AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **80(3)** (2021), Pages 305–321

A group theoretical approach to the partitioning of integers: Application to triangular numbers, squares, and centered polygonal numbers

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Abstract

The main result of this work is a group theoretic approach to partitioning integers into an arbitrary number of members of an arbitrary set of increasing non-negative integers. This work builds upon the existing literature on partitioning of numbers into triangular numbers and square numbers. Generating function formulae for the number of partitions and distinct partitions of the integers into arbitrarily many triangular numbers, squares, and centered polygonal numbers are presented as applications. These formulae arise from the connection between such partitioning and the structure of the symmetric group, namely the size and cycle structure of the conjugacy classes. Expressions in terms of Young's tableaux and several illustrative examples are given.

1 Introduction

The main result of this paper (Theorem 2.2) provides a group theoretic approach to determining the number of partitions and distinct partitions of the given integer n

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into the sum of m members of a set S of increasing non-negative real integers. In a typical setting, S would be a special set of interest such as the triangular numbers, $S = \Delta$. Indeed, the triangular numbers will be explored as will the square numbers, $S = \Box$, and the centered polygonal numbers, $S = C$.

A secondary goal of the current work is to directly extend the results of an interesting study in 2004 by Hirschhorn and Sellers [9]. In that work, the authors build off of their previous work on the *representations* of an integer as a summation of three triangular numbers [10] to develop generating function formulae for the *partitions* and *distinct partitions* of an integer into three triangular numbers [9]. The current work shows how to get the partitions and distinct partitions for an arbitrary number of triangular numbers.

In a more general context, the partitioning of integers into triangular numbers remains a topic of investigation. Ono, Robbins, and Wahl [11] have utilized the theory of modular forms to get formulae in terms of the Dedekind η-function for the representations of cases 2, 3, and even cases up to 12. Cañadas [3, 4] has employed the use of posets in the study of general partitions, including with triangular numbers and other polygonal numbers. Additionally, several works have studied combinations of square and triangular numbers [1, 2, 17, 18, 21].

The current work does not invoke the "heavy machinery" of modular forms, but rather it takes a more elementary approach by making a connection to group theory, particularly the properties of the symmetric group, $\mathfrak{S}_m$ (also called the permutation group). The fact that the result extends to centered polygonal numbers leads to several ancillary theorems which are proven and discussed in Section 4.

A typical approach in the study of partitioning of integers is to introduce a generating function of the form,

$$\Psi_S(q) = \sum_{j=0}^{\infty} q^{S(j)},$$

where $S(j)$ is referring to the j^{th} member of set S. For the case of the triangular numbers [9],

$$\psi(q) := \Psi_\Delta(q) = \sum_{j=0}^{\infty} q^{\frac{j(j+1)}{2}} = \sum_{j=0}^{\infty} q^{T(j)}, \tag{1.1}$$

(where $T(j)$ represents the j^{th} triangular number) is used. This serves as the basis for a generating function approach to obtaining the representations of a number in terms of m triangular numbers, t_m. A representation is an ordered summation ($1 + 1 + 3 = 5$ is a separate representation from $3 + 1 + 1 = 5$ or $1 + 3 + 1 = 5$) of m triangular numbers to obtain a target number, n.

One can readily see that the m-fold product,

$$\psi(q)^m = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_m=0}^{\infty} q^{T(j_1)+T(j_2)+\cdots+T(j_m)}$$

will serve as the generating function formula for the representations. Creating a single summation leads to coefficients that are precisely the number of representations. That is,

$$\psi(q)^m = \sum_{n=0}^{\infty} t_m(n) q^n. \tag{1.2}$$

It is also important to note another feature of ψ which is,

$$\psi(q^r) = \sum_{n=0}^{\infty} q^{rT(n)}. \tag{1.3}$$

The main idea of the work of Hirschhorn and Sellers was to determine, for the case $m = 3$, the generating function formula, in terms of ψ, when the summation is no longer ordered ($1 + 1 + 3 = 5$, $3 + 1 + 1 = 5$, and $1 + 3 + 1 = 5$ are all the same). It is noted that adding zero is allowed (e.g., $0 + 1 + 4 = 5$). These values are called partitions, $p_{3\Delta}$, where 3 represents the number of triangular numbers, and were shown to be [9]

$$P_{3\Delta} \equiv \sum_{n=0}^{\infty} p_{3\Delta}(n) q^n = \frac{1}{6}\left(\psi(q)^3 + 3\psi(q^2)\psi(q) + 2\psi(q^3)\right). \tag{1.4}$$

They further derived the generating function formula for the case when only three distinct numbers made up the summation ($1 + 1 + 3 = 5$ would not be counted because 1 appears twice). This eventually leads to $P^d_{3\Delta}$ which produces the number of ways a given number can be formed using 3 distinct triangular numbers [9]. This was shown to be

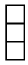
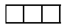
$$P^d_{3\Delta} \equiv \sum_{n=0}^{\infty} p^d_{3\Delta}(n) q^n = \frac{1}{6}\left(\psi(q)^3 - 3\psi(q^2)\psi(q) + 2\psi(q^3)\right). \tag{1.5}$$

The above generating function formulae will be recovered below along with the general expressions for $P_{m\Delta}(n)$ and $P^d_{m\Delta}(n)$ that hold for any m. One notes in Eqs. (1.4) and (1.5) that the expression for $P_{m\Delta}(n)$ and $P^d_{m\Delta}(n)$ are made up of terms of the form $\prod_i \psi(q^{\lambda_i})^{\nu_i}$, where $\sum_i \lambda_i \cdot \nu_i = m$ for each term. The main result of this work is to show that one can obtain a general version of this, both in terms of arbitrary m and S. One can, in fact, obtain an expression for P_{mS} and P^d_{mS}.

2 General partitioning of integers with the working example triangular numbers

Several simple ideas come into play in establishing the general case. First, determination of representations, partitions, and distinct partitions are, at the heart of the matter, counting problems and, in particular, counting permutations. Second, the choice of looking for representations, partitions, or distinct partitions effectively

Table 1: Group theory data for $\mathfrak{S}_3$.

Int. part. 3	C_i	$\lvert C_i \rvert$	$\chi^{(2)}$	Young's	$\left(\psi(q^{\lambda_i})\right)^{\nu_i}$
1+1+1	1^3	1	1		$\Psi_S(q)^3$
1+2	$1^1 2^1$	3	-1		$\Psi_S(q)\Psi_S(q^2)$
3	3^1	2	1		$\Psi_S(q^3)$

establishes constraints on the permutations allowed to be counted. For example, should $1+1+3=5$ be allowed or not? Finally, these constraints establish equivalence relations. For example, are $1+1+3=5$, $3+1+1=5$, and $1+3+1=5$ all the same or are they to be counted separately?

Taken together, this leads one to anticipate that the symmetric group of m elements might well govern the nature of how the different choices of constraints and equivalence classes impact the counting. Indeed there is a deep connection to the symmetric group and, in fact, the constraints are connected to the structure of the group. The upshot is that the full power of group theory can be brought to bear on this problem. To help matters, the symmetric group is arguably the most well-studied group, so making such a connection anchors to a strong foundation. The constraints imposed when one seeks representations, partitions, and distinct partitions are carried in the cycle structure, conjugacy classes, and characters of $\mathfrak{S}_m$. This sets the stage for a couple of theorems which are presented in this section.

Theorem 2.1. *The terms making up $P_{mS}(n)$ and $P^d_{mS}(n)$ map to the conjugacy classes of $\mathfrak{S}_m$ as*

$$\Psi_S\left(q^{\lambda_1}\right)^{\nu_1}\Psi_S\left(q^{\lambda_2}\right)^{\nu_2}\cdots\Psi_S\left(q^{\lambda_l}\right)^{\nu_l}\rightarrow(\lambda_1^{\nu_1},\lambda_2^{\nu_2},\ldots,\lambda_l^{\nu_l})\equiv C_i.$$

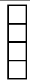
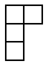
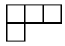
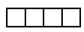
Proof. The overall power is $\sum\lambda_i\nu_i=m$. The $\Psi_S^{\nu_i}$ gives the number of independent terms to permute over, while q^{λ_i} fixes λ_i terms. Thus a cycle type of the permutations is determined. A fundamental theorem associated with $\mathfrak{S}_m$ is that the cycle type determines the conjugacy classes [8, 15]. □

It is helpful to consider a couple of illustrative examples of applications of this theorem. It is also helpful, although not required, to express the conjugacy classes in the form of Young's tableaux [8, 12, 15]. The conjugacy classes of $\mathfrak{S}_m$ are associated with the possible integer partitions of m [8, 12]. Young's tableaux are also representations of the integer partitions of m [12, 15]. These are collected in Tables 1 and 2 for the cases $m=3$ and $m=4$.

Consider the case $\Psi_\Delta=\psi$ (the triangular numbers) and $m=3$. Here,

$$\psi(q)^3=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}q^{T(a)+T(b)+T(c)}$$

Table 2: Group theory data for $\mathfrak{S}_4$.

Int. part. 4	C_i	$\lvert C_i \rvert$	$\chi^{(2)}$	Young's	$\left(\Psi_S(q^{\lambda_i})\right)^{\nu_i}$
1+1+1+1	1^4	1	1		$\Psi_S(q)^4$
1+1+2	$1^2 2^1$	6	-1		$\Psi_S(q)^2 \Psi_S(q^2)$
2+2	2^2	3	1		$\Psi_S(q^2)^2$
1+3	$1^1 3^1$	8	1		$\Psi_S(q)\Psi_S(q^3)$
4	4^1	6	-1		$\Psi_S(q^4)$

gives all permutations, (hence all the representations). To get to the desired partition expressions, one must separately account for cases where two of the dummy indices are equal ($a = b$),

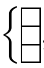
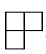
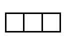
$$\psi(q^2)\psi(q) = \sum_{a=0}^{\infty}\sum_{c=0}^{\infty} q^{2T(a)+T(c)}.$$

(This is one of the constraints imposed in going from representations to partitions.) One must also account for the case when all three dummy indices are equal,

$$\psi(q^3) = \sum_{a=0}^{\infty} q^{3T(a)}.$$

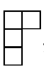
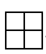
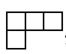
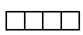
(This is another one of the constraints imposed in going from representations to partitions.)

In accordance with Theorem 2.1, this set then maps as

$$\left\{\psi(q)^3, \psi(q)\psi(q^2), \psi(q^3)\right\} \to \Big\{$$ , ,  $$\Big\} \to \{1^3, 1^1 2^1, 3^1\}.$$

Similarly for the case $m = 4$,

$$\left\{\psi(q)^4, \psi(q)^2\psi\left(q^2\right),\ \psi\left(q^2\right)^2, \psi(q)\psi\left(q^3\right), \psi\left(q^4\right)\right\} \to$$

$$\Big\{$$ , , , ,  $$\Big\} \to \{1^4, 1^2 2^1, 2^2, 1^1 3^1, 4^1\}.$$

For notational convenience (albeit abusive), the symbol C_i will be used at times in place of a product of ψ's and/or the Young's Tableaux (YT):

$$\prod_i \psi(q^{\lambda_i})^{\nu_i} \leftrightarrow \text{YT} \leftrightarrow C_i$$

Of course, the remaining problem is to determine how to construct the generating function formulae out of the set of conjugacy classes. This leads to the following theorem and the main result of the paper.

Theorem 2.2. *Writing the set of* $\{\ \prod_i \Psi_S(q^{\lambda_i})^{\nu_i}\ \}$ *as conjugacy classes* $\{C_i\}$,

$$P_{mS} = \frac{1}{m!}\sum_i \chi_i^{(1)} \cdot |C_i| C_i \tag{2.1}$$

and

$$P^d_{mS} = \frac{1}{m!}\sum_i \chi_i^{(2)} \cdot |C_i| C_i, \tag{2.2}$$

$\chi^{(1)}$ *is the trivial irreducible representation (hence* $\chi_i^{(1)} = 1$*), where* $\chi^{(2)}$ *is the sign irreducible representation (hence* $\chi_i^{(2)} = \pm 1$*), and the summation is over the conjugacy classes.*

Remark 2.3. It is unfortunate wording that "representation" is used in the group sense and also used in the number of representations of an integer. The group sense case is known as "irreducible representation" and for clarity will be written as "irrep".

Remark 2.4. The triangular numbers and $m = 2$ and $m = 3$ will be used as concrete examples to assist the presentation of the proof. The $m = 2$ case is too simple to stand alone and the $m = 3$ is to complicated too write out completely and explicitly. Together the two examples help to manifest the ideas of the proof.

Proof. First consider the explicit expansion of $\Psi_S(q)^m$:

$$\sum_{j_1}^{\infty}\sum_{j_2}^{\infty}\cdots\sum_{j_m}^{\infty} q^{S(j_1)+S(j_2)+\cdots S(j_m)}.$$

In this expansion there are $m!$ terms in which the set of indices, $\{j_i\}$, are all distinct. For the example of $S = \triangle$ and $m = 2$,

$$\psi(q)^2 = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} q^{T(a)+T(b)} \tag{2.3}$$

and for the example of $S = \Delta$ and $m = 3$,

$$\begin{aligned}\psi(q)^3 = &\cdots + q^{T(1)+T(2)+T(3)} + \cdots + q^{T(2)+T(1)+T(3)} \\ &+ \cdots + q^{T(1)+T(3)+T(2)} \cdots q^{T(2)+T(3)+T(1)} \\ &+ \cdots + q^{T(3)+T(1)+T(2)} + \cdots + q^{T(3)+T(2)+T(1)} + \cdots .\end{aligned}$$

Six (3!) and only six terms of a particular set of distinct indices are in the summation.

The $m!$ terms produce distinct representations but only a single partition, thus one divides by $m!$. However, there are only $\frac{m!}{\lambda_1!\lambda_2!\cdots\lambda_l!}$ terms in the expansion of $\Psi_S(q)^m$ when the set of indices $\{j_i\}$ have repeated values (λ_i is the multiplicity of a particular value). Using the example again,

$$\begin{aligned}\psi(q)^3 = &\cdots + q^{T(1)+T(1)+T(3)} + \cdots + q^{T(1)+T(3)+T(1)} \\ &+ \cdots + q^{T(3)+T(1)+T(1)} + \cdots .\end{aligned}$$

As a consequence, the division by $m!$ has overcompensated for these types of "self-terms" and, as such, more of these terms must be added in. A particular self-term $\{\lambda_1, \lambda_2, \ldots, \lambda_l\}$ is generated by

$$\Psi_S\left(q^{\lambda_1}\right)^{\nu_1}\Psi_S\left(q^{\lambda_2}\right)^{\nu_2}\cdots\Psi_S\left(q^{\lambda_l}\right)^{\nu_l}.$$

For the example of $S = \triangle$ and $m = 2$, one sees that, when $a \neq b$ in Eq. (2.3), this double sum produces twice as many representations as partitions. Dividing by 2 then gives

$$\frac{1}{2}\psi(q)^2 = \frac{1}{2}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty} q^{T(a)+T(b)}.$$

However, this overcompensates for the "self-terms" $(a = b)$. To alleviate this, half of the self-terms must be added back in to yield the final form of the expression,

$$\frac{1}{2}\psi(q)^2 + \frac{1}{2}\psi\left(q^2\right) = \frac{1}{2}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty} q^{T(a)+T(b)} + \frac{1}{2}\sum_{a=0}^{\infty} q^{2T(a)}.$$

For example,

$$\psi(q^2)\psi(q) = \cdots + q^{2T(1)+T(3)} + \cdots.$$

In this case, one and only one term of this particular set of indices appear in the summation.

Theorem 2.1 established a correspondence between these products of Ψ_S functions and conjugacy classes of $\mathfrak{S}_m$. Each term (e.g., $q^{2T(1)+T(3)}$) is a representative of a conjugacy class. Therefore one corrects the overcompensation by adding in $|C_i|$ terms of this type. In the example, $|C_{1^1 2^1}| = 3$ so $3q^{2T(1)+T(3)}$ is added in. In total,

$$\frac{1}{3!}\left(\psi(q)^3 + 3\psi(q^2)\psi(q)\right) = \cdots + \frac{3+3}{6}q^{2T(1)+T(3)} + \cdots,$$

which gives one distinct partition of the form 1+1+6 as expected. Note that in this example, C_{3^1} is not needed. Of course, in general all conjugacy classes are required. Thus, Eq. (2.1) provides the correct counting in general. Finally, while trivial, the factor from the trivial irrep, $\chi_1^{(1)}$, multiplies each term, it is explicitly included in the formula to better compare with the case of distinct partitions and for the completeness of connection with group theory.

Similar lines of reasoning can be made for the distinct partition case. Starting again with $\Psi_S(q)^3$, there are $m!$ terms for a given set of distinct indices, $\{j_i\}$, appearing in the expansion. These alone give the single distinct partition associated with that set of indices. Thus, as above, one must divide by $m!$. In addition however, $\Psi_S(q)^3$ contains the various self-terms that need to be subtracted out. This cannot, however, be done in one step, but instead, as an iterative process of adding and subtracting odd and even permutations of elements. Group theoretically, this corresponds to multiplying by the appropriate character from the sign irrep.

Returning to the working example, $-3\psi(q^2)\psi(q)$ will correctly subtract $3q^{2T(1)+T(3)}$ but it will also subtract the full self-terms, such as $q^{2T(1)+T(1)}$. Thus one must add that term back in as $|C_{3^1}|\psi(q^3) = + \cdots + 2q^{3T(1)}$. Taken together results in

$$\frac{1}{3!}\left(\psi(q)^3 - 3\psi(q^2)\psi(q) + 2\psi(q^3)\right).$$

In general the systematic subtraction and addition is determined by the sign irrep, $\chi_i^{(2)}$ of $\mathfrak{S}_m$. This accounts for the multiplication by $\chi_i^{(2)}$ and completes the proof. □

The proof of Theorem 2.2 relies on the fact that a particular conjugacy class is obtained from a particular *integer* partition of m [8]. All such integer representations will occur within the collection of terms for a given product of Ψ_S functions.

Here too, examples are helpful. Consider again the case $m = 3$. One can now quickly recover the formulae of Hirschhorn and Sellers (Eqs. (1.4) and (1.5)) The integer representations of 3 are $\{1+1+1,\, 1+2,\, 3\}$. The $1+1+1$ integer partition has the cycle structure (1), the $1+2$ partition has a cycle structure of (12), and the 3 partition has a cycle structure of (123). Knowing the cycle structure immediately gives the conjugacy classes and their respective orders. For $\mathfrak{S}_3$, $|C_{(1)}| = 1$, $|C_{(12)}| = 3$ and $|C_{(123)}| = 2$. Thus, from Theorem 2.2, one has

$$\begin{aligned} P_{3\Delta} &= \frac{1}{3!}\left(1\,\ydiagram{1,1,1} + 3\,\ydiagram{2,1} + 2\,\ydiagram{3}\right) \\ &= \frac{1}{6}\left(1\,\psi(q)^3 + 3\psi(q)\psi(q^2) + 2\psi(q^3)\right), \end{aligned}$$

which is indeed Eq. (1.4). The distinct partition case is immediately obtained by replacing $\chi^{(1)}$ with $\chi^{(2)} = (1, -1, 1)$,

$$\begin{aligned} P_{3\Delta} &= \frac{1}{3!}\left(1\,\ydiagram{1,1,1} - 3\,\ydiagram{2,1} + 2\,\ydiagram{3}\right) \\ &= \frac{1}{6}\left(1\,\psi(q)^3 - 3\psi(q)\psi(q^2) + 2\psi(q^3)\right), \end{aligned}$$

which is indeed Eq. (1.5).

Theorem 2.2 is now very simple to use for any order of m. The case $m = 4$ is,

$$\begin{aligned} P_{4\Delta} &= \frac{1}{4!}\left(1\,\ydiagram{1,1,1,1} + 6\,\ydiagram{2,1,1} + 3\,\ydiagram{2,2} + 8\,\ydiagram{3,1} + 6\,\ydiagram{4}\right) \\ &= \frac{1}{4!}\left(1\psi(q)^4 + 6\psi(q)^2\psi\left(q^2\right) + 3\psi\left(q^2\right)^2 + 8\psi(q)\psi\left(q^3\right) + 6\psi\left(q^4\right)\right). \end{aligned} \tag{2.4}$$

and, using $\chi^{(2)} = (1, -1, 1, 1, -1)$,

$$\begin{aligned} P^d_{4\Delta} &= \frac{1}{4!}\left(1\,\ydiagram{1,1,1,1} - 6\,\ydiagram{2,1,1} + 3\,\ydiagram{2,2} + 8\,\ydiagram{3,1} - 6\,\ydiagram{4}\right) \\ &= \frac{1}{4!}\left(1\psi(q)^4 - 6\psi(q)^2\psi\left(q^2\right) + 3\psi\left(q^2\right)^2 + 8\psi(q)\psi\left(q^3\right) - 6\psi\left(q^4\right)\right). \end{aligned} \tag{2.5}$$

The cases $m = 5$ through $m = 8$ are given in the Appendix. So, from data like those shown in Tables 1 and 2, one can construct the desired formula immediately. The most convenient place to find these data is online at the *Group Properties Wiki* [7]. MATHEMATICA has this information built into its database up to the case $\mathfrak{S}_{17}$ and one can use the built-in functions to write simple code to go beyond $m = 17$ [20].

As a quick concrete example, consider the partitioning of the integer 52 when $m = 3$, $m = 4$, and $m = 5$. First for the case $m = 3$, Eq. (1.4) gives $p_{3\Delta}(52) = 5$. Explicitly those five partitions are the summation of each of the rows in the array,

$$\begin{array}{ccc} 1 & 6 & 45 \\ 1 & 15 & 36 \\ 3 & 21 & 28 \\ 6 & 10 & 36 \\ 10 & 21 & 21 \end{array}.$$

The value of $p^d_{3\Delta}(52) = 4$ from Eq. (1.5) is obtained by eliminating the last row.

For the case $m = 4$ one uses Eqs. (2.4) and (2.5) to get $p_{4\Delta}(52) = 10$ and $p^d_{4\Delta}(52) = 6$ respectively. The first five of $p_{4\Delta}(52)$ are obtained from the array above by simply adding 0. The remaining five are

$$\begin{array}{cccc} 1 & 3 & 3 & 45 \\ 1 & 15 & 15 & 21 \\ 3 & 3 & 10 & 36 \\ 3 & 6 & 15 & 28 \\ 6 & 10 & 15 & 21 \end{array}.$$

Elimination of the non-distinct partitions gives the six distinct partitions.

Finally, consulting the Appendix for the case $m = 5$, one obtains $p_{5\Delta}(52) = 20$ and $p^d_{5\Delta}(52) = 2$. The first five are obtained by adding 0 twice to the first array above. This means none of those partitions will contribute to $p^d_{5\Delta}(52)$. To obtain the next five, a 0 is added to the second array above. The last ten are

$$\begin{array}{ccccc|ccccc} 1 & 1 & 1 & 21 & 28 & 1 & 10 & 10 & 10 & 21 \\ 1 & 3 & 6 & 6 & 36 & 3 & 3 & 3 & 15 & 28 \\ 1 & 3 & 6 & 21 & 21 & 3 & 3 & 10 & 15 & 21 \\ 1 & 3 & 10 & 10 & 28 & 6 & 6 & 6 & 6 & 28 \\ 1 & 6 & 15 & 15 & 15 & 6 & 6 & 10 & 15 & 15 \end{array}.$$

All of these last ten are non-distinct, leaving only $0 + 3 + 6 + 15 + 28 = 52$ and $0 + 6 + 10 + 15 + 21 = 52$ as the distinct partitions.

3 Square partitions

The most powerful thing about Theorem 2.2 is that it is relying on a correspondence from the set of generating functions making up $P_{m\Delta}$ to the set of conjugacy classes. As such, the precise nature of ψ is not important. In fact Theorem 2.2 works for partitioning into squares ($S = \square$) as well which is briefly shown here and into centered polygonal numbers which is discussed in more detail in Section 4.

Let

$$\sigma(q) := \Psi_{\square}(q) = \sum_{j=0}^{\infty} q^{j^2} \tag{3.1}$$

be the generating function for the representations of a target integer n as the sum of squares. Note: only non-negative component integers are considered. For an m-partition one begins with,

$$\sigma(q)^m = \sum_{n=0}^{\infty} s_m(n) q^n,$$

where $s_m(n)$ is the number of ways to represent n as the sum of m squares. One can apply Theorem 2.2, now using the notation $P_{m\square}$ to indicate partitioning into squares. So for example,

$$P_{3\square} = \frac{1}{6}\left(1\,\sigma(q)^3 + 3\sigma(q)\sigma(q^2) + 2\sigma(q^3)\right)$$

and

$$P^d_{3\square} = \frac{1}{6}\left(1\,\sigma(q)^3 - 3\sigma(q)\sigma(q^2) + 2\sigma(q^3)\right),$$

and so on for larger m.

4 Centered polygonal numbers

The centered polygonal numbers are presented as an example because the parameter, k, which gives the number of sides of the particular polygon, provides some versatility in some situations. It is interesting to determine what behavior is k-dependent and what is k-independent. The centered polygonal numbers have an intimate relation to the triangular numbers which provides some connection between these two sets. In addition to partitioning by centered polygonal numbers, several ancillary lemmas and theorems are presented. The set of centered k-gonal numbers where $j \in \mathbb{N}^+$ is given by [5, 6, 13, 19]

$$C^{(k)} = \left\{\frac{kj^2 - kj + 2}{2}\right\}. \tag{4.1}$$

When it is necessary to identify a particular member of $C^{(k)}$, the notation $C^{(k)}(j)$ will be used for the j^{th} member of the set. Centered polygonal numbers are intimately related to triangular numbers as given by the following lemma.

Lemma 4.1.

$$\frac{C^{(k)}(j+1) - 1}{k} = T(j). \tag{4.2}$$

Proof.

$$\begin{aligned}
&\frac{1}{k}\left(\frac{k(j+1)^2-k(j+1)+2}{2}-1\right)\\
&=\frac{1}{k}\left(\frac{kj^2+2kj+k-kj-k+2-2}{2}\right)\\
&=\frac{1}{k}\left(\frac{kj^2+kj}{2}\right)=T(j).
\end{aligned}$$

□

The generating function is 1 plus the centered polygonal lacunary function [16],

$$\Gamma^{(k)}(q) := \Psi_C(q) = 1 + \sum_{j=1}^{\infty} q^{C^{(k)}(j)}, \tag{4.3}$$

such that the number of representations of an integer by m centered polygonal numbers is given by $\Gamma^{(k)}(q)$ in analogy with Eq. (1.2).

The generating function $\Gamma^{(k)}(q)$ is related to $\psi(q)$ as shown by the following theorem.

Theorem 4.2.

$$\Gamma^{(k)}(q) - 1 = q\psi(q^k).$$

Proof. Starting from the right hand side of the theorem and using Eq. (4.2), one sees

$$\begin{aligned}
q\psi(q^k) &= q\sum_{n=0}^{\infty}\left(q^k\right)^{\frac{n^2+n}{2}} = \sum_{n=0}^{\infty} q^{\frac{kn^2+kn+2}{2}}\\
&= \sum_{n=1}^{\infty} q^{\frac{kn^2-kn+2}{2}} = \Gamma^{(k)}(q) - 1,
\end{aligned}$$

where the adjustment of dummy index, $n \to n+1$, was used in going from the top line to the bottom line. The last expression is the centered polygonal lacunary function which completes the proof. □

Definition 4.3. Let the set of numbers that can be represented by the sum of r triangular numbers be $\mathfrak{T}_r$. Further define $\mathfrak{T}_0 \equiv \{0, 0, \ldots\}$ (not the empty set). The j^{th} member of the set will be $\tau_r(j)$ and $\tau_r(0) \equiv 0$.

Remark 4.4. $\mathfrak{T}_1$ is just the triangular numbers themselves, $\mathfrak{T}_{r\geq}3$ are the non-negative integers by Gauss' Eureka Theorem. So $\mathfrak{T}_2$ is the only non-trivial set which is sequence A020756 in the OEIS [14].

Lemma 4.5. *The exponents present in* $\left(\Gamma^{(k)}(q)\right)^b$ *are of the form* $r + k\tau_r(j)$*, where* $j \geq 0 \in \mathbb{Z}$ *and* $1 \leq r \leq b$.

Proof. The proof will be done by induction, where $b = 2$ is the base case. Here,

$$\left(\Gamma^{(k)}(q)\right)^2 = \left(1 + \sum_{i=1}^{\infty} q^{C^{(k)}(i)}\right)\left(1 + \sum_{j=1}^{\infty} q^{C^{(k)}(j)}\right).$$

Expressing $C^{(k)}(i)$ allows the summation to be written as

$$\sum_{i=1}^{\infty} q^{C^{(k)}(i)} = \sum_{i=1}^{\infty} q^{\frac{ki^2+ki+2}{2}} = \sum_{i=1}^{\infty} q^{\frac{ki(i-1)}{2}+1} = \sum_{i=1}^{\infty} q^{kT(i-1)+1}.$$

Shifting the dummy index gives $\sum_{i=0}^{\infty} q^{kT(i)+1}$. So,

$$\left(\Gamma^{(k)}(q)\right)^2 = \left(1 + \sum_{i=0}^{\infty} q^{kT(i)+1}\right)\left(1 + \sum_{j=1}^{\infty} q^{kT(j)+1}\right).$$

Multiplying this out and remembering that one is only concerned with the exponents present gives

$$\left(\Gamma^{(k)}(q)\right)^2 = 1 + x_1 \sum_{i=0}^{\infty} q^{kT(i)+1} + x_2 \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} q^{k(T(i)+T(j))+2}.$$

Indeed the first term gives an exponent of 0, the middle term give exponents of the form $1 + k\tau_1 = 1 + kT(i)$ and the last term gives exponents of the form $2 + k\tau_2$. Thus the base case holds.

Now assume the $b = m$ case holds which is written out as

$$\begin{aligned}\left(\Gamma^{(k)}(q)\right)^m = 1 &+ x_1 \sum_{i=0}^{\infty} q^{kT(i)+1} + x_2 \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} q^{k(T(i)+T(j))+2} + \cdots \\ &+ x_l \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\cdots\sum_{u=1}^{\infty} q^{k(T(i)+T(j)+\cdots+T(u))+l} + \cdots + \\ &+ x_m \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\cdots\sum_{v=1}^{\infty} q^{k(T(i)+T(j)+\cdots+T(v))+m}.\end{aligned} \tag{4.4}$$

Moving to the $m + 1$ case,

$$\begin{aligned}\left(\Gamma^{(k)}(q)\right)^{m+1} = \Bigg(1 &+ x_1 \sum_{i=0}^{\infty} q^{kT(i)+1} + x_2 \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} q^{k(T(i)+T(j))+2} + \cdots \\ &+ x_l \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\cdots\sum_{u=1}^{\infty} q^{k(T(i)+T(j)+\cdots+T(u))+l} + \cdots \\ &+ x_m \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\cdots\sum_{v=1}^{\infty} q^{k(T(i)+T(j)+\cdots+T(v))+m}\Bigg) \times \\ &\left(1 + \sum_{w=1}^{\infty} q^{kT(j)+1}\right).\end{aligned}$$

Upon multiplying out this becomes

$$\begin{aligned}\left(\Gamma^{(k)}(q)\right)^{m+1} = \Bigg(1 &+ x_1' \sum_{i=0}^{\infty} q^{kT(i)+1} + x_2' \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} q^{k(T(i)+T(j))+2} + \cdots \\ &+ x_l \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\cdots\sum_{u=1}^{\infty}\sum_{w=1}^{\infty} q^{k(T(i)+T(j)+\cdots+T(u)+T(w))+l+1} + \cdots \\ &+ x_m \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\cdots\sum_{v=1}^{\infty}\sum_{w=1}^{\infty} q^{k(T(i)+T(j)+\cdots+T(v)+T(w)+m+1}\Bigg).\end{aligned}$$

This indeed holds and completes the proof. ☐

Lemma 4.6. *The exponents present in $\Gamma^{(k)}(q^a)$ are of the form $a(1+kT(n))$.*

Proof. Expressing the summation in Eq. (4.3) as was done in the proof of Lemma 4.5 and substituting q^a for q gives

$$\begin{aligned}\left(\Gamma^{(k)}(q^a)\right) &= 1 + \sum_{i=0}^{\infty}(q^a)^{kT(i)+1} \\ &= 1 + \sum_{i=0}^{\infty} q^{a(kT(i)+1)}\end{aligned}$$

and the proof is complete. ☐

Corollary 4.7. *The exponents present in $\left(\Gamma^{(k)}(q^a)\right)^b$ are of the form $a(r+k\tau_r(j))$.*

Proof. The proof follows immediately Lemmas 4.5 and 4.6. The details are omitted in the interest of space. ☐

Theorem 4.8. *The exponents present in P_{mC} are of the form $r+k\tau_r(j)$, where $j \geq 0 \in \mathbb{Z}$ and $1 \leq n \leq m$.*

Proof. Note, again, that this theorem is not concerned with the coefficients of the terms. By Theorem 2.2, $\left(\Gamma^{(k)}(q)\right)^m$ will always be present because this corresponds to the identity conjugacy class which is obviously present in any $\mathfrak{S}_m$ group. Then by Lemma 4.5, exponents of the form $r+k\tau_r(j)$ are present. Again by Theorem 2.2 and now Corollary 4.7, the other conjugacy classes will simple give rise to multiples of $r+k\tau_r(j)$. Hence only those exponents appear. ☐

As a concrete example, consider the case P_{3C}. Here the exponents present will be of the form $1+\tau_1(j)k$, $2+\tau_2(j)k$, and $3+\tau_3(j)k$. For the case $k=3$, $1+\tau_3(j)3$ generates $\{1,4,10,19,31,46,\ldots\}$, $2+\tau_2(j)3$ generates $\{2,5,8,11,14,20,\ldots\}$, and $1+\tau_3(j)3$ generates $\{3,6,9,12,15,\ldots\}$. Their union along with $\mathfrak{T}_0 = \{0\}$ is $\{0,1,2,3,4,5,6,8,9,10,11,12,\ldots\}$ (only 7 is absent for numbers less than 12). Expressing the first few terms of P_{3C} indeed gives

$$P_{3C} = 1 + q + q^2 + q^3 + q^4 + q^5 + q^6 + q^8 + q^9 + q^{10} + q^{11} + 2q^{12}.$$

Corollary 4.9. *(**a**) The set of non-zero partition values, $\{p_{mC}\}$, are independent of k for $k \geq m$. (**b**) The set of non-zero distinct partitions values, $\{p^d_{mC}\}$, are independent of k for $k \geq m$.*

Remark 4.10. Note for this corollary one is considering only the *values* of p_{mC} as a set that is stripped of any connection to the target integer n.

Proof. (**a**) By Theorem 4.8 exponents of the form $r+k\tau_r(j)$ are the only ones present. Let $a = \tau_r(j) \in \mathbb{N}$. Now, $r \in \{1, 2, \dots, m\}$. Let $r_1 \neq r_2$ and one seeks to show that the intersection of the sets of exponents generated by r_1 and by r_2 is the empty set. To begin, assume $r_1 + ka_1 = r_2 + ka_2$ which rearranges to $r_1 - r_2 = k(a1 - a2)$. Now for $k \geq m$ and $a_1 \neq a_2$, the right hand side is greater than or equal to m where as the left hand side is less than m and the equation cannot be satisfied. The case $a_1 = a_2$ gives $r_1 = r_2$. The intersection of the sets of exponents generated by r_1 and by r_2 is, indeed, the empty set. Part (**b**) follows immediately. □

Connecting with example of P_{3C} above, this corollary manifests, for $k = 4, 5, 6$ respectively, as

$$P_{4C} = 1 + q + q^2 + q^3 + q^5 + q^6 + q^7 + q^{10} + q^{11} + q^{13} + q^{14} + 2q^{15},$$

$$P_{5C} = 1 + q + q^2 + q^3 + q^6 + q^7 + q^8 + q^{12} + q^{13} + q^{16} + q^{17} + 2q^{18},$$

and

$$P_{6C} = 1 + q + q^2 + q^3 + q^7 + q^8 + q^9 + q^{14} + q^{15} + q^{19} + q^{20} + 2q^{21}.$$

One sees the non-zero coefficients are the same for all the k values shown. The exponents associated with those coefficients are, of course, different.

5 Conclusion

This work built upon the 2004 work of Hirschhorn and Sellers [9] in developing general generating function formulae for the number partitions and distinct partitions of integers into arbitrarily many triangular numbers. These formulae arose from the connection between such partitioning and the structure of the symmetric group, namely the size and cycle structure of the conjugacy classes. A simple procedure was given to use data from the symmetry group to construct $P_{m\Delta}$ and $P^d_{m\Delta}$. The use of Young's tableaux added additional insight.

Because this technique did not depend in a fundamental way on the precise nature of the generating functions, it was able to be generalized. Square partitions and centered polygonal number partitions were explored. It is hoped that this work will aid in further explorations of partitioning, perhaps along the lines of looking for relations between terms and among combinations of generating functions [1, 2, 3, 17, 18, 21].

6 Appendix

Generating function formulae (in Young's tableaux form) for P^d_{mS} are collected here for the cases $m = 5$ through $m = 8$. Formulae for P_{mS} are immediately obtained from those of P^d_{mS} by simply changing all the minus signs to plus signs. In the interest of space savings, tables for $\mathfrak{S}_5$ through $\mathfrak{S}_8$ like those in the body of the text are omitted. The interested reader can readily find the information in standard texts [8] or online [7].

$$P^d_{5S} = \frac{1}{5!}\Big(1\,\ydiagram{1,1,1,1,1} - 10\,\ydiagram{2,1,1,1} + 15\,\ydiagram{2,2,1} + 20\,\ydiagram{3,1,1} - 20\,\ydiagram{3,2} - 30\,\ydiagram{4,1} + 24\,\ydiagram{5}\Big)$$

$$P^d_{6S} = \frac{1}{6!}\Big(1\,\ydiagram{1,1,1,1,1,1} - 15\,\ydiagram{2,1,1,1,1} + 45\,\ydiagram{2,2,1,1} - 15\,\ydiagram{2,2,2} + 40\,\ydiagram{3,1,1,1} - 120\,\ydiagram{3,2,1} + 40\,\ydiagram{3,3} - 90\,\ydiagram{4,1,1} + 90\,\ydiagram{4,2} + 144\,\ydiagram{5,1} - 120\,\ydiagram{6}\Big)$$

$$P^d_{7S} = \frac{1}{7!}\Big(1\,\ydiagram{1,1,1,1,1,1,1} - 21\,\ydiagram{2,1,1,1,1,1} + 105\,\ydiagram{2,2,1,1,1} - 105\,\ydiagram{2,2,2,1} + 70\,\ydiagram{3,1,1,1,1} - 420\,\ydiagram{3,2,1,1} + 210\,\ydiagram{3,2,2} + 280\,\ydiagram{3,3,1} - 210\,\ydiagram{4,1,1,1} + 630\,\ydiagram{4,2,1} - 420\,\ydiagram{4,3} + 504\,\ydiagram{5,1,1} + 504\,\ydiagram{5,2} - 840\,\ydiagram{6,1} + 720\,\ydiagram{7}\Big)$$

$$
\begin{aligned}
P^d_{8S} = \frac{1}{8!}\Bigg(& 1\,\ydiagram{1,1,1,1,1,1,1,1} - 28\,\ydiagram{2,1,1,1,1,1,1} + 210\,\ydiagram{2,2,1,1,1,1} - 420\,\ydiagram{2,2,2,1,1} + 105\,\ydiagram{2,2,2,2} + 112\,\ydiagram{3,1,1,1,1,1} \\
& - 1120\,\ydiagram{3,2,1,1,1} + 1680\,\ydiagram{3,2,2,1} + 1120\,\ydiagram{3,3,1,1} - 1120\,\ydiagram{3,3,2} - 420\,\ydiagram{4,1,1,1,1} \\
& + 2520\,\ydiagram{4,2,1,1} - 1260\,\ydiagram{4,2,2} - 3360\,\ydiagram{4,3,1} + 1260\,\ydiagram{4,4} \\
& + 1344\,\ydiagram{5,1,1,1} - 4032\,\ydiagram{5,2,1} + 2688\,\ydiagram{5,3} - 3360\,\ydiagram{6,1,1} \\
& + 3360\,\ydiagram{6,2} + 5760\,\ydiagram{7,1} - 5040\,\ydiagram{8} \Bigg)
\end{aligned}
$$

Acknowledgements

We thank Douglas R. Anderson and Nathan Axvig for valuable discussion. This work was supported by the Concordia College Chemistry Research Endowment and by the Concordia College Office of Undergraduate Research.

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(Received 30 Mar 2020; revised 24 Feb 2021)