assume that $b \notin C$ and hence $(v,b) \notin E(D') - E(D)$.

Subcase III.8.2.a.1. If $(v,b) \in E(D)$, then either $b \in S$ or b is a whole vertex. Thus, by the losing relation $xv \to zy$, we get that $b \in N_D^{++}(x)$.

Subcase III.8.2.a.2. If $(v,b) \in E(T)-E(D')$, then $b \in K$ and (v,b) is a convenient orientation with respect to D'. But this exactly the same as Subcase II.6.1.c.

Subcase III.8.2.b. Suppose that $x \to a \to b \to x$ in T' with $b \neq y$ and $a \neq v$. Then $(x,a) \in E(D)$ and thus $a \notin K \cup \{x,y,v,u\}$. We may assume that $b \notin \{x,y,v,z\}$. But this case is already treated in Case III.7.2.

Case III.9. $f = y$. Clearly, f gains no new first out-neighbor in T' by comparison to D. We prove that it gains no new second out-neighbor in T' by comparison to D. Note that $(z,y) \in E(T')$ and u and x are first and second out-neighbors of y, respectively. Suppose that $f \to a \to b \to f$ in T'. Then $(f,a) \in E(D)$ and $a \notin K \cup \{x,y,v,z\}$.

Subcase III.9.a. If $(a,b) \in E(D)$, then $b \in N_D^{++}(f)$.

Subcase III.9.b. If $(a,b) \in E(D') - E(D)$, then either (a,b) is a convenient orientation with respect to D and hence $b \in N_D^{++}(f)$, or there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N_D^{++}(s)$. So either $rs, ab \in E(C)$, or $rs, ab \in E[Y_j, M_j]$ for some j. If $ab \in E(C)$, then $(a,b) = (x,y)$ or $(a,b) = (y,z)$ or $(a,b) = (z,y)$ and hence $a \in \{x,y,z\}$, a contradiction. Thus $rs, ab \in E[Y_j, M_j]$ for some j. Since $a \notin K$, then $a \in M_j$ and thus $s \in M_j$. By the definition of G, fs is not a missing edge of D. But $(f,a) \in E(D)$ and $a \notin N_D^{++}(s)$, then by Lemma 4.2 we get $b \in N_D^{++}(f)$.

Subcase III.9.c. If $(a,b) \in E(T) - E(D')$, then $b \in K$ and (a,b) is a convenient orientation with respect to D'. But $(f,a) \in E(D)$, then $(f,a) \in E(D')$. This gives the existence of a vertex v' such that $f \to v' \to b$ in D'. Since $(v',b) \in E(D')$ and $b \in K$, then $v' \notin C$. Since $v' \notin C$ and $(f,v') \in E(D')$, then $(f,v') \in E(D)$. But this is already treated in Subcase III.9.a and Subcase III.9.b.

Case III.10. $f = z$. Exactly same as Case III.9, with one difference that yz is reoriented so that $(y,z) \in E(T')$.

Case III.11. $f \in V(G) - (Y \cup M \cup C) = A \cup (X - X_1) = A \cup (X - Y_1)$. Exactly same as Case I.6, with only one difference. The difference is that in Subcase III.11.b there are two possibilities for the edges rs, ab: Either $rs, ab \in E[Y_j, M_j]$ for some j, or $rs, ab \in E(C)$. The first case is treated in Subcase I.6.b. However, for the second case, f must belong to A since otherwise fa is a missing edge because $G[X \cup C]$ is a complete split graph, which contradicts the fact that $(f,a) \in E(D)$. Thus fs is not missing edge of D because $E[A, C] = \emptyset$ by the definition of G. Since $f \to a$ in D and $a \notin N_D^{++}(s)$, then by Lemma 4.2 we get that $b \in N_D^{++}(f)$.

Therefore, due to all the discussions above, f has the SNP in D in case that $\Delta[E(C)]$ contains only the two arcs (uv, xy) and (vx, yz). $\diamondsuit$

Case IV. $\Delta[E(C)]$ *has exactly two arcs, say* $uv \to xy \to zu$.

Then $v \to y \to u \to x \to z$ in D and the edges uv, yz, vx are good missing edges of D. Assume without loss of generality that (u, v) is a convenient orientation with respect to D. Add the arcs (u, v), (x, y) and $(z, u)$to D. Then assign to the good missing edges vx and zy a convenient orientation and add them to D. The obtained oriented graph D' is missing $G' = G - (\cup E[Y_j, M_j] \cup E(C))$ which is a threshold graph. Due to Theorem 3.3, all the missing edges of D' are good. We assign to them a convenient orientation and we add them to get a tournament T. Let L be a median order of T and let f denote its feed vertex. Reorient all the missing edges incident to f towards f except those whose out-degree in Δ is not zero. The same order L is again a median order of the obtained tournament T' and thus f has the SNP in T'. We will prove that f has the SNP in D. We have the following cases.

Case IV.1. f *is a whole vertex.* This is the same as Case I.1.

Case IV.2. *There exists* $1 \leq t \leq l$ *such that* $f \in M_t$. This is similar to Case III.2. Note that, in this case, if rs, $ab \in E(C)$, then $(r, s) = (u, v)$ and $(a, b) = (x, y)$ or $(r, s) = (x, y)$ and $(a, b) = (z, u)$.

Case IV.3. *There exists* $1 \leq t \leq l$ *such that* $f \in Y_t$ *and* $M_t \neq \emptyset$. This case is exactly the same as Case III.3, so the proof is left to the reader.

Case IV.4. *There exists* $1 \leq t \leq l + 1$ *such that* $f \in Y_t$ *and* $M_t = \emptyset$. The proof of this case is omitted, because it can be easily done by imitating the demonstration of Case III.4.

Case IV.5. $f \in Y_{l+2}$. This case is exactly the same as Case III.5.

Case IV.6. $f = u$. Clearly, u gains only v as a first out-neighbor and it gains y as a second out-neighbor in T' by comparison to D. To verify that f has the SNP in D, it is sufficient to prove that u gains no new second out-neighbors other than y in T'. To this end, we proceed in the same manner as that of Case III.6.

Case IV.7. $f = v$. It is clear that v gains no new first out-neighbor in T' by comparison to D. We prove that it gains no new second out-neighbor in T' by comparison to D. Suppose that $v \to a \to b \to v$ in T'. Then $(v, a) \in E(D)$ and $a \notin K \cup \{x, v, u\}$. Since $v \to y \to u$ in D, we may assume that $b \notin \{u, v, y\}$ and $a \neq y$.

Subcase IV.7.a. If $(a, b) \in E(D)$ or $(a, b) \in E(D) - E(D')$ such that it is a convenient orientation with respect to D, then clearly $b \in N_D^{++}(f)$.

Subcase IV.7.b. If $(a, b) \in E(D) - E(D')$ and it is not convenient with respect to D, then there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N_D^{++}(s)$. Thus either

$rs, ab \in E(C)$ or $rs, ab \in E[Y_j, M_j]$ for some j. If $ab \in E(C)$, then $(a, b) = (x, y)$ or $(a, b) = (z, u)$ and hence $b \in \{u, y\}$, a contradiction. Thus $rs, ab \in E[Y_j, M_j]$ for some j. Since $a \notin K$, then $a \in M_j$ and thus $s \in M_j$. By the definition of G, fs is not a missing edge of D. But $(f, a) \in E(D)$ and $a \notin N_D^{++}(s)$, then by Lemma 4.2 we get that $b \in N_D^{++}(f)$.

Subcase IV.7.c. If $(a, b) \in E(T) - E(D')$, then $b \in K$ and (a, b) is a convenient orientation with respect to D'. But $(f, a) \in E(D)$, then $(f, a) \in E(D')$. This induces the existence of a vertex v' such that $f \to v' \to b$ in D'. Since $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C \cup K$. But $(f, v') \in E(D')$, then $(f, v') \in E(D)$. But this is already treated in Subcase IV.7.a and Subcase IV.7.b.

Case IV.8. $f = x$. Then x gains only y as a first out-neighbor and gains u as a second out-neighbor in T' by comparison to D. We prove that it gains only u as a new second out-neighbor in T' by comparison to D.

Subcase IV.8.a. Suppose that $x \to y \to b \to x$ in T' with $b \neq u$. Then $(y, b) \in E(T) - E(T')$. Since $x \to z$ and $v \to y$ in D, then we may assume that $b \notin C$ and hence $(y, b) \notin E(D') - E(D)$. If $(y, b) \in E(D)$, then b is a whole vertex or $b \in S$. Thus, by the losing relation $xy \to zu$, we get that $b \in N_D^{++}(x)$. Else if $(y, b) \in E(T) - E(D')$, then $b \in K$ and (y, b) is a convenient orientation with respect to D'. So there is a vertex v' such that $x \to v' \to b$ in D'. Since $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C \cup K$. Since $v' \notin C$ and $(f, v') \in E(D')$, then $(f, v') \in E(D)$. As usual, we can prove that $b \in N_D^{++}(x)$ in D.

Subcase IV.8.b. Suppose that $x \to a \to b \to x$ in T' with $a \neq y$ and $b \neq u$. Then $(x, a) \in D$ and thus $a \notin K$. Since $x \to z$ in D and $x \to y$ in T, then $b \notin \{x, y, u, z\}$. We proceed exactly as in Case IV.7.

Case IV.9. $f = y$. Clearly, y gains no new first out-neighbor in T' by comparison to D. We prove that it gains no new second out-neighbor in T' by comparison to D. Suppose that $y \to a \to b \to y$ in T'. Then $(y, a) \in E(D)$ and $a \notin K$. Since $y \to u \to x$ in D, then we may assume $b \notin \{x, y, u\}$. We continue exactly as in Case IV.7.

Case IV.10. $f = z$. It is clear that z gains no new vertex as a first out-neighbor in T' by comparison to D. We prove that it gains no new vertex as a second out-neighbor in T' by comparison to D. Suppose that $z \to a \to b \to z$ in T'. Then $(z, a) \in E(D)$ and $a \notin K$. Since $x \to z$ in D, we may assume that $a \notin K \cup \{x, z\}$.

Subcase IV.10.a. If $(a, b) \in E(D)$ or $(a, b) \in E(D) - E(D')$ and it is a convenient orientation with respect to D, then $b \in N_D^{++}(z)$.

Subcase IV.10.b. If $(a, b) \in E(D) - E(D')$ and it is not convenient with respect to D, then there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N_D^{++}(s)$. So either $rs, ab \in E(C)$ or $rs, ab \in E[Y_j, M_j]$ for some j. If $ab \in E(C)$, then $(a, b) = (x, y)$ or $(a, b) = (z, u)$ and hence $a \in \{x, z\}$, a contradiction. Thus $rs, ab \in E[Y_j, M_j]$ for some j. Since $a \notin K$, then $a \in M_j$ and thus $s \in M_j$. So by the definition of G, fs

is not a missing edge of D. But $(f, a) \in E(D)$ and $a \notin N_D^{++}(s)$, then by Lemma 4.2 we get $b \in N_D^{++}(f)$.

Subcase IV.10.c. If $(a, b) \in E(T) - E(D')$, then $b \in K$ and (a, b) is a convenient orientation with respect to D'. Since $(f, a) \in E(D)$, then $(f, a) \in E(D')$. This gives the existence of a vertex v' such that $f \to v' \to b$ in D'. Since $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C \cup k$. Since $v' \notin C$ and $(f, v') \in E(D')$, then $(f, v') \in E(D)$. But this is already treated in Subcase IV.10.a and Subcase IV.10.b.

Case IV.11. $f \in V(G) - (Y \cup M \cup C) = A \cup (X - X_1) = A \cup (X - Y_1)$. Exactly same as Case III.11.

Therefore, in the light of the preceding, f satisfies the SNP given that $\Delta[E(C)]$ has exactly the two arcs (uv, xy) and (xy, zu). $\diamondsuit$

Case V. $\Delta[E(C)]$ *has exactly three arcs, say* $uv \to xy \to zu \to vx$.

Then $u \to x \to z \to v \to y$ in D and the edges uv, yz are good missing edges of D. Assume without loss of generality that (u, v) is a convenient orientation with respect to D. Add first the arcs (u, v), (x, y), (z, u) and (v, x) to D. Assign to yz a convenient orientation and add it to D. The obtained oriented graph D' is missing $G' = G - (\cup E[Y_j, M_j] \cup E(C))$ which is a threshold graph. Due to Theorem 3.3, all the missing edges of D' are good. We assign to each of them a convenient orientation and we add them till getting a tournament T. Let L be a median order of T and let f denote its feed vertex. Reorient all the missing edges incident to f towards f except those whose out-degree in Δ is not zero. The same order L is again a median order of the obtained tournament T' and f has the SNP in T'. We will prove that f has the SNP in D. We have the following cases.

Case V.1. *f is a whole vertex.* This is is the same as Case I.1.

Case V.2. *There exists $1 \leq t \leq l$ such that $f \in M_t$.* The proof of this case is similar to that of Case III.2. Note that, in this case, if rs, $ab \in E(C)$, then $(r, s) = (u, v)$ and $(a, b) = (x, y)$, or $(r, s) = (x, y)$ and $(a, b) = (z, u)$, or $(r, s) = (z, u)$ and $(a, b) = (v, x)$.

Case V.3. *There exists $1 \leq t \leq l$ such that $f \in Y_t$ and $M_t \neq \emptyset$.* This case is exactly the same as Case III.3, so the proof is left to the reader.

Case V.4. *There exists $1 \leq t \leq l + 1$ such that $f \in Y_t$ and $M_t = \emptyset$.* The proof of this case is omitted, because it can be easily done by imitating the demonstration of Case III.4.

Case V.5. $f \in Y_{l+2}$. This case is exactly the same as Case III.5.

Case V.6. $f \in \{u, x, z\}$. The proof of this case is left to the reader, because it is exactly the same as Case III.6.

Case V.7. $f = y$. Exactly same as Case III.9, with only one difference. The difference is that in Subcase V.7.b when $ab \in E(C)$ then (a, b) can be either (x, y) or (z, u) or (v, x) and so $a \in \{x, z, v\}$, a contradiction.

Case V.8. $f = v$. Similar to the case where $f = y$, that is to Case V.7.

Case V.9. $f \in V(G) - (Y \cup M \cup C) = A \cup (X - X_1) = A \cup (X - Y_1)$. The proof of this case is similar to that of Case III.11.

Therefore, in view of all the observations above, f satisfies the SNP in D, given that $\Delta[E(C)]$ has exactly the three arcs (uv, xy), (xy, zu) and (zu, vx). $\lozenge$

Case VI. $\Delta[E(C)]$ *has exactly three arcs, say* $uv \to xy \to zu$ *and* $xv \to zy$.

Then $u \to x \to z \to v \to y \to u$ in D and uv and xv are good missing edges. Assume without loss of generality that (u, v) is a convenient orientation of uv. Add (u, v), (x, y) and (z, u) to D. If (x, v) is a convenient orientation the good missing edge xv, then add (x, v) and (z, y) to D, otherwise add (v, x) and (y, z) to D. The obtained oriented graph D' is missing $G' = G - (\cup E[Y_j, M_j] \cup E(C))$ which is a threshold graph. So all the missing edges of D' are good. We assign to them a convenient orientation and we add them to get a tournament T. Let L be a median order of T and let f denote its feed vertex. Reorient all the missing edges incident to f towards f except those whose out-degree in Δ is not zero. The same order L is again a median order of the obtained tournament T' and f has the SNP in T'. We will prove that f has the SNP in D. We have the following cases.

Case VI.1. *f is a whole vertex.* This is the same as Case I.1.

Case VI.2. *There exists* $1 \leq t \leq l$ *such that* $f \in M_t$. This case is similar to Case III.2. Note that, in this case, if $rs, ab \in E(C)$ then $(r, s) = (u, v)$ and $(a, b) = (x, y)$, $(r, s) = (x, y)$ and $(a, b) = (z, u)$, $(r, s) = (v, x)$ and $(a, b) = (y, z)$ if (v, x) is a convenient orientation of vx, or $(r, s) = (x, v)$ and $(a, b) = (z, y)$ if (x, v) is a convenient orientation of vx.

Case VI.3. *There exists* $1 \leq t \leq l$ *such that* $f \in Y_t$ *and* $M_t \neq \emptyset$. This case is exactly same as Case III.3.

Case VI.4. *There exists* $1 \leq t \leq l + 1$ *such that* $f \in Y_t$ *and* $M_t = \emptyset$. This case can be proved by imitating the demonstration of Case III.4.

Case VI.5. $f \in Y_{l+2}$. This case is exactly the same as Case III.5.

Case VI.6. $f = u$. This case is exactly the same as Case III.6, so it is safely left to the reader.

Case VI.7. $f = y$. Exactly same as Case III.9, with only one difference. The difference is that in Subcase VI.7.b if $ab \in E(C)$ then (a, b) can be either (x, y), (z, u) (y, z) or (z, y) and so $a \in \{x, y, z\}$, a contradiction.

Case VI.8. $f = z$. Exactly same as Case VI.7, with one difference that yz and uz are reoriented so that $(y, z) \in E(T')$ and $(u, z) \in E(T')$, respectively.

Case VI.9. $f = v$. Exactly same as Case III.7, with only one difference. In Subcase VI.9.2.b, if $ab \in E(C)$ then (a, b) can be either (x, y), (z, u) or (z, y) and so $a \in \{x, z\}$. This a contradiction because $z \to v$ in D and $x \to v$ in D' while $v \to a$ in D.

Case VI.10. $f = x$. Here we consider two main cases:

Case VI.10.1. $(x, v) \in E(D')$. Clearly, x gains only v and y as new first out-neighbors in T' by comparison to D. However, x loses v as a second out-neighbor and gains u as a new second out-neighbor in T' by comparison to D. We prove that x gains only u as a new second out-neighbor in T' by comparison to D.

Case VI.10.1.1. *$x \to a \to b \to x$ in T', with $a \neq y$, $a \neq v$ and $b \neq u$.* Then $(a, b) \in E(T)$, $(x, a) \in E(D)$ and thus $a \notin K$. Since $x \to z \to v$ in D and $x \to y$ in T, then we may assume that $b \notin C$.

Subcase VI.10.1.1.a. If $(a, b) \in E(D)$, then clearly $b \in N_D^{++}(x)$.

Subcase VI.10.1.1.b. If $(a, b) \in E(D') - E(D)$, then either $ab \in E(C)$ or $ab \in E[Y_j, M_j]$ for some j. If $ab \in E(C)$, then $b \in C$, a contradiction. Thus $ab \in E[Y_j, M_j]$ for some j. It follows that either (a, b) is a convenient orientation with respect to D and hence $b \in N_D^{++}(x)$, or there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N_D^{++}(s)$. Since $a \notin K$, then $a \in M_j$ and thus $s \in M_j$. By the definition of G, xs is not a missing edge of D. But $(x, a) \in E(D)$ and $a \notin N_D^{++}(s)$, then by Lemma 4.2 we get $b \in N_D^{++}(x)$.

Subcase VI.10.1.1.c. If $(a, b) \in E(T) - E(D')$, then $b \in K$ and (a, b) is a convenient orientation with respect to D'. Since $x \to a$ in D and so in D', there exists a vertex v' such that $x \to v' \to b$ in D'. Since $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C$. Since $v' \notin C$ and $(x, v') \in E(D')$, then $(x, v') \in E(D)$. But this is already treated in Subcase VI.10.1.1.a and Subcase VI.10.1.1.b.

Case VI.10.1.2. *$x \to y \to b \to x$ in T' with $b \neq u$.* Since $x \to z \to v$ in D, then we may assume that $b \notin C$ and hence $(y, b) \notin E(D') - E(D)$.

Subcase VI.10.1.2.a. If $(y, b) \in E(D)$, then either $b \in S$ or b is a whole vertex. Then, by the losing relation $xy \to zu$, we get that $b \in N_D^{++}(x)$.

Subcase VI.10.1.2.b. If $(y, b) \in E(T) - E(D')$, then $b \in K$ and (y, b) is a convenient orientation with respect to D'. But this is exactly the same as Subcase VI.10.1.1.a and Subcase VI.10.1.1.b.

Case VI.10.1.3. *$x \to v \to b \to x$ in T' with $b \neq u$.* Since $x \to z \to v$ in D and $x \to y$ in T, then we may assume that $b \notin C$. We proceed similarly to Case VI.10.1.2 by replacing the losing relation $xy \to zu$ in Subcase VI.10.1.2.a by the losing relation $xv \to zy$ in Subcase VI.10.1.3.a.

Case VI.10.2. $(v, x) \in E(D')$. Clearly, x gains only y as a new first out-neighbor and u as a new second out-neighbor in T' by comparison to D. We prove that it gains only u as a new second out-neighbor in T' by comparison to D.

Subcase VI.10.2.a. Suppose that $x \to y \to b \to x$ in T' with $b \neq u$. Then $(y, b) \in E(T) - E(T')$. Since $x \to z \to v$ in D, then we may assume that $b \notin C$ and hence $(y, b) \notin E(D') - E(D)$. This is exactly the same as Case VI.10.1.2.

Subcase VI.10.2.b. Suppose that $x \to a \to b \to x$ in T' with $a \neq y$ and $b \neq u$. Then $(a, b) \in E(T) - E(T')$, $(x, a) \in E(D)$ and thus $a \notin K$. Since $x \to z \to v$ in D and $x \to y$ in T, then $b \notin C$. We proceed exactly as in Case VI.10.1.1.

Case VI.11. $f \in V(G) - (Y \cup M \cup C) = A \cup (X - X_1) = A \cup (X - Y_1)$. This case is similar to Case III.11.

Therefore, f has the SNP in case that $\Delta[E(C)]$ has exactly the three arcs (uv, xy), (xy, zu) and (zu, vx). $\diamondsuit$

Case VII. *$\Delta[E(C)]$ is a directed cycle of length 5, say $xy \to zu \to vx \to yz \to uv \to xy$.*

In this case, add to D the arcs (x, y), (z, u), (v, x), (y, z) and (u, v). The obtained oriented graph D' is missing $G' = G - (\cup E[Y_j, M_j] \cup E(C))$ which is a threshold graph. Thus, due to Theorem 3.3, all the missing edges of D' are good. Assign to each missing edge of D' a convenient orientation and add them to D' to get a tournament T. Let L be a median order of T and let f denote its feed vertex. Reorient all the missing edges incident to f towards f, except those whose out-degree in Δ is not zero. The same order L is a median order of the obtained tournament T', f is also a feed vertex of L and thus f has the SNP in T'. We will prove that f has the SNP in D also. For this purpose, we consider the following cases.

Case VII.1. *f is a whole vertex.* This is is the same as Case I.1.

Case VII.2. *There exists $1 \leq t \leq l$ such that $f \in M_t$.* The proof of this case is similar to that of Case III.2.

Case VII.3. *There exists $1 \leq t \leq l$ such that $f \in Y_t$ and $M_t \neq \emptyset$.* This case is exactly the same as Case III.3, so the proof is left to the reader.

Case VII.4. *There exists $1 \leq t \leq l + 1$ such that $f \in Y_t$ and $M_t = \emptyset$.* The proof of this case is similar to that of Case III.4.

Case VII.5. $f \in Y_{l+2}$. This case is exactly the same as Case III.5.

Case VII.6. $f \in C$. Due to symmetry, assume that $f = u$. Clearly, u gains only v as a new first out-neighbor in T' by comparison to D. We prove that it gains only y as a new second out-neighbor in T' by comparison to D.

Case VII.6.1. *$u \to v \to b \to u$ in T' with $b \neq y$.* Then $(v, b) \in E(T) - E(T')$. Since $u \to x$ and $z \to v$ in D, then we may assume that $b \notin C$ and hence $(v, b) \notin E(D') - E(D)$.

Subcase VII.6.1.a. If $(v, b) \in E(D)$, then either $b \in S$ or b is a whole vertex. Then, by the losing relation $uv \to xy$, we get $b \in N^{++}_D(u)$.

Subcase VII.6.1.b. If $(v, b) \in E(T) - E(D')$, then $b \in K$ and (v, b) is a convenient orientation with respect to D'. Then there exists v' such that $u \to v' \to b \to u$ in D'. Since $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C$. Since $v' \notin C$ and $(f, v') \in$

$E(D')$. The rest of the proof of this case is exactly the same as Subcase II.6.1.c.1 and Subcase II.6.1.c.2.

Case VII.6.2. *$u \to a \to b \to u$ in T', with $a \neq v$ and $b \neq y$.* Then $(u, a) \in E(D)$ and $(a, b) \in E(T)$. Note that $a \notin K \cup \{u, v, y, z\}$ and $b \notin \{u, v, x, y\}$.

Subcase VII.6.2.a. If $(a, b) \in E(D)$, then $b \in N_D^{++}(u)$.

Subcase VII.6.2.b. If $(a, b) \in E(D') - E(D)$, then either $ab \in E(C)$ or $ab \in E[Y_j, M_j]$ for some j. If $ab \in E(C)$, then $a = x$ because $a \notin \{u, v, y, z\}$ and so $(a, b) = (x, y)$, a contradiction to the fact that $b \neq y$. Thus $ab \in E[Y_j, M_j]$ for some j. Then either (a, b) is a convenient orientation with respect to D and hence $b \in N_D^{++}(u)$, or there is $rs \to ab$ in Δ, namely $s \to b$ and $a \notin N_D^{++}(s)$. Since $a \notin K$, then $a \in M_j$ and thus $s \in M_j$. By the definition of G, us is not a missing edge of D. But $(u, a) \in E(D)$ and $a \notin N_D^{++}(s)$, then by Lemma 4.2 we get that $b \in N_D^{++}(u)$.

Subcase VII.6.2.c. If $(a, b) \in E(T) - E(D')$, then $b \in K$ and (a, b) is a convenient orientation with respect to D'. Since $u \to a$ in D and so in D', there exists a vertex v' such that $u \to v' \to b$ in D'. Since $(v', b) \in E(D')$ and $b \in K$, then $v' \notin C$. Since $v' \notin C$ and $(u, v') \in E(D')$, then $(u, v') \in E(D)$. But this is already treated in Subcase II.6.1.c.1 and Subcase II.6.1.c.2.

Case VII.7. *$f \in V(G) - (Y \cup M \cup C) = A \cup (X - X_1) = A \cup (X - Y_1)$.* This is the same as Case III.11, so the proof is left to the reader.

Therefore, f has the SNP in D when $\Delta[E(C)]$ is a directed cycle of length 5. $\lozenge$

In view of the preceding, f has the SNP in D. This completes the proof. $\square$

As immediate consequences of the previous theorem, we may conclude the following:

Corollary 4.6. *(Ghazal [5]) Every oriented graph missing a comb satisfies the SNC.*

Corollary 4.7. *(Ghazal [7]) Every oriented graph missing a threshold graph satisfies the SNC.*

Corollary 4.8. *Every oriented graph missing a generalized comb satisfies the SNC.*

Corollary 4.9. *Every oriented threshold graph satisfies the SNC.*

Proof. Let D be an oriented graph whose underlying graph G is a threshold graph. Then the missing graph of D is the complement of G minus its isolated vertices, which is in its turn a threshold graph. Thus, due to Corollary 4.7, D has a vertex with the SNP. $\square$

Since threshold graphs and generalized combs are $\overrightarrow{\mathcal{P}}$-forcing graphs, C_5 is a $\overrightarrow{\mathcal{P}} \cup \overrightarrow{\mathcal{C}_5}$-forcing graph, $\{C_4, \overline{C_4}, S_3,$ *chair* and $\overline{\text{chair}}\}$-free graphs are $\overrightarrow{\mathcal{P}} \cup \overrightarrow{\mathcal{C}_5}$-forcing graphs and any oriented graph missing one of the graphs mentioned before satisfies the SNC, we end this article by asking the following:

Problem 2. *Does every oriented graph missing a $\overrightarrow{\mathcal{P}} \cup \overrightarrow{\mathcal{C}_5}$-forcing graph satisfy the SNC?*

Acknowledgments

The authors would like to thank the referees for their valuable comments.

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(Received 27 Mar 2020; revised 10 Mar 2021, 4 July 2021)