AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **80(3)** (2021), Pages 342–360

A new polynomial for polymatroids

Laura Chávez-Lomelí

Unidad Azcapotzalco, Universidad Autónoma Metropolitana
Ciudad de México, 02200, Mexico
lelc@azc.uam.mx

Criel Merino

Instituto de Matemáticas, Universidad Nacional Autónoma de México
Ciudad de México, 04510, Mexico
merino@matem.unam.mx

Guadalupe Rodríguez Sánchez

Unidad Azcapotzalco, Universidad Autónoma Metropolitana
Ciudad de México, 02200, Mexico
rsmg@azc.uam.mx

Geoffrey Whittle

Victoria University of Wellington, Wellington 6140
New Zealand
geoff.whittle@vuw.ac.nz

Abstract

We present a polynomial invariant for polymatroids. This polynomial is a natural generalization of the Tutte polynomial of matroids. We explore some basic properties of this invariant in an effort to establish a solid basis for the systematic study of the polynomial, in a similar fashion as with the Tutte polynomial. We also reveal some combinatorial interpretations of evaluations of this invariant with the intent of encouraging further research on polymatroid invariants.

1 Introduction

Polymatroids were originally conceived in [5] as polytopes that generalized the concept of matroid polytope; however, our approach follows the work in [16] that considers them as generalizations of matroids. Since their introduction in 1969, polymatroids have been very important in combinatorial optimization, e.g., the polymatroid

ISSN: 2202-3518 ©The author(s). Released under the CC BY 4.0 International License

parity problem [13]. In cryptology, polymatroid representability over a finite field has a direct impact on the problem of secret sharing schemes [7]. More recently, polymatroid decompositions are at the core of a variety of problems: the study of tropical linear spaces [17]; compactifying fine Schubert cells in the Grassmannian [12]; and the study of many important invariants on polymatroids as valuations [1]. There are currently other attempts at generalising the Tutte polynomial to polymatroids, see [2, 4].

The paper is organized as follows: In Section 2, the definition and basic properties of polymatroids are given; most of these results have appeared before in [10, 18]. The polynomial $N(P; u, v)$ associated with a polymatroid P, the main object of study of this work, is defined in Section 3, and some basic properties are proved. Examples of polymatroids and polymatroid invariants coming from hypergraphs are provided in Section 4. Some easy evaluations of $N(P; u, v)$ are given in Section 5; and two more evaluations are presented in Section 6 that were considered recently in [20] in the context of hypergraphs. The last section contains a polynomial that also seems interesting to explore.

2 Polymatroids

Let E be a finite set and let r be a function $r : 2^E \to \mathbb{Z}$. We say that r is *normalized* if $r(\emptyset) = 0$, r is *increasing* if $A \subseteq B \subseteq E$ implies $r(A) \leq r(B)$ and r is *submodular* if $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$ for all subsets A and B of E. The ordered pair $P = (E, r)$ is a *polymatroid* if r is normalized, increasing and submodular. We say that E is the *ground set* of P and r is the *rank* function of P. Let k be a positive integer. Then the polymatroid P is a *k-polymatroid* if $r(\{e\}) \leq k$, and it is a *strict k-polymatroid* if $r(\{e\}) = k$ for all elements $e \in E$. A *1-polymatroid* is a matroid.

We define the concept of minor on polymatroids. Let $P = (E, r)$ be a polymatroid, and let A be a subset of E. The *deletion* of A from P is the polymatroid $P \setminus A = (E \setminus A, r_{\setminus A})$ such that $r_{\setminus A}(X) = r(X)$ for each $X \subseteq E \setminus A$. The *contraction* of A from P is $P/A = (E \setminus A, r_{/A})$ where $r_{/A}(X) = r(X \cup A) - r(A)$ for each $X \subseteq E \setminus A$. When $A = \{e\}$, we denote by $P \setminus e$ the deletion of $\{e\}$ and by P/e the contraction of $\{e\}$. Clearly, for any two elements e and f in the polymatroid P, $P \setminus e/f = P/f \setminus e$, as the corresponding rank functions are the same; thus, after doing a sequence of contractions and deletions, the resulting polymatroid is independent of the order of the operations in the sequence. The polymatroid P' is a *minor* of P if $P' = (P \setminus A)/B$ for some disjoint subsets A and B of E.

2.1 Loops, coloops and compactification

Let $P = (E, r)$ be a polymatroid. As with matroids, an element $e \in E$ is a *loop* in P if $r(e) = 0$. For an element $e \in E$, let $d_r(e) = r(E) - r(E \setminus e)$. We call an element e *compact* if $d_r(e) = 0$. In a matroid, the non-compact elements are precisely the coloops of the matroid. A polymatroid P is compact if every element of P is compact. The definition of compact polymatroid is from [9].

For a non-compact element e of a polymatroid $P = (E, r)$, we define the *compactification* of the element e as the polymatroid $\mathrm{com}_e(P)$ over the set E with rank function r_e such that for all $X \subseteq E$,

$$r_e(X) = \begin{cases} r(X), & \text{if } e \notin X, \\ r(X) - d_r(e), & \text{otherwise.} \end{cases} \tag{1}$$

Proposition 2.1. *The compactification $com_e(P)$ is a polymatroid and the element e is compact in $com_e(P)$.*

Proof. Clearly, $r_e(\emptyset) = r(\emptyset) = 0$. The only interesting case in the proof of the inequality $r_e(A) \leq r_e(B)$ for $A \subseteq B$ is when $e \in B \setminus A$. This is a consequence of the following.

$$\begin{aligned} r_e(A \cup \{e\}) &= r(A \cup \{e\}) - r(E) + r(E - \{e\}) \\ &\geq r(E) + r(A) - r(E) \\ &= r_e(A), \end{aligned}$$

where the inequality holds by submodularity. The submodularity of r_e, $r_e(A) + r_e(B) \geq r_e(A \cup B) + r_e(A \cap B)$ for all subsets A and B of E, is trivial, since in the three possible cases, $e \in A \cap B$, $e \in A \setminus B$ and $e \notin A \cup B$, we add two, one or zero times the constant $-r(E) + r(E - e)$ to both sides.

The definition of r_e guarantees that the element e is compact in $\mathrm{comp}_e(P)$. □

Observe that in $\mathrm{comp}_e(P)$, $d_{r_e}(f) = d_r(f)$ for all $f \in E \setminus e$; thus the set of non-compact elements in $\mathrm{comp}_e(P)$ is the same as in $P \setminus e$. When every non-compact element of P is compacted, we obtain the *compactification* of P, denoted by *com*(P). The following result appears in [9] but with a different proof.

Proposition 2.2. *The polymatroid $com(P)$ is well defined.*

Proof. It is enough to prove that *com*(P) is independent of the order in which the compactification of the elements is done. Let e and f be different elements in $P = (E, r)$. Let P_e, P_f, $P_{e,f}$ and $P_{f,e}$ be the compactifications of P by e, f, e and then f, and f and then e, respectively; also, let r_e, r_f, $r_{e,f}$ and $r_{f,e}$ be the corresponding rank functions.

Observe that $d_{r_e}(f) = d_r(f)$ and $d_{r_f}(e) = d_r(e)$. Then, we have three cases. If $e, f \notin A$, then clearly, $r_{e,f}(A) = r_{f,e}(A)$. If $e \in A$ but $f \notin A$, then $r_{e,f}(A) = r(A) - d_r(e) = r_{f,e}(A)$. If $e, f \in A$, $r_{e,f}(A) = r(A) - d_r(e) - d_r(f) = r_{f,e}(A)$. Thus, $P_{e,f} = P_{f,e}$. □

From the previous proof it is not difficult to infer that if X is the set of non-compact elements in the polymatroid $P = (E, r)$, then for $A \subseteq E$,

$$r_{com(P)}(A) = r(A) - \sum_{a \in A \cap X} d_r(a). \tag{2}$$

2.2 Duality

For a matroid $M = (r, E)$ its dual matroid, $M^* = (r^*, E)$, where $r^*(X) = r(E - X) + |X| - r(E)$, behaves very nicely. It is well known that this dual is an involution (that is, $(M^*)^* = M$) that interchanges the deletion and contraction of an element, that is, $(M \setminus e)^* = M^*/e$ and $(M/e)^* = M^* \setminus e$ for all $e \in E$. This is a rare property in an involution of matroids, as shown in the following Theorem from [10].

Theorem 2.3. *The dual is the only involution on the class of matroids that interchanges the deletion with the contraction of an element.*

Let k be an integer and let $P = (E, r)$ be a k-polymatroid. We define the polymatroid $P^{*k} = (E, r^*)$, the *k-dual* of P, where $r^{*k}(X) = r(E-X)+k|X|-r(E)$, for all subsets X of E. We have the following extension of the previous theorem from [18].

Theorem 2.4. *The k-dual is the only involution on the class of k-polymatroids that interchanges the deletion and contraction of an element.*

Because the k-dual and the $(k+1)$-dual of a given k-polymatroid P are not equal, the previous result prohibits a general involution on polymatroids.

Corollary 2.5. *There is no involution on the class of polymatroids that interchanges deletion and contraction of an element.*

Let $P = (E, r)$ be a polymatroid. We define the *dual* of P, denoted P^*, as the polymatroid (E, r^*), where, for $X \subseteq E$,

$$r^*(X) = r(E - X) + ||X||_r - r(E)$$

and $||X||_r = \sum_{x \in X} r(\{x\})$. We use $||X||$, when the polymatroid is clear from the context. This notion of duality and the following results in this subsection are from [9].

Theorem 2.6. *If P is a polymatroid, then P^* is a polymatroid.*

If P is a strict k-polymatroid, its dual is the k-dual mentioned above. And, as noted before, this dual is not an involution on the class of polymatroids. Notice that for a polymatroid $P = (E, r)$, compact or not, the dual P^* is always compact. This is because for any element $x \in E$,

$$r^*(E) - r^*(E \setminus \{x\}) = (r(\emptyset) + ||E|| - r(E)) - (r(x) + ||E \setminus \{x\}|| - r(E)) = 0.$$

Thus, taking $(P^*)^*$ might not be equal to P, but we have the following.

Proposition 2.7. *If P is a polymatroid, then $(P^*)^* = com(P)$.*

Proof. We can do the computation of $(r^*)^*$ for a set $A \subseteq E$;

$$\begin{aligned}(r^*)^*(A) &= ||A||_{r^*} + r^*(E \setminus A) - r^*(E) \\ &= \sum_{x\in A} r^*(x) + ||E \setminus A|| + r(A) - r(E) - ||E|| - r(\emptyset) + r(E) \\ &= \sum_{x\in A}(r(x) + r(E \setminus x) - r(E)) + \sum_{x\in E\setminus A} r(x) - \sum_{x\in E} r(x) + r(A) \\ &= \sum_{x\in A} r(x) - \sum_{x\in A} d_r(x) - \sum_{x\in A} r(x) + r(A) \\ &= r_{com(P)}(A),\end{aligned}$$

where the last equality follows from Equation (2). □

Observe that if $P = (E, r)$ is compact and $e \in E$, then $P/e = (E \setminus \{e\}, r_{/e})$ is compact because for any element $f \in E \setminus \{e\}$, $r_{/e}(E \setminus \{e\}) - r_{/e}(E \setminus \{e, f\})=r(E) - r(e) - r(E \setminus \{f\}) + r(e)=0$. However, $P \setminus e$ is not necessarily compact. Thus, the relation of this new dual and contraction and deletion cannot be as straightforward as that in matroids. This relation is explained in the following.

Proposition 2.8. *If P is a polymatroid, then*

$$(P \setminus e)^* = P^*/e \text{ and } (P/e)^* = com(P^* \setminus e).$$

In [15], duality and compactification are used as stepping stones to prove a splitter theorem for 3-connected 2-polymatroids.

3 An invariant for polymatroids

Associated with any 2-polymatroid, there is a well-known polynomial defined in [16]. The polynomial $S(P; u, v)$ of the 2-polymatroid P is defined as:

$$S(P; u, v) = \sum_{A\subseteq E} u^{r(E)-r(A)} v^{2|A|-r(A)}.$$

We now introduce a variation on the polynomial $S(P; u, v)$; this polynomial $N(P; u, v)$ is the focus of the rest of the paper.

Definition 3.1. *Given a polymatroid $P = (E, r)$ we define its polynomial $N(P; u, v)$ as:*

$$\begin{aligned}N(P; u, v) &= \sum_{A\subseteq E} u^{r(E)-r(A)} v^{||A||_r-r(A)} \\ &= \sum_{A\subseteq E} u^{r(E)-r(A)} v^{r^*(E)-r^*(E\setminus A)}.\end{aligned}$$

The equivalence between the two expressions follows from the equality $r^*(E) - r^*(A) = ||E \setminus A||_r - r(E \setminus A)$. Observe that when P is a strict 2-polymatroid, $N(P; u, v) = N(S; u, v)$. We notice that this polynomial also appears in [11], and most likely has been considered many times before.

The classical relation for the Tutte polynomial of a matroid M and its dual M^* is that $T(M^*; x, y) = T(M; y, x)$. The relation between $N(P)$ and $N(P^*)$ is given by the following.

Proposition 3.2. *If $P = (E, r)$ is a polymatroid, $com(P) = (E, r_{com})$ is its compactification and $P^* = (E, r^*)$ is its dual, then*

$$N(P^*; u, v) = \sum_{A \subseteq E} v^{r(E)-r(A)+r_{com}(E\setminus A)-r(E\setminus A)} u^{||A||_r - r(A)}.$$

In particular, if P is compact, then

$$N(P^*; u, v) = N(P; v, u).$$

Proof. By definition we have that

$$\begin{aligned} N(P^*; u, v) &= \sum_{A \subseteq E} u^{r^*(E)-r^*(A)} v^{(r^*)^*(E)-(r^*)^*(E\setminus A)} \\ &= \sum_{A \subseteq E} u^{||E\setminus A||_r - r(E\setminus A)} v^{r_{com}(E)-r_{com}(E\setminus A)}. \end{aligned}$$

The second equality follows from the proof of Proposition 2.7. If P is compact, $r(A) = r_{com}(A)$ for all $A \subseteq E$ and it is clear that $N(P^*; u, v) = N(P; v, u)$. If P is not compact, $r_{com}(E) - r_{com}(E \setminus A) = r(E) - r(E \setminus A) + r_{com}(A) - r(A)$. By making a change of variable we obtain the result. □

The Tutte polynomial can be computed using deletion-contraction. For a matroid M and an element e that is neither a loop nor a coloop, $T(M; x, y) = T(M \setminus e; x, y) + T(M/e; x, y)$. If e is a loop, $T(M; x, y) = yT(M \setminus e; x, y)$, and if it is a coloop, $T(M; x, y) = xT(M/e; x, y)$. In our case, we also have similar recursions stated in Propositions 3.3, 3.4, and 3.6. Notice that unlike the case of Tutte polynomials of matroids, the recurrence relations here do not cover all cases.

Proposition 3.3. *If $P = (E, r)$ is a polymatroid and $e \in E$ is a loop, then*

$$N(P; u, v) = 2N(P \setminus e; u, v).$$

Proof. This follows from the observation that, for a loop e in P, $r(A \cup \{e\}) = r(A)$ and $||A \cup \{e\}||_r = ||A||_r$. □

Proposition 3.4. *If $P = (E, r)$ is a polymatroid and $e \in E$ is a non-compact element, then*

$$N(P; u, v) = (u^{d_r(e)} - 1)N(P \setminus e; u, v) + N(com_e(P); u, v).$$

Proof. Let $\mathrm{com}_e(P)$=(E, r_e). Notice that for $A \subseteq E \setminus e$, $r_e(A) = r(A)$ and $r_e(E) = r(E \setminus e)$. Then,

$$\begin{aligned} N(com_e(P); u, v) - N(P \setminus e; u, v) &= \sum_{A \subseteq E \setminus e} u^{r_e(E) - r_e(A \cup \{e\})} v^{||A \cup \{e\}||_{r_e} - r_e(A \cup \{e\})} \\ &= \sum_{A \subseteq E \setminus e} u^{r(E) - r(A \cup \{e\})} v^{||A \cup \{e\}||_r - r(A \cup \{e\})}. \end{aligned}$$

Also, as $r(E) = r(E \setminus e) + d_r(e)$, we have that

$$u^{d_r(e)} N(P \setminus e; u, v) = \sum_{A \subseteq E \setminus e} u^{r(E) - r(A)} v^{||A||_r - r(A)}.$$

By adding both expressions we get the result. □

The above proposition also holds when e is a compact element. Observe that if e and f are non-compact elements in P, then e is non-compact in $P \setminus f$. Also, $\mathrm{com}_f(P \setminus e)$ and $\mathrm{com}_f(P) \setminus e$ are not necessarily equal, but in many cases, for example if $r(E \setminus f) + r(E \setminus e) = r(E) + r(E \setminus \{e, f\})$, they will be equal.

From the previous two propositions, we can concentrate on compact polymatroids with no loops. For a matroid M and element e that is neither a loop nor a coloop, the combinatorial information in $T(M \setminus e)$ and $T(M/e)$ is enough to recover all the information of $T(M)$. In polymatroids, this might not be the case; see the last example in Subsection 3.1. But in some cases, this is possible. For this purpose we define for a polymatroid $P = (E, r)$ and an element e, $\rho(e) = \min_{a \in E \setminus e}\{r(\{a, e\}) - r(a)\}$. Note that by submodularity, $\rho(e) \leq r(e)$. We define the *partial contraction* of e as the polymatroid $P/\rho(e) = (E \setminus e, r_{\rho(e)})$, where for $X \subseteq E \setminus e$

$$r_{\rho(e)}(X) = \begin{cases} 0, & \text{if } X = \emptyset, \\ r(X \cup \{e\}) - \rho(e), & \text{otherwise.} \end{cases} \tag{3}$$

Proposition 3.5. *If $P = (E, r)$ is a polymatroid and $e \in E$, then $P/\rho(e)$ is a polymatroid.*

Proof. Clearly $r_{\rho(e)}$ is normalized and increasing. Now, for any pair of subsets A and B of $E \setminus e$,

$$\begin{aligned} r_{\rho(e)}(A) + r_{\rho(e)}(B) &= r(A \cup \{e\}) - \rho(e) + r(B \cup \{e\}) - \rho(e) \\ &\geq r(A \cup B \cup \{e\}) - \rho(e) + r((A \cap B) \cup \{e\}) - \rho(e) \\ &= r_{\rho(e)}(A \cup B) + r_{\rho(e)}(A \cap B). \end{aligned}$$

□

When $\rho(e) = r(e)$, $P/\rho(e)$ is P/e. The *raison d'être* of $P/\rho(e)$ is the following. An element e is *near-skew* if $r(\{a, e\}) - r(a) = \rho(e) > 0$ for all $a \in E \setminus e$. It is called *skew* if it is near-skew and $\rho(e) = r(e)$.

Proposition 3.6. *If $P = (E, r)$ is a compact polymatroid and $e \in E$ is a near-skew element, then*

$$N(P; u, v) = N(P \setminus e; u, v) + v^{r(e)-\rho(e)} N(P/\rho(e); u, v).$$

Proof. We proceed by computing $N(P \setminus e; u, v)$ and $N(P/\rho(e); u, v)$. In $P \setminus e = (E \setminus e, r_{\setminus e})$, we have that for all $A \subseteq E \setminus e$, $r_{\setminus e}(E \setminus e) - r_{\setminus e}(A) = r(E) - r(A)$, as P is compact. Also, $||A||_{r_{\setminus e}} - r_{\setminus e}(A) = ||A||_r - r(A)$. Thus,

$$\begin{aligned} N(P \setminus e; u, v) &= \sum_{A \subseteq E \setminus e} u^{r_{\setminus e}(E \setminus e) - r_{\setminus e}(A)} v^{||A||_{r_{\setminus e}} - r_{\setminus e}(A)} \\ &= \sum_{A \subseteq E \setminus e} u^{r(E)-r(A)} v^{||A||_r - r(A)}. \end{aligned}$$

In $P/\rho(e) = (E \setminus e, r_{\rho(e)})$, we have that for all $A \subseteq E \setminus e$,

$$\begin{aligned} r_{\rho(e)}(E \setminus e) - r_{\rho(e)}(A) &= r(E) - \rho(e) - (r(A \cup \{e\}) - \rho(e)) \\ &= r(E) - r(A \cup \{e\}). \end{aligned}$$

Also, when e is a near-skew element of P, $||A||_{r_{\rho(e)}} = ||A||_r$ for all $A \subseteq E \setminus e$. Then,

$$\begin{aligned} ||A||_{r_{\rho(e)}} - r_{\rho(e)}(A) &= ||A||_r - r(A \cup \{e\}) + \rho(e) \\ &= ||A \cup \{e\}||_r - r(A \cup \{e\}) + (\rho(e) - r(e)). \end{aligned}$$

Thus,

$$\begin{aligned} N(P/\rho(e); u, v) &= \sum_{A \subseteq E \setminus e} u^{r_{\rho(e)}(E \setminus e) - r_{\rho(e)}(A)} v^{||A||_{r_{\rho(e)}} - r_{\rho(e)}(A)} \\ &= v^{\rho(e)-r(e)} \sum_{A \subseteq E \setminus e} u^{r(E)-r(A \cup \{e\})} v^{||A \cup \{e\}||_r - r(A \cup \{e\})}. \end{aligned}$$

By adding the polynomials $N(P \setminus e; u, v)$ and $v^{r(e)-\rho(e)} N(P/\rho(e); u, v)$ we get $N(P; u, v)$. □

3.1 Examples

The empty matroid $P_\emptyset = (\emptyset, 0)$ has $N(P_\emptyset) = 1$. A polymatroid with one element, $P_1 = (\{a\}, r_1)$, has rank function of the form $r_1(a) = k$ and then, $N(P_1) = u^k + 1$. A loopless compact polymatroid with two elements, $P_2 = (\{a, b\}, r_2)$, has rank function of the form $r_2(a) = r_2(b) = r_2(\{a, b\}) = k$ and then, $N(P_2) = u^k + v^k + 2$. A loopless compact polymatroid with three elements, $P_3 = (\{a, b, c\}, r_3)$, has rank function of the form $r_3(a) = k$, $r_3(b) = l$, $r_3(c) = m$, and $r_3(A) = n$ for $|A| > 1$, where $k, l, m \leq n \leq k + l, k + m, l + m$, and then, $N(P_3) = u^n + u^{n-k} + u^{n-l} + u^{n-m} + v^{k+l-n} + v^{k+m-n} + v^{l+m-n} + v^{k+l+m-n}$.

There are compact polymatroids that have no near-skew elements, for example the polymatroid $P = (\{a, b, c\}, r)$, where $r(\emptyset) = 0$, $r(a) = 3$, $r(b) = 4$, $r(c) = 5$

and $r(\{a,b\}) = r(\{a,c\}) = r(\{b,c\}) = r(\{a,b,c\}) = 5$. In this case $N(P;u,v) = u^5+u^2+u+1+v^2+v^3+v^4+v^7$. No element e of P is near-skew as $r(\{x,e\})-r(x) = 5-r(x)$, and for each e, there are two unequal options for $r(x)$. Also, $N(P\setminus a;u,v) = u^5+u+1+v^4$, $N(P/a;u,v) = u^2+2+v^2$ and $N(P/\rho(a);u,v) = u^5+2+v^5$. The polynomials are similar for b. For the element c, $N(P/\rho(c);u,v) = u^4+2+v^4$, $N(P/c;u,v) = 4$ and $N(P\setminus c;u,v) = u^5+u^2+u+v^2$. Thus, a decomposition using just deletion and (partial-)contraction seems more complicated in the absence of near-skew elements.

4 Polymatroids and hypergraphs

Here we consider a hypergraph as a triple, $\mathcal{H} = (V,E,\phi)$, where $\phi : E \to 2^V$, that is, $\phi(e)$ is a subset of vertices. For $A \subseteq E$, we write $\phi(A)$ for $\cup_{e\in A}\phi(e)$. When $|\phi(e)|$ is always at most k, for all $e \in E$, we say that $\mathcal{H}$ is a k-hypergraph.

For this paper we consider graphs as 2-hypergraphs. Given a fixed graph $G = (V,E,\phi_G)$, there are also two natural 3-hypergraphs associated with G. First, the *apex hypergraph* of G, $\mathcal{H}_{\bullet G} = (V\cup\{w\},E,\phi_{\bullet G})$, where $w \notin V$ and $\phi_{\bullet G}(e) = \phi_G(e)\cup\{w\}$. The second is the *edge-vertex hypergraph* of G, $\mathcal{H}_G = (V\cup\hat{E},E,\phi_E)$, where $\hat{E} = \{e'_1,\ldots,e'_m\}$ if $E = \{e_1,\ldots,e_m\}$ (that is, $\hat{E}$ is a disjoint copy of E) and $\phi_E(e) = \phi_G(e)\cup\{e'\}$. Both of these hypergraphs associated with graphs were considered in [20] and they will become relevant in Section 6.

Associated with any hypergraph $\mathcal{H}$ we have its chromatic polynomial, $\chi(\mathcal{H};\lambda)$ that counts the number of proper colorings of $\mathcal{H}$ with λ colours. A λ-colouring $c : V \to \{1,\ldots,\lambda\}$ is a proper colouring if no hyperedge of $\mathcal{H}$ is monochromatic. This clearly generalizes the colouring polynomial of graphs. Also, it is clear, that if the hypergraph has a hyperedge with just one vertex, then its chromatic polynomial equals 0.

There are two natural polymatroids associated with a hypergraph $\mathcal{H}$. First, the Boolean polymatroid $P_{\mathcal{H},B} = (E,r_{\mathcal{H},B})$, given by the submodular function $r_{\mathcal{H},B}(A) = |\phi(A)|$. The second is the generalization of the rank function for graphs (and matroids), $P_{\mathcal{H}}(E,r_{\mathcal{H}})$, where $r_{\mathcal{H}}(A)= |\phi(A)|-\kappa(\mathcal{H}|A)$. The value $\kappa(\mathcal{H}|A)$ is the number of connected components of the hypergraph $(\phi(A),A,\phi_{|A})$.

Associated with a polymatroid $P = (E,r)$, we have its characteristic polynomial,

$$\chi(P;x) = \sum_{A\subseteq E}(-1)^{|A|}x^{r(E)-r(A)}.$$

There is a classical result of Helgason [8] (also in Whittle [19]), that elucidates the relation between these two polynomials in the case of hypergraphs.

Theorem 4.1 (Helgason)**.**

$$\chi(\mathcal{H};\lambda) = \lambda^{\kappa(\mathcal{H})}\chi(P_{\mathcal{H}};\lambda).$$

Now, we relate the characteristic polynomial of a polymatroid P and the polynomial $N(P)$.

Proposition 4.2. *If P is a strict k-polymatroid, then*

$$\chi(P;\lambda) = \frac{1}{\alpha^{r(E)}} N(P;\alpha\lambda,\frac{1}{\alpha}), \tag{4}$$

for $\alpha = \sqrt[k]{-1}$.

Proof. If $P = (E, r)$ is a strict k-polymatroid, then $||A|| = k|A|$, for all $A \subseteq E$. Then, by expanding the right-hand side of Equation (4), we get

$$\begin{aligned} N(P;\alpha\lambda,\frac{1}{\alpha}) &= \sum_{A\subseteq E} \lambda^{r(E)-r(A)} \frac{\alpha^{r(E)}}{\alpha^{r(A)}} \frac{\alpha^{r(A)}}{\alpha^{k|A|}} \\ &= \alpha^{r(E)} \sum_{A\subseteq E} \lambda^{r(E)-r(A)} (-1)^{|A|}. \end{aligned}$$

□

Notice that the previous result can be used with any k-th root of -1. Proposition 4.2 will be relevant in Section 6.

5 Some easy evaluations

For a matroid $M = (E, r)$, its Tutte polynomial has some trivial evaluations: $T(0, 0) = 0$, $T(2, 2) = 2^{|E|}$ and in general along the hyperbola $(x − 1)(y − 1) = 1$, $T(x, y) = y^{|E|}(y − 1)^{−r(E)}$. Also, it has some well-known combinatorial interpretations: $T(2, 1)$ equals the number of independent sets, $T(1, 2)$ equals the number of spanning sets, $T(1, 1)$ equals the number of bases of the matroid. It is straightforward to prove similar evaluations for $N(P; u, v)$.

Proposition 5.1. *For any polymatroid $P = (E, r)$, we have the following evaluations for the polynomial N.*

- *Along the hyperbola $uv = 1$, $N(P; u, v) = v^{-r(E)} \prod_{a\in E}(v^{r(a)} + 1)$.*
- *In particular,*

$$N(P;-1,-1) = \begin{cases} 0, & \text{if } \exists e \in E,\ r(e) \equiv 1 \pmod 2, \\ (-1)^{r(E)} 2^{|E|}, & \text{otherwise.} \end{cases}$$

  Also, $N(P; 1, 1) = 2^{|E|}$.
- *$N(P; 0, 1)$ equals the number of spanning sets, that is, subsets $A \subseteq E$ such that $r(A) = r(E)$.*
- *$N(P; 1, 0)$ equals the number of subsets $A \subseteq E$ such that $r(A) = ||A||_r$.*

In a 2-polymatroid $P = (E, r)$ a subset X of E is a *2-matching* if $r(X) = 2|X|$. In a graph G, a *free loop* is an edge that is not incident to any vertex. For a graph $G = (V, E)$, where free loops are allowed, let $P_{G,B} = (E, r_{G,B})$ be the boolean 2-polymatroid associated with G when seen as a hypergraph. Free loops in G correspond to loops in $P_{G,B}$, but loops in G correspond to elements of rank 1 in $P_{G,B}$. With these definitions, a set A is a 2-matching in $P_{G,B}$ if and only if A is a matching in G. Notice that $P_{G,B}$ is a strict 2-polymatroid if and only if G has no loops or free loops.

Clearly, $P_{G,B} \setminus e = P_{G\setminus e,B}$. However, the contraction of an element in $P_{G,B}$ behaves in a different way. Define the graph $G/\!\!/e$ as follows. First, we delete the edge e and its incident vertices. Then, for an edge f, $f \neq e$, either f has no common vertex with e and stays the same; or has one common vertex with e and becomes a loop at its other endpoint; or it is a parallel edge to e and becomes a free loop. In order for the reader to visualize the operation $/\!\!/e$, we give an example in Figure 1. With this operation we have that $P_{G,B}/e = P_{G/\!\!/e,B}$. Let us denote by $m(G)$ the number of matchings of G. It follows from the classical theory of matchings that $m(G) = m(G \setminus e) + m(G/\!\!/e)$.

For the graph G in Figure 1 we have that $N(P_{G,B}; u, v) = u^4 + 5u^2 + 8uv + 2uv^3 + 8v^2 + 5v^4 + v^6 + 2$. As $P_{G,B}$ is a strict 2-polymatroid, $N(P_{G,B}; 1, 0) = 8$, the number of 2-matchings of $P_{G,B}$. However, this is not true for $P_{G/\!\!/e,B}$, as $N(P_{G/\!\!/e,B}; u, v) = u^2 + 3u + 3 + uv + 4v + 3v^2 + v^3$, then when evaluated at (1,0) we get 7, but $P_{G/\!\!/e,B}$ has only two 2-matching. The reason for this is that at (1,0) we are counting the family of sets $\emptyset$,$\{a\}$, $\{c\}$, $\{d\}$, $\{b\}$, $\{a, c\}$, $\{a, d\}$. This suggests considering the following definition: a *matching* in a (hyper)graph G is a set of (hyper)edges without common vertices that does not contain free loops.

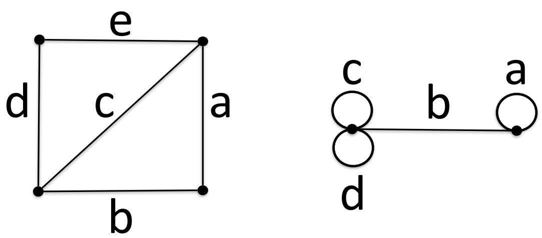

Figure 1: On the left is the graph G and on the right the graph $G/\!\!/e$

For a polymatroid $P = (E, r)$, we define a *hypermatching* as any set $X \subseteq E$ such that $r(X) = ||X||_r$ and it does not contain any loops. If X is a hypermatching that satisfies $r(X) = r(E)$, we called it a *perfect hypermatching*. We have the following.

Proposition 5.2. *For any loopless polymatroid $P = (E, r)$,*

- $N(1, 0)$ *equals the number of hypermatchings.*
- $N(0, 0)$ *equals the number of perfect hypermatchings.*

Observe that if P is compact the number of perfect hypermatchings is the same for P and P^*.

Notice that it is not true in general that $N(P;1,0) = N(P\setminus e;1,0)+N(P/e;1,0)$, as it can be checked in the last example of Subsection 3.1. Also, it is not the case that $N(P_{G,B};1,0) = N(P_{G,B}\setminus e;1,0) + N(P_{G,B}/e;1,0)$, as it can be checked in the example of Figure 1.

When P is matroid, $N(P;1,0)$ is the number of independent sets and also the number of hypermatchings in this 1-polymatroid. We call these 1-matchings. Thus, when the matroid is graphic, $P \cong M(G)$, we can interpret a forest in G as a 1-matching in P.

For a loopless $P = (E,r)$, we have that

$$N(P;u,0) = \sum_{A \text{ hypermatching}} u^{r(E)-r(A)}.$$

For the boolean polymatroid $P_{\mathcal{H},B} = (E, r_{\mathcal{H},B})$ of the hypergraph $\mathcal{H} = (V,E,\phi)$ with n non-isolated vertices, if we denote by m_k the number of hypermatchings A with $r_{\mathcal{H},B}(A) = k$, we have that

$$N(P_{\mathcal{H},B};u,0) = \sum_{k=0}^{n} m_k u^{n-k}.$$

Thus, in the case that $\mathcal{H}$ is the 2-hypergraph of a loopless graph G, we get as an evaluation the *matching defect polynomial*, that is,

$$(-i)^n N(P_{G,B};it,0) = \sum_{k\geq 0}(-1)^k \Phi_k(G)t^{n-2k},$$

where $i^2 = -1$ and $\Phi_k(G)$ is the number of matchings in G with k edges.

Similarly, for a $P = (E,r)$, we have that

$$N(P;0,v) = \sum_{A \text{ spanning}} v^{||A||_r - r(A)}.$$

For the boolean polymatroid $P_{\mathcal{H},B} = (E, r_{\mathcal{H},B})$ of the hypergraph $\mathcal{H} = (V,E,\phi)$ with n vertices, if we denote by c_k the number of spanning sets A with $k = \sum_{a\in A}|a| - n$, we have that

$$N(P_{\mathcal{H},B};0,v) = \sum_{k\geq 0} c_k v^k.$$

Thus, in the case that $\mathcal{H}$ is the 2-hypergraph of a loopless graph G, we get as an evaluation the *edge cover polynomial*, that is,

$$t^{n/2} N(P_{G,B};0,\sqrt{t}) = \sum_{k\geq 0} c_k(G)t^k,$$

where $c_k(G)$ is the number of edge covers of G with k edges.

6 Another two interpretations of $N(P)$

Recently, Zhang and Dong [20] proved the following theorems about the chromatic polynomial of the two hypergraphs associated with the graph $G = (V, E, \phi_G)$ mentioned in Section 4. For a graph $G = (V, E)$, the polynomial

$$I(G; x) = \sum_{U \subseteq V \text{ stable}} x^{|U|}$$

is the enumerator of stable sets in G. Here a stable set is a set of vertices in G, no two of which are adjacent.

Theorem 6.1 (Zhang-Dong)**.**

$$\chi(\mathcal{H}_{\bullet G}; \lambda) = \lambda(\lambda - 1)^n I(G; \frac{1}{\lambda - 1}).$$

The enumerator of stable sets has been considered in other works with an equivalent definition. The most relevant definition for us is the one given by Farr [6],

$$A(G; x) = \sum_{U \subseteq V \text{ stable}} x^{|U|}(1 - x)^{|V \setminus U|},$$

from which it is clear that

Proposition 6.2.

$$A(G; x) = (1 - x)^{|V|} I(G; \frac{x}{1 - x}).$$

This invariant was already related to the boolean polymatroid of the 2-hypergraph of a graph G. In Oxley and Whittle [16], they proved that

Proposition 6.3 (Oxley and Whittle)**.**

$$A(G; x) = x^{r_{G,B}(E)} \chi(P_{G,B}; \frac{1}{x}).$$

We now give a different proof of Zhang and Dong's result using the theory of polymatroid invariants.

New proof of 6.1. The fundamental observation is the following. Given G and the hypergraph $\mathcal{H}_{\bullet G}$, the rank function of the polymatroid associated with $\mathcal{H}_{\bullet G}$ is

$$\begin{aligned} r_{\mathcal{H}_{\bullet G}}(A) &= |\phi(A)| - \kappa(\mathcal{H}_{\bullet G}|A) \\ &= |\phi_G(A)| + 1 - 1 \\ &= r_{G,B}(A). \end{aligned}$$

Notice that $\kappa(\mathcal{H}_{\bullet G}|A) = 1$ because the apex vertex makes any set of hyperedges a connected hypergraph. Now, by Theorem 4.1 and Proposition 6.2 and 6.3

$$\begin{aligned}
\chi(\mathcal{H}_{\bullet G};\lambda) &= \lambda^{\kappa(\mathcal{H}_{\bullet G})}\chi(P_{\mathcal{H}_{\bullet G}};\lambda) \\
&= \lambda\chi(P_{G,B};\lambda) \\
&= \lambda^{|V|+1}A(G;\frac{1}{\lambda}) \\
&= \lambda^{|V|+1}\left(1-\frac{1}{\lambda}\right)^{|V|} I(G;\frac{1}{\lambda-1}) \\
&= \lambda(\lambda-1)^{|V|}I(G;\frac{1}{\lambda-1}).
\end{aligned}$$

□

The second theorem of Zhang and Dong, also in [20], relates the chromatic polynomial of the hypergraph $\mathcal{H}_G$ associated with the graph G and the Tutte polynomial of G and it is the following.

Theorem 6.4 (Zhang-Dong)**.**

$$\chi(\mathcal{H}_G;\lambda) = \lambda^{|E|-|V|+2\kappa(G)}(-1)^{|V|-\kappa(G)}T(G;1-\lambda^2,\frac{\lambda-1}{\lambda}).$$

New proof of 6.4. The fundamental observation is that if $G = (V, E, \phi_G)$ then

$$\begin{aligned}
r_{\mathcal{H}_G}(A) &= |\phi_E(A)| - \kappa(\mathcal{H}_G|A) \\
&= |A| + |\phi_G(A)| - \kappa(G|A) \\
&= |A| + |V(G|A)| - \kappa(G|A) \\
&= |A| + r(A),
\end{aligned}$$

where $r(A)$ is the (matroidal) rank of A. Now, a sequence of change of variables produces the result. On one hand, for a graph $G = (V, E)$, the cardinality-rank polynomial

$$B(G;x,y) = \sum_{A\subseteq E} x^{r(A)}y^{|A|}$$

was define in Brylaswki [3]. This polynomial is clearly equivalent to the Tutte polynomial, because

$$(x-1)^{r(E)}B(G;\frac{1}{(x-1)(y-1)},y-1) = T(G;x,y).$$

Then,

$$T(G;1-\lambda^2,\frac{\lambda-1}{\lambda}) = (-\lambda^2)^{r(E)}B(G;\frac{1}{\lambda},\frac{-1}{\lambda}).$$

On the other hand,

$$\begin{aligned}
\lambda^{|E|+r(E)}B(G;\frac{1}{\lambda},\frac{-1}{\lambda}) &= \lambda^{|E|+r(E)}\sum_{A\subseteq E}(-1)^{|A|}\left(\frac{1}{\lambda}\right)^{|A|+r(A)} \\
&= \sum_{A\subseteq E}(-1)^{|A|}\lambda^{r_{\mathcal{H}_G}(E)-r_{\mathcal{H}_G}(A)} \\
&= \chi(P_{\mathcal{H}_G};\lambda).
\end{aligned}$$

The result follows from our fundamental observation and Theorem 4.1,

$$\begin{aligned}
\chi(\mathcal{H}_G;\lambda) &= \lambda^{\kappa(\mathcal{H}_G)}\chi(P_{\mathcal{H}_G};\lambda) \\
&= \lambda^{|E|+|V|}B(G;\frac{1}{\lambda},\frac{-1}{\lambda}) \\
&= \lambda^{|E|+|V|}(-\lambda^2)^{-r(E)}T(G;1-\lambda^2,\frac{\lambda-1}{\lambda}) \\
&= \lambda^{|E|-|V|+2\kappa(G)}(-1)^{r(E)}T(G;1-\lambda^2,\frac{\lambda-1}{\lambda}).
\end{aligned}$$

□

To conclude this section, we give two corollaries from the previous results. For a loopless graph G, the polymatroid $P_{\mathcal{H}_{\bullet G}}$ is a strict 2-polymatroid. Thus, we can recast Theorem 6.1 as follows.

Corollary 6.5. *For a loopless graph G with n vertices,*

$$(-i)^n N(P_{\mathcal{H}_{\bullet G}};i\lambda,-i) = (\lambda-1)^n I(G;\frac{1}{\lambda-1}).$$

Proof. Using Proposition 4.2, where in this case $\alpha=\sqrt[2]{-1}=i$, we get

$$\begin{aligned}
\lambda(\lambda-1)^n I(G;\frac{1}{\lambda-1}) &= \chi(\mathcal{H}_{\bullet G};\lambda) \\
&= \lambda\chi(P_{\mathcal{H}_{\bullet G}};\lambda) \\
&= \lambda\frac{1}{i^{r_{\mathcal{H}_{\bullet G}}(E)}}N(P_{\mathcal{H}_{\bullet G}};i\lambda,\frac{1}{i}).
\end{aligned}$$

□

Also, for a loopless graph G, the polymatroid $P_{\mathcal{H}_G}$ is a strict 2-polymatroid, then we obtain from Theorem 6.4 the following.

Corollary 6.6. *For a loopless graph $G=(V,E)$ with m edges and rank $r(E)$,*

$$N(P_{\mathcal{H}_G};i\lambda,-i) = (i\lambda)^{|E|-r(E)}T(G;1-\lambda^2,\frac{\lambda-1}{\lambda}).$$

Proof. From the proof of Theorem 6.4, we know that

$$\begin{aligned}\chi(P_{\mathcal{H}_G};\lambda) &= \lambda^{|E|+r(E)}B(G;\frac{1}{\lambda},\frac{-1}{\lambda}) \\ &= \frac{\lambda^{|E|+r(E)}}{(-\lambda^2)^{r(E)}}T(G;1-\lambda^2,\frac{\lambda-1}{\lambda}) \\ &= \lambda^{|E|-r(E)}(-1)^{r(E)}T(G;1-\lambda^2,\frac{\lambda-1}{\lambda}).\end{aligned}$$

Using Proposition 4.2, where in this case $\alpha = \sqrt[2]{-1} = i$, we get

$$\begin{aligned}N(P_{\mathcal{H}_G};i\lambda,-i) &= i^{|E|+r(E)}\chi(P_{\mathcal{H}_G};\lambda) \\ &= i^{|E|+r(E)}\lambda^{|E|-r(E)}(-1)^{r(E)}T(G;1-\lambda^2,\frac{\lambda-1}{\lambda}).\end{aligned}$$

□

7 Further results and conclusion

McDiarmid [14] considered a family of duals for polymatroids by assigning weights to the elements and using them to replace the term $||A||_r$. It is possible that this could lead to other interesting polynomials associated with polymatroids, although we have not investigated this possibility. In the absence of any other information, it seems that the most natural weight to assign an element e is $r(\{e\})$, which is what we have done here.

However, there is also another generalization of $S(P)$. Let $\mathcal{P}$ be a class of polymatroids that is closed under minors. Given a polymatroid $P \in \mathcal{P}$ and an element $e \in P$, we can consider that $E(P \setminus e) \subset E(P)$ and $E(P/e) \subset E(P)$. Thus, we define $\mathcal{E} = \cup_{P\in\mathcal{P}}E(P)$ and set an arbitrary but fixed integer function $\omega : \mathcal{E} \to \mathbb{Z}$. We call ω the weight function of $\mathcal{P}$.

Now, for a polymatroid $P \in \mathcal{P}$, $P = (r, E)$, and a weight function ω we define $N(P,\omega;u,v)$ as:

$$N(P,\omega;u,v) = \sum_{A\subseteq E} u^{r(E)-r(A)}v^{||A||_\omega - r(A)},$$

where $||A||_\omega = \sum_{x\in A}\omega(x)$. This in general is not a polynomial, unless $||A||_\omega - r_P(A) \geq 0$ for all $A \in \mathcal{E}$ and $P \in \mathcal{P}$. In particular, $\omega(a) \geq r_P(a)$ for all $a \in \mathcal{E}$ and $P \in \mathcal{P}$. However, when $\mathcal{P}$ consists of all the minors of a fixed polymatroid P, or if it is the class of k-polymatroids, the above condition can be satisfied for infinitely many integer functions ω.

The relevance of $N(P,\omega;u,v)$ is that one can recover a deletion-contraction formula, while there appears to be no reasonable deletion-contraction formula for $N(P;u,v)$.

Proposition 7.1. *If $P = (E,r)$ is a polymatroid and $e \in E$, then*

$$N(P,\omega;u,v) = u^{d_r(e)}N(P\setminus e,\omega;u,v) + v^{\omega(e)-r(e)}N(P/e,\omega;u,v).$$

Proof. As in the proof of Proposition 3.6, in $P \setminus e = (E \setminus e, r_{\setminus e})$, we have that for all $A \subseteq E \setminus e$,

$$\begin{aligned} r_{\setminus e}(E \setminus e) - r_{\setminus e}(A) &= r(E \setminus e) - r(A) \\ &= r(E) - r(A) - (r(E) - r(E \setminus e)) \\ &= r(E) - r(A) - d_r(e). \end{aligned}$$

Also, $||A||_\omega - r_{\setminus e}(A) = ||A||_\omega - r(A)$. Thus

$$\begin{aligned} N(P \setminus e, \omega; u, v) &= \sum_{A \subseteq E \setminus e} u^{r_{\setminus e}(E \setminus e) - r_{\setminus e}(A)} v^{||A||_\omega - r_{\setminus e}(A)} \\ &= u^{-d_r(e)} \sum_{A \subseteq E \setminus e} u^{r(E) - r(A)} v^{||A||_\omega - r(A)}. \end{aligned}$$

Again, for $P/e = (E \setminus e, r_{/e})$, we have that for all $A \subseteq E \setminus e$,

$$\begin{aligned} r_{/e}(E \setminus e) - r_{/e}(A) &= r(E) - r(e) - (r(A \cup \{e\}) - r(e)) \\ &= r(E) - r(A \cup \{e\}). \end{aligned}$$

Also,

$$\begin{aligned} ||A||_\omega - r_{/e}(A) &= ||A||_\omega - r(A \cup \{e\}) + r(e) \\ &= ||A \cup \{e\}||_\omega - r(A \cup \{e\}) + r(e) - \omega(e). \end{aligned}$$

Thus

$$\begin{aligned} N(P/e, \omega; u, v) &= \sum_{A \subseteq E \setminus e} u^{r_{/e}(E \setminus e) - r_{/e}(A)} v^{||A||_\omega - r_{/e}(A)} \\ &= v^{r(e) - \omega(e)} \sum_{A \subseteq E \setminus e} u^{r(E) - r(A \cup \{e\})} v^{||A \cup \{e\}||_\omega - r(A \cup \{e\})}. \end{aligned}$$

By adding the polynomials $u^{d_r(e)} N(P \setminus e, \omega; u, v)$ and $v^{\omega(e) - r(e)} N(P/e, \omega; u, v)$ we get $N(P, \omega; u, v)$.

□

Notice that, if $\omega(e) = r_P(e)$ for all $e \in E$, we get $N(P; u, v) = N(P, \omega; u, v)$; however, since ω cannot depend on P, the equality $\omega(e) = r_P(e)$ can hold for P but fail if P is replaced by one of its proper minors. We do not know of any relationship between $N(P, \omega; u, v)$ and $N(P^*, \omega; u, v)$, in the case that both expressions are polynomials.

Clearly, the polynomials $N(P; u, v)$ and $N(P, \omega; u, v)$ are very general and more families of polymatroids need to be explored in order to find interesting evaluations. For example, in [11], the authors use the matroids $M = (E, r)$ and $M^* = (E, r^*)$ to construct the polymatroid $P = (\rho, E)$, where $\rho = r + r^*$. This polymatroid has rank $n = |E|$ and the authors consider also the quantity $s = \max\{r(E), r^*(E)\}$. They find that $N(P; x + 1, 1) = \sum_{i=0}^{s} H_i x^{n-i}$ is the generating function for the number H_i of bi-independent sets, that is, sets independent in both M and M^*, of size i. Similarly, they find that $y^{-n/2}N(P;1,\sqrt{y}+1) = \sum_{i=0}^{n} S_i y^i$ is the generating function for the number S_i of bi-spanning sets, that is, sets spanning in both M and M^*, of size i.

As pointed out in [20], evaluating at $\lambda = -1$ in Theorem 6.4 we get $T(G;0,2)$ that has a combinatorial interpretation as the number of totally cyclic orientations of G. A combinatorial proof of this fact could be interesting.

As we mentioned in the introduction, there are currently other attempts at generalising the Tutte polynomial to polymatroids in [2] and [4]. In principle, our approach is different as both of the aforementioned papers investigate the maximal facet of the polytope associated with a polymatroid, while our work stays in the realm of polymatroids. There are obvious differences between the definition of duality in our work and the other two papers. However, for compact strict k-polymatroids, this difference becomes less obvious. Thus, it would be very interesting to check if the polynomial presented here is related to any of the polynomials in [2] or [4].

Acknowledgements

The authors would like to thank the referees for carefully reading our manuscript and for giving constructive comments which substantially helped improve the quality of the paper. The second author also thanks grant CONACYT A1-S-8195.

References

[1] F. Ardila, A. Fink and F. Rincón, Valuations for matroid polytope subdivisions, *Canad. J. Math.* 62(6) (2010), 1228–1245.

[2] O. Bernardi, T. Kálmán and A. Postnikov, Universal Tutte polynomial, (preprint), (2020): arXiv:2004.00683.

[3] T. Brylawski, The Tutte polynomial I, General theory, in: *Matroid theory and its applications*, Liguori, Naples (1982), 125–275.

[4] A. Cameron and A. Fink, The Tutte polynomial via lattice point counting, (preprint), (2018): arXiv:1802.09859.

[5] J. Edmonds, Submodular functions, matroids, and certain polyhedra, in *Combinatorial Structures and their Applications* (Proc. Calgary Internat. Conf., Calgary, Alta., 1969) Gordon and Breach, New York (1970), 69–87.

[6] G. E. Farr, A correlation inequality involving stable set and chromatic polynomials, *J. Combin. Theory Ser. B* 58(1) (1993), 14–21.

[7] O. Farràs, J. Martí-Farré and C. Padró, Ideal multipartite secret sharing schemes, in: *Advances in cryptology—EUROCRYPT* 2007, Lec. Notes in Comp. Sci. 4515, Springer, Berlin (2007), 448–465.

[8] T. Helgason, Aspects of the theory of hypermatroids, Lec. Notes in Math. 411 (1974), 191–213.

[9] S. Jowett, S. Mo and G. Whittle, Connectivity functions and polymatroids, *Adv. Appl. Math.* 81 (2016), 1–12.

[10] J. P. S. Kung, A characterization of orthogonal duality in matroid theory, *Geom. Dedicata* 15(1) (1983), 69–72.

[11] J. P. S. Kung and G. Whittle, Matroid invariants constructed from the connectivity function, (in preparation), (2020), 10 pp.

[12] L. Lafforgue, *Chirurgie des grassmanniennes*, CRM Monograph Series, vol. 19, Amer. Math. Soc., Providence, RI, 2003.

[13] E. L. Lawler, *Combinatorial optimization: networks and matroids*, Holt, Rinehart and Winston, New York-Montreal, Que.-London, 1976.

[14] C. J. H. McDiarmid, Rado's theorem for polymatroids, *Math. Proc. Cambridge Philos. Soc.* 78(2) (1975), 263–281.

[15] J. Oxley, C. Semple and G. Whittle, A splitter theorem for 3-connected 2-polymatroids, *Electron. J. Combin.* 26(2) (2019), #P2.37, 95 pp.

[16] J. Oxley and G. Whittle, A characterization of Tutte invariants of 2-polymatroids, *J. Combin. Theory Ser. B* 59(2) (1993), 210–244.

[17] D. E. Speyer, Tropical linear spaces, *SIAM J. Discrete Math.* 22(4) (2008), 1527–1558.

[18] G. Whittle, Duality in polymatroids and set functions, *Combin. Probab. Comput.* 1(3) (1992), 275–280.

[19] G. Whittle, A geometric theory of hypergraph colouring, *Aequationes Math.* 43(1) (1992), 45–58.

[20] R. Zhang and F. Dong, Properties of chromatic polynomials of hypergraphs not held for chromatic polynomials of graphs, *European J. Combin.* 64 (2017), 138–151.

(Received 9 May 2020; revised 2 Mar 2021, 22 May 2021)