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Orthogonal Latin square graphs based on groups of order 8

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Abstract

An orthogonal Latin square graph is a graph whose vertices are Latin squares of the same order, adjacency being synonymous with orthogonality. We are interested in the orthogonal Latin square graph in which each square is orthogonal to the Cayley table M of a group G and is obtained from M by permuting columns. The structure of this graph is completely determined by the structure of $\mathrm{Orth}(G)$, the orthomorphism graph of G. The structure of $\mathrm{Orth}(G)$, for $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, D_8, or Q_8, has been determined through computer searches only; we will present a theoretical determination of these structures.

1 Introduction

A *Latin square* of order n is an $n \times n$ matrix in which each symbol from an n-element set appears exactly once in each row and each column, and two Latin squares of the same order are *orthogonal* if each ordered pair of symbols appears exactly once when the squares are superimposed. A *mutually orthogonal set of Latin squares* (MOLS) of order n is a set of Latin squares of order n, each pair of which is orthogonal, and a set of MOLS is *maximal* if it is not contained in a larger set of MOLS. An *orthogonal Latin square graph* is a graph whose vertices are Latin squares of the same order, adjacency being synonymous with orthogonality. For more information on Latin squares see [6].

An *orthomorphism* of a group G is a bijection $\theta\colon G \to G$ for which the mapping $x \mapsto x^{-1}\theta(x)$ is also a bijection, and two orthomorphisms θ, ϕ of G are *orthogonal*, written $\theta \perp \phi$, if the mapping $x \mapsto \theta(x)^{-1}\phi(x)$ is a bijection: this relation is symmetric. For an orthomorphism θ of G, the mapping θ_a, defined by $x \mapsto \theta(x)a$, is also an orthomorphism of G and, if θ and ϕ are orthomorphisms of G, then $\theta \perp \phi$ if and only if $\theta_a \perp \phi_b$ for $a, b \in G$. Note that, if $a = \theta(1)^{-1}$, then $\theta_a(1) = 1$: we say that an orthomorphism θ of G is *normalized* if $\theta(1) = 1$. The *orthomorphism*

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graph of G, $\mathrm{Orth}(G)$, has as vertices the normalized orthomorphisms of G, adjacency being synonymous with orthogonality. An *r-clique* of $\mathrm{Orth}(G)$ is a set of r pairwise orthogonal orthomorphisms in $\mathrm{Orth}(G)$, and the *clique number* of $\mathrm{Orth}(G)$, denoted $\omega(G)$, is the largest r for which an r-clique of $\mathrm{Orth}(G)$ exists. A pairwise orthogonal set of orthomorphisms in $\mathrm{Orth}(G)$ is *maximal* if it cannot be extended to a larger pairwise orthogonal set of orthomorphisms in $\mathrm{Orth}(G)$. For more information on orthomorphisms of groups see [9].

For a group $G = \{g_1, \ldots, g_n\}$, the *Cayley table* of G is the Latin square, M, of order n, with ijth entry $g_i g_j$. For θ a mapping $G \to G$, let M_θ denote the $n \times n$ matrix with ijth entry $g_i\theta(g_j)$. Then M_θ is a Latin square if and only if θ is a bijection, and M_θ is orthogonal to M if and only if θ is an orthomorphism of G and, if θ and ϕ are orthomorphisms of G, then M_θ is orthogonal to M_ϕ if and only if $\theta \perp \phi$. If $\theta_1, \ldots, \theta_r$ is an r-clique of $\mathrm{Orth}(G)$, then $M, M_{\theta_1}, \ldots, M_{\theta_r}$ is a set of $r+1$ MOLS: this r-clique is maximal if and only if the corresponding set of $r+1$ MOLS is maximal. We leave it to the reader to verify that the structure of the Latin square graph whose vertices are Latin squares, obtained from M by permuting columns, is completely determined by the structure of $\mathrm{Orth}(G)$.

In this paper, we are going to make extensive use of automorphisms and congruences of $\mathrm{Orth}(G)$. A bijection $A\colon \mathrm{Orth}(G) \to \mathrm{Orth}(G)$ is an *automorphism* of $\mathrm{Orth}(G)$ if $A[\theta] \perp A[\phi]$ if and only if $\theta \perp \phi$. Known automorphisms are the *translations*, T_g, $g \in G$, defined by $T_g[\theta](x) = \theta(xg)\theta(g)^{-1}$; the *homologies*, H_f, $f \in \mathrm{Aut}(G)$, defined by $H_f[\theta] = f\theta f^{-1}$; and the *reflection*, R, defined by $R[\theta](x) = x\theta(x^{-1})$. A bijection $C\colon \mathrm{Orth}(G) \to \mathrm{Orth}(G)$ is a *congruence* of $\mathrm{Orth}(G)$ if the neighbourhood of θ is isomorphic to the neighbourhood of $C[\theta]$. A known congruence, that is not an automorphism, is the *inversion*, I, defined by $I[\theta] = \theta^{-1}$. If $\theta \in \mathrm{Orth}(G)$ and $\phi_1, \ldots, \phi_k$ are the orthomorphisms in $\mathrm{Orth}(G)$ orthogonal to θ, then $\phi_1\theta^{-1}, \ldots, \phi_k\theta^{-1}$ are the orthomorphisms in $\mathrm{Orth}(G)$ orthogonal to $I[\theta]$, and $\phi_i\theta^{-1} \perp \phi_j\theta^{-1}$ if and only if $\phi_i \perp \phi_j$. Note that $I^2 = R^2 = 1$.

The structure of $\mathrm{Orth}(G)$ has been theoretically determined for groups of order 7 or less and for two of the abelian groups of order 8 (see Chapter 13 in [9]). The structures of the remaining groups of order 8, $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$, D_8, the dihedral group of order 8; and Q_8, the quaternion group of order 8, have yet to be explained theoretically.

In 1961, Johnson, Dulmage, and Mendelsohn [10] showed, via a computer search, that $|\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)| = 48$: they found that $\omega(\mathbb{Z}_2 \times \mathbb{Z}_4) = 2$. In 1964, through exhaustive computation, Chang, Hsiang, and Tai [4] also found that $\omega(\mathbb{Z}_2 \times \mathbb{Z}_4) = 2$; and in 1986, via a computer search, Jungnickel and Grams [11] found that the only maximal cliques in $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ are 2-cliques. The structure of $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ was determined by Evans and Perkel using Cayley (a forerunner of the computer algebra system Magma [3]) and was reported in [7]: they found that $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ consists of twelve 4-cycles. In 1964, through exhaustive computation, Chang and Tai [5] found that D_8 and Q_8 both have 48 normalized orthomorphisms, and $\omega(D_8) = \omega(Q_8) = 1$; and in 1986, via a computer search, Jungnickel and Grams [11] confirmed that $\omega(D_8) = \omega(Q_8) = 1$: this was further confirmed by Evans and Perkel using Cayley (see [7]).

The number of orthomorphisms of $\mathbb{Z}_2 \times \mathbb{Z}_4$, D_8, and Q_8, has been explained theoretically. In 1991, Bedford [1] established a correspondence between the orthomorphisms of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_4$, and also between the orthomorphisms of D_8 and Q_8; and in 1999, Bedford and Whitaker [2] proved that all non-cyclic groups of order eight have 48 normalized orthomorphisms.

Problem 16.48 in [9] asks for a theoretical proof that $\omega(D_8) = 1$, and Problem 16.49 in [9] asks for a theoretical proof that $\omega(Q_8) = 1$. Problem 16.70 in [9] asks for a theoretical determination of the structure of $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$: it is suggested there that ideas and methods used in this determination might yield insight into the structure of $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_8)$ and, more generally, $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_q)$, $q = 2^n$. We will solve these three problems from [9] in this paper. We will lay the ground work in Section 2. In Section 3 we will classify the orthomorphisms in $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$: this classification will then be used to theoretically determine the structure of $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$. We will use the same approach in Section 4 to theoretically determine the structure of $\mathrm{Orth}(D_8)$, and in Section 5 to theoretically determine the structure of $\mathrm{Orth}(Q_8)$.

2 Some general results

The groups $\mathbb{Z}_2 \times \mathbb{Z}_4$, D_8, and Q_8 can be given similar presentations with generators p and q as follows

$$\begin{aligned} \mathbb{Z}_2 \times \mathbb{Z}_4 &= \langle p, q \mid p^4 = q^2 = 1, pq = qp \rangle, \\ D_8 &= \langle p, q \mid p^4 = q^2 = 1, pq = qp^{-1} \rangle, \text{ and} \\ Q_8 &= \langle p, q \mid p^4 = 1, q^2 = p^2, pq = qp^{-1} \rangle. \end{aligned}$$

While the multiplicative group $\langle p \rangle$ is $\mathbb{Z}_4$, in what follows, we will use $\mathbb{Z}_4$ to denote the additive group $\{0, 1, 2, 3\}$ with the operation addition modulo 4. For G one of $\mathbb{Z}_2 \times \mathbb{Z}_4$, D_8, or Q_8, any mapping $\theta\colon G \to G$ can be written in the form:

$$\theta(x) = \begin{cases} p^{\alpha_\theta(i)} & \text{if } x = p^i,\ i \in A_\theta, \\ qp^{\beta_\theta(i)} & \text{if } x = p^i,\ i \in A'_\theta, \\ p^{\gamma_\theta(i)} & \text{if } x = qp^i,\ i \in B_\theta, \\ qp^{\delta_\theta(i)} & \text{if } x = qp^i,\ i \in B'_\theta, \end{cases}$$

for some partitions $\{A_\theta, A'_\theta\}$ and $\{B_\theta, B'_\theta\}$ of $\mathbb{Z}_4$ and some mappings $\alpha_\theta\colon A_\theta \to \mathbb{Z}_4$, $\beta_\theta\colon A'_\theta \to \mathbb{Z}_4$, $\gamma_\theta\colon B_\theta \to \mathbb{Z}_4$, and $\delta_\theta\colon B'_\theta \to \mathbb{Z}_4$. We will call α_θ the 00-*mapping* for θ, β_θ the 01-*mapping* for θ, γ_θ the 10-*mapping* for θ, and δ_θ the 11-*mapping* for θ. We will call A_θ the 00-*set* for θ, A'_θ the 01-*set* for θ, B_θ the 10-*set* for θ, and B'_θ the 11-*set* for θ. A characterization of the 00-sets, 01-sets, 10-sets, 11-sets, 00-mappings, 01-mappings, 10-mappings, and 11-mappings that correspond to orthomorphisms of G is given next: this characterization is adapted from a characterization in [8] of these sets and mappings that correspond to strong complete mappings of dihedral and quaternion groups.

Lemma 2.1 *Let G be one of $\mathbb{Z}_2 \times \mathbb{Z}_4$, D_8, or Q_8. Let A_θ be the 00-set, A'_θ the 01-set, B_θ the 10-set, B'_θ the 11-set, α_θ the 00-mapping, β_θ the 01-mapping, γ_θ the 10-mapping, and δ_θ the 11-mapping for $\theta\colon G \to G$.*

1. *θ is a bijection if and only if*
   - (a) *$\{\alpha_\theta(i) \mid i \in A_\theta\}$ and $\{\gamma_\theta(i) \mid i \in B_\theta\}$ partition $\mathbb{Z}_4$, and*
   - (b) *$\{\beta_\theta(i) \mid i \in A'_\theta\}$ and $\{\delta_\theta(i) \mid i \in B'_\theta\}$ partition $\mathbb{Z}_4$.*
2. *If θ is a bijection and $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, then θ is an orthomorphism of G if and only if the following hold.*
   - (a) *$\{\alpha_\theta(i) - i \mid i \in A_\theta\}$ and $\{\delta_\theta(i) - i \mid i \in B'_\theta\}$ partition $\mathbb{Z}_4$, and*
   - (b) *$\{\beta_\theta(i) - i \mid i \in A'_\theta\}$ and $\{\gamma_\theta(i) - i \mid i \in B_\theta\}$ partition $\mathbb{Z}_4$.*
3. *If θ is a bijection and $G = D_8$, then θ is an orthomorphism of G if and only if the following hold.*
   - (a) *$\{\alpha_\theta(i) - i \mid i \in A_\theta\}$ and $\{\delta_\theta(i) - i \mid i \in B'_\theta\}$ partition $\mathbb{Z}_4$, and*
   - (b) *$\{\beta_\theta(i) + i \mid i \in A'_\theta\}$ and $\{\gamma_\theta(i) + i \mid i \in B_\theta\}$ partition $\mathbb{Z}_4$.*
4. *If θ is a bijection and $G = Q_8$, then θ is an orthomorphism of G if and only if the following hold.*
   - (a) *$\{\alpha_\theta(i) - i \mid i \in A_\theta\}$ and $\{\delta_\theta(i) - i \mid i \in B'_\theta\}$ partition $\mathbb{Z}_4$, and*
   - (b) *$\{\beta_\theta(i) + i \mid i \in A'_\theta\}$ and $\{\gamma_\theta(i) + i + 2 \mid i \in B_\theta\}$ partition $\mathbb{Z}_4$.*

Further, if θ is an orthomorphism of G, then $|A_\theta| = |A'_\theta| = |B_\theta| = |B'_\theta| = 2$.

Proof: This is a routine adaptation of the proof of Theorem 9 in [8]. □

We will find it useful to represent each orthomorphism of $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, D_8, or Q_8 by an array of four tables, an α-table, a β-table, a γ-table, and a δ-table. A 2-element subset, $X = \{x_1, x_2\}$, of $\mathbb{Z}_4$ can be converted to a 2-vector in $\mathbb{Z}_4 \times \mathbb{Z}_4$ by choosing an ordering of the elements of X, and so X can be converted to either (x_1, x_2) or (x_2, x_1): it should become clear that the choice of ordering is of no relevance. By an abuse of notation we will use X to denote both the set and the vector that X is converted into. We will call a 2-vector in $\mathbb{Z}_4 \times \mathbb{Z}_4$ *set-like* if its components are distinct, and we will call two set-like 2-vectors in $\mathbb{Z}_4 \times \mathbb{Z}_4$ *complementary* if the union of their sets of components is $\mathbb{Z}_4$. The array of tables representing a mapping $\theta\colon G \to G$ is as follows.

$$\theta \sim \left[\begin{array}{cc} \begin{array}{c} A_\theta \\ C_\theta \\ \hline E_\theta \end{array} & \begin{array}{c} A'_\theta \\ D_\theta \\ \hline F_\theta \end{array} \\ \\ \begin{array}{c} B_\theta \\ C'_\theta \\ \hline F'_\theta \end{array} & \begin{array}{c} B'_\theta \\ D'_\theta \\ \hline E'_\theta \end{array} \end{array}\right]$$

In this array of tables, $C_\theta = \alpha_\theta(A_\theta)$, $E_\theta = C_\theta - A_\theta$, $D_\theta = \beta_\theta(A'_\theta)$, $C'_\theta = \gamma_\theta(B_\theta)$, $D'_\theta = \delta_\theta(B'_\theta)$, and $E'_\theta = D'_\theta - B'_\theta$. If $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, then $F_\theta = D_\theta - A'_\theta$ and $F'_\theta = C'_\theta - B_\theta$. If $G = D_8$, then $F_\theta = D_\theta + A'_\theta$ and $F'_\theta = C'_\theta + B_\theta$. If $G = Q_8$, then $F_\theta = D_\theta + A'_\theta$ and $F'_\theta = C'_\theta + B_\theta + \vec{2}$, where $\vec{2} = (2, 2)$. The table in the top-left is the *α-table* for θ, the table in the top-right is the *β-table* for θ, the table in the bottom-left is the *γ-table* for θ, and the table in the bottom-right is the *δ-table* for θ. It follows from Lemma 2.1, that θ is an orthomorphism of G if and only if each vector in its array of tables is set-like, and the pair $\{X, X'\}$ is complementary for $X = A_\theta$, B_θ, C_θ, D_θ, E_θ, and F_θ.

For $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, D_8, or Q_8, there are two congruences of $\mathrm{Orth}(G)$ that we will use in the construction of all elements of $\mathrm{Orth}(G)$; the inversion I; and the reflection R. Their effects on $\theta \in \mathrm{Orth}(G)$ are easily described in the array of tables representation for θ: the effects do depend on the choice of G.

If $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, then for $I[\theta]$:

$$\theta \sim \left[\begin{array}{cc} \begin{array}{c} A_\theta \\ C_\theta \\ \hline E_\theta \end{array} & \begin{array}{c} A'_\theta \\ D_\theta \\ \hline F_\theta \end{array} \\ \\ \begin{array}{c} B_\theta \\ C'_\theta \\ \hline F'_\theta \end{array} & \begin{array}{c} B'_\theta \\ D'_\theta \\ \hline E'_\theta \end{array} \end{array}\right] \quad \Rightarrow \quad I[\theta] \sim \left[\begin{array}{cc} \begin{array}{r} C_\theta \\ A_\theta \\ \hline -E_\theta \end{array} & \begin{array}{r} C'_\theta \\ B_\theta \\ \hline -F'_\theta \end{array} \\ \\ \begin{array}{r} D_\theta \\ A'_\theta \\ \hline -F_\theta \end{array} & \begin{array}{r} D'_\theta \\ B'_\theta \\ \hline -E'_\theta \end{array} \end{array}\right],$$

and for $R[\theta]$:

$$\theta \sim \left[\begin{array}{cc} \begin{array}{c} A_\theta \\ C_\theta \\ \hline E_\theta \end{array} & \begin{array}{c} A'_\theta \\ D_\theta \\ \hline F_\theta \end{array} \\ \\ \begin{array}{c} B_\theta \\ C'_\theta \\ \hline F'_\theta \end{array} & \begin{array}{c} B'_\theta \\ D'_\theta \\ \hline E'_\theta \end{array} \end{array}\right] \quad \Rightarrow \quad R[\theta] \sim \left[\begin{array}{cc} \begin{array}{r} -A_\theta \\ E_\theta \\ \hline C_\theta \end{array} & \begin{array}{r} -A'_\theta \\ F_\theta \\ \hline D_\theta \end{array} \\ \\ \begin{array}{r} -B'_\theta \\ E'_\theta \\ \hline D'_\theta \end{array} & \begin{array}{r} -B_\theta \\ F'_\theta \\ \hline C'_\theta \end{array} \end{array}\right].$$

If $G = D_8$, then for $I[\theta]$:

$$\theta \sim \begin{bmatrix} A_\theta & A'_\theta \\ \underline{C_\theta} & \underline{D_\theta} \\ E_\theta & F_\theta \\ & \\ B_\theta & B'_\theta \\ \underline{C'_\theta} & \underline{D'_\theta} \\ F'_\theta & E'_\theta \end{bmatrix} \quad \Rightarrow \quad I[\theta] \sim \begin{bmatrix} C_\theta & C'_\theta \\ \underline{A_\theta} & \underline{B_\theta} \\ -E_\theta & F'_\theta \\ & \\ D_\theta & D'_\theta \\ \underline{A'_\theta} & \underline{B'_\theta} \\ F_\theta & -E'_\theta \end{bmatrix},$$

and for $R[\theta]$:

$$\theta \sim \begin{bmatrix} A_\theta & A'_\theta \\ \underline{C_\theta} & \underline{D_\theta} \\ E_\theta & F_\theta \\ & \\ B_\theta & B'_\theta \\ \underline{C'_\theta} & \underline{D'_\theta} \\ F'_\theta & E'_\theta \end{bmatrix} \quad \Rightarrow \quad R[\theta] \sim \begin{bmatrix} -A_\theta & -A'_\theta \\ \underline{E_\theta} & \underline{F_\theta} \\ C_\theta & D_\theta \\ & \\ B'_\theta & B_\theta \\ \underline{E'_\theta} & \underline{F'_\theta} \\ D'_\theta & C'_\theta \end{bmatrix}.$$

If $G = Q_8$, then for $I[\theta]$:

$$\theta \sim \begin{bmatrix} A_\theta & A'_\theta \\ \underline{C_\theta} & \underline{D_\theta} \\ E_\theta & F_\theta \\ & \\ B_\theta & B'_\theta \\ \underline{C'_\theta} & \underline{D'_\theta} \\ F'_\theta & E'_\theta \end{bmatrix} \quad \Rightarrow \quad I[\theta] \sim \begin{bmatrix} C_\theta & C'_\theta \\ \underline{A_\theta} & \underline{B_\theta} \\ -E_\theta & F'_\theta + \vec{2} \\ & \\ D_\theta & D'_\theta \\ \underline{A'_\theta} & \underline{B'_\theta} \\ F_\theta + \vec{2} & -E'_\theta \end{bmatrix},$$

and for $R[\theta]$:

$$\theta \sim \begin{bmatrix} A_\theta & A'_\theta \\ \underline{C_\theta} & \underline{D_\theta} \\ E_\theta & F_\theta \\ & \\ B_\theta & B'_\theta \\ \underline{C'_\theta} & \underline{D'_\theta} \\ F'_\theta & E'_\theta \end{bmatrix} \quad \Rightarrow \quad R[\theta] \sim \begin{bmatrix} -A_\theta & -A'_\theta \\ \underline{E_\theta} & \underline{F_\theta} \\ C_\theta & D_\theta \\ & \\ B'_\theta + \vec{2} & B_\theta + \vec{2} \\ \underline{E'_\theta} & \underline{F'_\theta} \\ D'_\theta & C'_\theta \end{bmatrix}.$$

These congruences will prove useful in classifying orthomorphisms as well as in determining the structure of orthomorphism graphs. We will next consider orthogonality.

Lemma 2.2 *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, D_8, or Q_8, and let $\theta, \phi \in \mathrm{Orth}(G)$.*

1. *If $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, then $\theta \perp \phi$ if and only if*

   (a) $\{\alpha_\phi(i) - \alpha_\theta(i) \mid i \in A_\theta \cap A_\phi\}$,
   $\{\beta_\phi(i) - \beta_\theta(i) \mid i \in A'_\theta \cap A'_\phi\}$,
   $\{\gamma_\phi(i) - \gamma_\theta(i) \mid i \in B_\theta \cap B_\phi\}$, *and*
   $\{\delta_\phi(i) - \delta_\theta(i) \mid i \in B'_\theta \cap B'_\phi\}$ *partition* $\mathbb{Z}_4$*; and*

   (b) $\{\beta_\phi(i) - \alpha_\theta(i) \mid i \in A_\theta \cap A'_\phi\}$,
   $\{\alpha_\phi(i) - \beta_\theta(i) \mid i \in A'_\theta \cap A_\phi\}$,
   $\{\delta_\phi(i) - \gamma_\theta(i) \mid i \in B_\theta \cap B'_\phi\}$, *and*
   $\{\gamma_\phi(i) - \delta_\theta(i) \mid i \in B'_\theta \cap B_\phi\}$ *partition* $\mathbb{Z}_4$.

2. *If $G = D_8$, then $\theta \perp \phi$ if and only if*

   (a) $\{\alpha_\phi(i) - \alpha_\theta(i) \mid i \in A_\theta \cap A_\phi\}$,
   $\{\beta_\phi(i) - \beta_\theta(i) \mid i \in A'_\theta \cap A'_\phi\}$,
   $\{\gamma_\phi(i) - \gamma_\theta(i) \mid i \in B_\theta \cap B_\phi\}$, *and*
   $\{\delta_\phi(i) - \delta_\theta(i) \mid i \in B'_\theta \cap B'_\phi\}$ *partition* $\mathbb{Z}_4$*; and*

   (b) $\{\beta_\phi(i) + \alpha_\theta(i) \mid i \in A_\theta \cap A'_\phi\}$,
   $\{\alpha_\phi(i) + \beta_\theta(i) \mid i \in A'_\theta \cap A_\phi\}$,
   $\{\delta_\phi(i) + \gamma_\theta(i) \mid i \in B_\theta \cap B'_\phi\}$, *and*
   $\{\gamma_\phi(i) + \delta_\theta(i) \mid i \in B'_\theta \cap B_\phi\}$ *partition* $\mathbb{Z}_4$.

3. *If $G = Q_8$, then $\theta \perp \phi$ if and only if*

   (a) $\{\alpha_\phi(i) - \alpha_\theta(i) \mid i \in A_\theta \cap A_\phi\}$,
   $\{\beta_\phi(i) - \beta_\theta(i) \mid i \in A'_\theta \cap A'_\phi\}$,
   $\{\gamma_\phi(i) - \gamma_\theta(i) \mid i \in B_\theta \cap B_\phi\}$, *and*
   $\{\delta_\phi(i) - \delta_\theta(i) \mid i \in B'_\theta \cap B'_\phi\}$ *partition* $\mathbb{Z}_4$*; and*

   (b) $\{\beta_\phi(i) + \alpha_\theta(i) \mid i \in A_\theta \cap A'_\phi\}$,
   $\{\alpha_\phi(i) + \beta_\theta(i) + 2 \mid i \in A'_\theta \cap A_\phi\}$,
   $\{\delta_\phi(i) + \gamma_\theta(i) \mid i \in B_\theta \cap B'_\phi\}$, *and*
   $\{\gamma_\phi(i) + \delta_\theta(i) + 2 \mid i \in B'_\theta \cap B_\phi\}$ *partition* $\mathbb{Z}_4$.

Proof: Routine derivation from the possible values of $\theta(x)^{-1}\phi(x)$. □

The orders of the intersections $A_\theta \cap A_\phi$, etc., will prove to be important.

Lemma 2.3 *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, D_8, or Q_8, and let $\theta, \phi \in \mathrm{Orth}(G)$, $\theta \perp \phi$.*

If $k = |A_\theta \cap A_\phi|$, then $k = 1$ or 2, and

$$|A_\theta \cap A_\phi| = |A'_\theta \cap A'_\phi| = |B_\theta \cap B'_\phi| = |B'_\theta \cap B_\phi| = k,$$

and

$$|A_\theta \cap A'_\phi| = |A'_\theta \cap A_\phi| = |B_\theta \cap B_\phi| = |B'_\theta \cap B'_\phi| = 2 - k.$$

Proof: As $0 \in A_\theta \cap A_\phi$, $k \geq 1$. As $|A_\theta \cap A_\phi| + |A_\theta \cap A'_\phi| = 2$, $|A_\theta \cap A'_\phi| = 2 - k$.

Similarly $|A'_\theta \cap A'_\phi| = |A_\theta \cap A_\phi| = k$, and $|A'_\theta \cap A_\phi| = |A_\theta \cap A'_\phi| = 2 - k$; and, for some u, $|B_\theta \cap B_\phi| = |B'_\theta \cap B'_\phi| = u$, and $|B'_\theta \cap B_\phi| = |B_\theta \cap B'_\phi| = 2 - u$.

Now $|A_\theta \cap A_\phi| + |A'_\theta \cap A'_\phi| + |B_\theta \cap B_\phi| + |B'_\theta \cap B'_\phi| = 4$. It follows that $2k + 2u = 4$. Hence $u = 2 - k$ and the result follows. $\square$

For $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, D_8, or Q_8, if $\theta, \phi \in \mathrm{Orth}(G)$, $\theta \perp \phi$, and $k = |A_\theta \cap A_\phi|$, we will say that θ is *k-orthogonal* to ϕ, written $\theta \perp_k \phi$. The congruences R, and I preserve k-orthogonality.

Lemma 2.4 *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, D_8, or Q_8. If $\theta, \phi \in \mathrm{Orth}(G)$, then*

1. *$\theta \perp_k \phi$ if and only if $R[\theta] \perp_k R[\phi]$, and*
2. *$\theta \perp_k \phi$ if and only if $I[\theta] \perp_k \phi\theta^{-1}$.*

Proof: Let $\theta, \phi \in \mathrm{Orth}(G)$, $\theta \perp \phi$. If $\theta' = R[\theta]$, then $A_{\theta'} = -A_\theta$. It follows that $\theta \perp_k \phi$ if and only if $R[\theta] \perp_k R[\phi]$.

If $\theta \perp_2 \phi$, then $I[\theta] \perp \phi\theta^{-1}$ and $A_{I[\theta]} = C_\theta = A_{\phi\theta^{-1}}$, and so $I[\theta] \perp_2 \phi\theta^{-1}$. If $I[\theta] \perp_2 \phi\theta^{-1}$, then $I[\theta](\phi\theta^{-1})^{-1} \perp_2 I[\phi\theta^{-1}]$, i.e., $\phi^{-1} \perp_2 \theta\phi^{-1}$. Then $I[\phi^{-1}] \perp_2 \theta\phi^{-1}\phi$, i.e., $\phi \perp_2 \theta$. Hence $\theta \perp_2 \phi$ if and only if $I[\theta] \perp_2 \phi\theta^{-1}$. As no pair of orthomorphisms can be both 2-orthogonal and 1-orthogonal, $\theta \perp_1 \phi$ if and only if $I[\theta] \perp_1 \phi\theta^{-1}$. $\square$

In determining the structure of orthomorphism graphs, 1-orthogonalities and 2-orthogonalities will be dealt with separately.

3 The structure of $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$

In this section we will derive a classsification of orthomorphisms in $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$. We will then use this classification to determine the structure of $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$.

Theorem 3.1 *If θ is a mapping $\mathbb{Z}_2 \times \mathbb{Z}_4 \to \mathbb{Z}_2 \times \mathbb{Z}_4$, then $\theta \in \mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ if and only if θ can be obtained from one of the orthomorphisms θ_1 and θ_2 below, using the congruences I, and/or R. Here $a = \pm 1$ and $b \in \mathbb{Z}_4$.*

1. *$\theta_1 \sim$*

$$\left[\begin{array}{cc}
\begin{array}{rr} 0 & 2 \\ 0 & a \\ \hline 0 & -a \end{array} & \begin{array}{rr} a & -a \\ b & b+a \\ \hline b-a & b+2 \end{array} \\
 & \\
\begin{array}{rr} a-b & -a-b \\ 2 & -a \\ \hline b+a & b \end{array} & \begin{array}{rr} b & b+2 \\ b+2 & b-a \\ \hline 2 & a \end{array}
\end{array}\right]$$

2. $\theta_2 \sim$

$$\left[\begin{array}{rrcrr} 0 & 2 & & a & -a \\ 0 & a & & b & b-a \\ \cline{1-2}\cline{4-5} 0 & -a & & b-a & b \\ & & & & \\ -b & 2-b & & b+a & b-a \\ 2 & -a & & b+2 & b+a \\ \cline{1-2}\cline{4-5} b+2 & b+a & & a & 2 \end{array}\right].$$

Proof: It is easy to check that the array of tables representations of θ_1 and θ_2 satisfy the conditions of Lemma 2.1, and so, $\theta_1, \theta_2 \in \mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$.

Let $\theta \in \mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ and let the α-table for θ be

$$\left[\begin{array}{cc} 0 & x \\ 0 & y \\ \hline 0 & z \end{array}\right],$$

where $z = y - x$. Note that $x, y, z \neq 0$. If x, y, z are distinct, then exactly one of x, y, or z equals 2. If x, y, z are not distinct, then, as none of x, y, or z can be 0, $x \neq y$ and $y \neq z$, and so $x = z \neq 0$ and $y = 2x = 0$ or 2 and, as $y \neq 0$, it must be that $y = 2$. Using I, and R we will ensure that $x = 2$ and $y = a$ via a two-step algorithm.

Step 1. If $y = 2$, then $I[\theta]$ has α-table

$$\left[\begin{array}{cr} 0 & 2 \\ 0 & x \\ \hline 0 & -z \end{array}\right].$$

Step 2. If $z = 2$, then $R[\theta]$ has α-table

$$\left[\begin{array}{cr} 0 & -x \\ 0 & 2 \\ \hline 0 & y \end{array}\right],$$

and we can apply Step 1.

Thus, we may assume that $\theta \in \mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ has α-table

$$\left[\begin{array}{cr} 0 & 2 \\ 0 & a \\ \hline 0 & -a \end{array}\right].$$

The β-table for θ must then be

$$\left[\begin{array}{rr} a & -a \\ b & c \\ \hline b-a & c+a \end{array}\right],$$

for some $b, c \in \mathbb{Z}_4$. As $c \neq b, b+2$, there are two cases to consider, $c = b+a$ and $c = b-a$.

Case 1. $c = b + a$.

The β-table for θ is

$$\left[\begin{array}{rr} a & -a \\ b & b+a \\ \hline b-a & b+2 \end{array}\right],$$

and the γ-table for θ is

$$\left[\begin{array}{cr} x & y \\ 2 & -a \\ \hline 2-x & -a-y \end{array}\right],$$

for some $x, y \in \mathbb{Z}_4$. Now $(2-x, -a-y) = (b, b+a)$ or $(b+a, b)$. If $(2-x, -a-y) = (b, b+a)$, then $x = 2-b = y$ which is not possible. Hence $(2-x, -a-y) = (b+a, b)$ and the γ-table for θ is

$$\left[\begin{array}{cr} a-b & -a-b \\ 2 & -a \\ \hline b+a & b \end{array}\right].$$

Similarly, we can show that the δ-table for θ is

$$\left[\begin{array}{rr} b & b+2 \\ b+2 & b-a \\ \hline 2 & a \end{array}\right].$$

This yields the array of tables representation for θ_1.

Case 2. $c = b - a$.

The derivation of the array of tables representation for θ_2 is similar to the derivation in Case 1. $\square$

Reversing the algorithm in the proof of Theorem 3.1 yields all orthomorphisms in $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ and gives us a classification of orthomorphisms in $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$. In this classification every orthomorphism will belong to one of three classes, class 1, class 2, or class 3; will be of one of two types, type s, or type t; and will have parameters (a, b), where $a = \pm 1$ and $b \in \mathbb{Z}_4$. The orthomorphisms, θ_1 and θ_2 in Theorem 3.1, are in class 1; θ_1 is of type s, and θ_2 is of type t; and both have parameters (a, b). The complete classification follows.

Theorem 3.2 (The classification of orthomorphisms in $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$) *Let $\theta \in \mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ have parameters (a, b), $a = \pm 1$, $b \in \mathbb{Z}_4$. Then θ is one of the following.*

Class 1, type s

$$\theta_{1s} \sim \left[\begin{array}{cc} \begin{array}{rr} 0 & \mathbf{2} \\ 0 & \mathbf{a} \\ \hline 0 & -a \end{array} & \begin{array}{rr} \mathbf{a} & -a \\ \mathbf{b} & \mathbf{b+a} \\ \hline b-a & b+2 \end{array} \\ \\ \begin{array}{rr} a-b & -a-b \\ 2 & -a \\ \hline b+a & b \end{array} & \begin{array}{rr} b & b+2 \\ b+2 & b-a \\ \hline 2 & a \end{array} \end{array}\right]$$

Class 1, type t

$$\theta_{1t} \sim \left[\begin{array}{cc} \begin{array}{rr} 0 & \mathbf{2} \\ 0 & \mathbf{a} \\ \hline 0 & -a \end{array} & \begin{array}{rr} \mathbf{a} & -a \\ \mathbf{b} & \mathbf{b-a} \\ \hline b-a & b \end{array} \\ \\ \begin{array}{rr} -b & 2-b \\ 2 & -a \\ \hline b+2 & b+a \end{array} & \begin{array}{rr} b+a & b-a \\ b+2 & b+a \\ \hline a & 2 \end{array} \end{array}\right].$$

Class 2, type s

$$\theta_{2s} \sim \left[\begin{array}{cc} \begin{array}{rr} 0 & \mathbf{a} \\ 0 & \mathbf{2} \\ \hline 0 & a \end{array} & \begin{array}{rr} \mathbf{2} & -a \\ \mathbf{b} & b+2 \\ \hline b+2 & b-a \end{array} \\ \\ \begin{array}{rr} 2-b & a-b \\ -a & a \\ \hline b+a & b \end{array} & \begin{array}{rr} -\mathbf{b} & -a-b \\ -a-b & a-b \\ \hline -a & 2 \end{array} \end{array}\right].$$

Class 2, type t

$$\theta_{2t} \sim \left[\begin{array}{cc} \begin{array}{rr} 0 & \mathbf{a} \\ 0 & \mathbf{2} \\ \hline 0 & a \end{array} & \begin{array}{rr} \mathbf{2} & -a \\ \mathbf{b} & b+2 \\ \hline b+2 & b-a \end{array} \\ \\ \begin{array}{rr} -\mathbf{b} & -a-b \\ a & -a \\ \hline b+a & b \end{array} & \begin{array}{rr} 2-b & a-b \\ a-b & -a-b \\ \hline -a & 2 \end{array} \end{array}\right].$$

Class 3, type s

$$\theta_{3s} \sim \left[\begin{array}{rrcrr} 0 & \mathbf{a} & & -\mathbf{a} & 2 \\ 0 & -\mathbf{a} & & \mathbf{b} & b+a \\ \cline{1-2}\cline{4-5} 0 & 2 & & b+a & b-a \\ & & & & \\ b-a & b+2 & & \mathbf{b} & b+a \\ a & 2 & & b-a & b+2 \\ \cline{1-2}\cline{4-5} 2-b & -b & & -a & a \end{array}\right].$$

Class 3, type t

$$\theta_{3t} \sim \left[\begin{array}{rrcrr} 0 & \mathbf{a} & & -\mathbf{a} & 2 \\ 0 & -\mathbf{a} & & \mathbf{b} & b+a \\ \cline{1-2}\cline{4-5} 0 & 2 & & b+a & b-a \\ & & & & \\ b+a & \mathbf{b} & & b+2 & b-a \\ a & 2 & & b-a & b+2 \\ \cline{1-2}\cline{4-5} -b & 2-b & & a & -a \end{array}\right].$$

Proof: Let θ_1 and θ_2 be as in Theorem 3.1. It is clear that $\theta_{1s} = \theta_1$ and $\theta_{1t} = \theta_2$ Now θ_{2s} can be obtained from $I[\theta_{1s}]$ by replacing b by $a-b$; θ_{2t} can be obtained from $I[\theta_{1t}]$ by replacing b by $-b$; θ_{3s} can be obtained from $R[\theta_{2s}]$ by replacing a by $-a$, and then b by $b-a$; and θ_{3t} can be obtained from $R[\theta_{2t}]$ by replacing a by $-a$, and then b by $b-a$. $\square$

We obtain the following as a corollary.

Corollary 3.1 $|\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)| = 48$.

Proof: It is routine to check the orthomorphisms in Theorem 3.2 are all distinct. The result then follows from the fact that $\theta \in \mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ is in one of 3 classes, is of one of 2 types, and has one of 8 parameter pairs. $\square$

In each of the arrays of tables in Theorem 3.2, five entries are shown in bold. In each array, two entries in the second column of the α-table, and two entries in the first column of the β-table are shown in bold: these entries determine the class and parameters of an orthomorphism. The fifth entry shown in bold distinguishes between types.

If $\theta \in \mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$, then we will write $\theta = xy(a, b)$, whenever θ is in class x, of type y, and has parameters (a, b). The actions of I and R are easily described with the classification of the elements of $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ given in Theorem 3.2.

Lemma 3.1 *For elements of* $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ *in class* x*, of type* y*, with parameters* (a, b)*, the following hold.*

1. $R[1y(a, b)] = 1y(-a, b - a)$,
2. $R[2y(a, b)] = 3y(-a, b - a)$,
3. $R[3y(a, b)] = 2y(-a, b - a)$,
4. $I[1y(a, b)] = \begin{cases} 2s(a, a - b) & \text{if } y = s, \\ 2t(a, -b) & \text{if } y = t, \end{cases}$
5. $I[2y(a, b)] = \begin{cases} 1s(a, a - b) & \text{i } y = s, \\ 1t(a, -b) & \text{if } y = t, \end{cases}$
6. $I[3y(a, b)] = \begin{cases} 3s(-a, b - a) & \text{if } y = s, \\ 3t(-a, a + b) & \text{if } y = t. \end{cases}$

Proof: Routine. □

We see that I, and R are all type-preserving, while the precise action of I does depend on the type.

We will next determine all orthogonalities in $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$. We will begin by determining the orthogonalities within class 1.

Lemma 3.2 *Let* $\theta, \phi \in \mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ *be in class 1. Then* $\theta \perp \phi$ *if and only if* $\theta \perp_2 \phi$*, if and only if, without loss of generality,* $\theta = 1s(a, b)$ *and* $\phi = 1t(-a, -b)$ *for some* $a = \pm 1$ *and some* $b \in \mathbb{Z}_4$*.*

Proof: Let $\theta, \phi \in \mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ be in class 1. Clearly $\theta \perp \phi$ if and only if $\theta \perp_2 \phi$. Suppose that the parameters for θ are (a, b), for some $a = \pm 1$ and some $b \in \mathbb{Z}_4$: then, as $\theta(0, 2) \neq \phi(0, 2)$, the parameters for ϕ must be $(-a, c)$, for some $c \in \mathbb{Z}_4$.

If θ and ϕ are of the same type, then

$$\beta(a) - \beta(a) = c - d - b = \beta(-a) - \beta(-a),$$

where $d = a$ if the type is s and $d = -a$ if the type is t. Hence, by Lemma 2.2, θ and ϕ cannot be orthogonal.

Thus we may assume, without loss of generality, that $\theta = 1s(a, b)$ and $\phi = 1t(-a, c)$. In this case, we have

$$B'_\theta = \{b, b + 2\} = B_\phi = \{-c, 2 - c\}.$$

Therefore $c = 2 - b$ or $-b$. If $c = 2 - b$, then

$$\gamma(b) - \delta(b) = -a - b = \gamma(b + 2) - \delta(b + 2).$$

Hence, by Lemma 2.2, θ and ϕ cannot be orthogonal.

It follows that, if $\theta \perp \phi$, then $\phi = 1t(-a,-b)$: in this case

$$\{\alpha_\phi(i) - \alpha_\theta(i) \mid i \in A_\theta = A_\phi\} \cup \{\beta_\phi(i) - \beta_\theta(i) \mid i \in A'_\theta = A'_\phi\}$$
$$= \{0,2\} \cup \{a-2b, -a-2b\} = \mathbb{Z}_4,$$

and

$$\{\gamma_\phi(i) - \delta_\theta(i) \mid i \in B_\theta = B'_\phi\} \cup \{\delta_\phi(i) - \gamma_\theta(i) \mid i \in B'_\theta = B_\phi\}$$
$$= \{-b, 2-b\} \cup \{a-b, -a-b\} = \mathbb{Z}_4,$$

from which it follows, by Lemma 2.2, that $\theta \perp \phi$. □

Using Lemma 3.2 and the actions of congruences, we can determine all the 2-orthogonalities in $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$.

Theorem 3.3 *Let $a = \pm 1$. The 2-orthogonalities in $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ are the following.*

1. $1s(a,b) \perp_2 1t(-a,-b)$ *if* $b \in \mathbb{Z}_4$;
2. $2s(a,b) \perp_2 3s(a,b+a)$ *if* $b = 0,2$;
3. $2s(a,b) \perp_2 3t(a,b+a)$ *if* $b = \pm a$;
4. $2t(a,b) \perp_2 3t(a,b+a)$ *if* $b = 0,2$; *and*
5. $2t(a,b) \perp_2 3s(a,b+a)$ *if* $b = \pm a$.

Proof: It follows from Lemmas 3.1 and 3.2 that each orthomorphism in $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ is 2-orthogonal to exactly one orthomorphism in $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$. By lemma 3.2, $1s(a,b) \perp_2 1t(-a,-b)$.

Applying Lemma 3.1,

$$I[1s(a,b)] = 2s(a,a-b) \perp_2 1t(-a,-b)1s(a,b)^{-1} = \begin{cases} 3s(a,2-b) & \text{if } b = \pm a, \\ 3t(a,2-b) & \text{if } b = 0,2. \end{cases}$$

Substituting $a-b$ for b, and noting that, if $b' \in \{\pm a \pm b\}$, then $b' \in \{a,-a\}$ if $b \in \{0,2\}$ and $b' \in \{0,2\}$ if $b \in \{a,-a\}$, we obtain

$$2s(a,b) \perp_2 3s(a,b+a) \text{ if } b = 0,2,$$

and

$$2s(a,b) \perp_2 3t(a,b+a) \text{ if } b = \pm a.$$

Now

$$R[2s(a,b)] = 3s(-a,b-a) \perp_2 R[3s(a,b+a)] = 2s(-a,b) \text{ if } b = 0,2,$$

and

$$R[2s(a,b)] = 3s(-a,b-a) \perp_2 R3t(a,b+a)] = 2t(-a,b) \text{ if } b = \pm a.$$

Substituting $-a$ for a yields

$$2s(a,b) \perp_2 3s(a,b+a) \text{ if } b = 0,2,$$

and

$$2t(a,b) \perp_2 3s(a,b+a) \text{ if } b = \pm a.$$

The last 2-orthogonality can be obtained similarly, starting with $1t(a,b) \perp_2 1s(-a,-b)$. $\square$

We will next determine all the 1-orthogonalities in $\mathrm{Orth}(G)$.

Theorem 3.4 *Let $a = \pm 1$. The 1-orthogonalities in $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ are the following.*

1. $2x(a,b) \perp_1 3x(-a,-a-b)$ *for* $b = 0,2$ *and* $x = s,t$,
2. $2s(a,b) \perp_1 1s(a,b+2)$ *for* $b = \pm a$,
3. $2t(a,b) \perp_1 1t(-a,b-a)$ *for* $b = \pm a$,
4. $3s(a,b) \perp_1 1s(a,b+2)$ *for* $b = 0,2$, *and*
5. $3t(a,b) \perp_1 1t(-a,b-a)$ *for* $b = 0,2$.

Proof: Let $\theta, \phi \in \mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ be in the union of class 2 and class 3. As $|A'_\theta \cap A'_\phi| = 1$, we may assume θ to have parameters (a,b) and ϕ to have parameters $(-a,c)$ for some $a = \pm 1$ and some $b,c \in \mathbb{Z}_4$.

If θ and ϕ are both in class 2, then $\beta_\phi(2) - \beta_\theta(2) = c - b$, and so $c \neq b$. Now $|\{-b,-a-b\} \cap \{-c,a-c\}| = 1$. We see that $-c \neq -b$ and $-c \neq -a-b$ as then $a-c = -b$. Thus, $a-c = -b$ or $-a-b$. Further $a-c \neq -b$ as then $-c = -a-b$. Hence, $a-c = -a-b$, and so $c = b+2$. But then

$$\theta(p^a)^{-1}\phi(p^a) = qp^{b+2} \neq qp^{-b} = \theta(p^{-a})^{-1}\phi(p^{-a}),$$

from which it follows that $b = 0,2$. Therefore

$$\{\theta(p^i)^{-1}\phi(p^i) \mid i \in \mathbb{Z}_4\} = \{1, p^2, q, qp^2\}.$$

As $e,f \in \{\pm a, 2 \pm a\}$ implies that $e - f \in \{0,2\}$, inspection of the potential γ-tables and δ-tables for θ and ϕ shows that

$$\theta(qp^a)^{-1}\phi(qp^a) \in \{1, p^2, q, qp^2\},$$

and so θ and ϕ cannot be 1-orthogonal.

If θ and ϕ are both in class 3 and $\theta \perp_1 \phi$, then $R[\theta] \perp_1 R[\phi]$. As $R[\theta]$ and $R[\phi]$ are both in class 2, this is not possible, and so θ and ϕ cannot be 1-orthogonal.

Thus, we may assume that θ is in class 2 and ϕ is in class 3. Now $\phi(0,2) - \theta(0,2) = (0, c-a-b)$, and so $c \neq b+a$. Now $|\{-b, -a-b\} \cap \{c, c-a\}| = 1$. We see that $c \neq -b$ as then $c-a = -a-b$, and $c-a \neq -a-b$ as then $c = -b$. If $c-a = -b$, then $c = a-b$ and

$$\theta(p^{-a})^{-1}\phi(p^{-a}) = qp^{-a-b} = \theta(p^{a})^{-1}\phi(p^{a}).$$

It follows that $c-a \neq -b$ and, hence $c = -a-b \neq b+a$, and so $2b \neq 2$, which implies that $b = 0, 2$.

Thus $\theta = 2x(a,b)$ and $\phi = 3y(-a, -a-b)$ for some $x, y = s, t$, some $a = \pm 1$, and some $b = 0, 2$. If $x = s$ and $y = t$, then $\theta(q^{2-b})^{-1}\phi(qp^{2-b}) = 1$, which is not possible. Hence θ and ϕ cannot be 1-orthogonal. As $R[2t(a,b)] = 3t(-a, b-a)$ and $R[3s(-a,-a-b)] = 2s(a,-b)$, we can conclude that θ and ϕ cannot be 1-orthogonal if $x = t$ and $y = s$. Therefore $x = y$. The proof that $2s(a,b) \perp_1 3s(-a,-a-b)$, for $b = 0, 2$, follows from Table 1, and the proof that $2t(a,b) \perp_1 3t(-a,-a-b)$, for $b = 0, 2$, follows from Table 2. These are the only 1-orthogonalities within the union of class 2 and class 3: other 1-orthogonalities can be obtained from these by using congruences.

As $2s(a,b) \perp_1 3s(-a,-a-b)$ for $b = 0, 2$,

$$1s(a, a-b) = I[2s(a,b)] \perp_1 3s(-a,-a-b)2s(a,b)^{-1} = 2s(a,-a-b),$$

for $b = 0, 2$. Substituting $-a-b$ for b yields $2s(a,b) \perp_1 1s(a, b+2)$ for $b = \pm a$.

As $2t(a,b) \perp_1 3t(-a,-a-b)$ for $b = 0, 2$,

$$1t(a,-b) = I[2t(a,b)] \perp_1 3t(-a,-a-b)2t(a,b)^{-1} = 2t(-a,-a-b),$$

for $b = 0, 2$. Substituting $-a-b$ for b and then $-a$ for a yields $2t(a,b) \perp_1 1t(-a, b-a)$ for $b = \pm a$.

As $2s(a,b) \perp_1 1s(a, b+2)$ for $b = \pm a$,

$$3s(-a, b-a) = R[2s(a,b)] \perp_1 R[1s(a,b+2)] = 1s(-a, b+a),$$

for $b = \pm a$. Substituting $-b+a$ for b and then $-a$ for a yields $3s(a,b) \perp_1 1s(a, b+2)$ for $b = 0, 2$.

As $2t(a,b) \perp_1 1t(-a, b-a)$ for $b = \pm a$,

$$3t(-a, b-a) = R[2t(a,b)] \perp_1 R[1t(-a, b-a)] = 1t(a,b),$$

for $b = \pm a$. Substituting $b+a$ for b and then $-a$ for a yields $3t(a,b) \perp_1 1t(-a, b-a)$ for $b = 0, 2$.

We will next show that there are no other 1-orthogonalities in $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$. As all 1-orthogonalities within the union of class 2 and class 3 have been determined, and as there can be no 1-orthogonalities within class 1, any excess 1-orthogonality must be between an orthomorphism in class 1 and an orthomorphism in the union of class 2 and class 3. Let $\theta_1, \ldots, \theta_{16}$ be the orthomorphisms in class 1, and set $\phi_i = I[\theta_i]$ and $\epsilon_i = I[\phi_i]$ for $i = 1, \ldots, 16$. In the subgraph Γ of $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ whose edges correspond to 1-orthogonalities, $\mathrm{degree}(\theta_i) = \mathrm{degree}(\phi_i) = \mathrm{degree}(\epsilon_i)$ for $i = 1, \ldots, 16$, from which it follows that the average degree of a vertex of Γ in class 1 is equal to the average degree of a vertex of Γ in the union of class 2 and class 3. Let e be the number of edges in Γ corresponding to excess 1-orthogonalities. Then the average degree of a vertex in class 1 is $1 + (e/16)$ and the average degree of a vertex in the union of class 2 and class 3 is $1 + (e/32)$. Hence, $e = 0$ and the result follows. $\square$

Table 1: Proof that $2s(a, b) \perp_1 3s(-a, -a - b)$, $a = \pm 1$, $b = 0, 2$

x	$\phi(x)$	$\theta(x)$	$\phi(x) - \theta(x)$
1	1	1	1
p^{-a}	p^a	qp^{b+2}	qp^{-a-b}
p^a	qp^{-a-b}	p^2	qp^{a-b}
p^2	qp^{2-b}	qp^b	p^2
qp^{-b}	p^{-a}	qp^{-a-b}	qp^b
qp^{a-b}	p^2	p^a	p^a
qp^{-a-b}	qp^{-b}	qp^{a-b}	p^{-a}
qp^{2-b}	qp^{a-b}	p^{-a}	qp^{2-b}

Table 2: Proof that $2t(a, b) \perp_1 3t(-a, -a - b)$, $a = \pm 1$, $b = 0, 2$

x	$\phi(x)$	$\theta(x)$	$\phi(x) - \theta(x)$
1	1	1	1
p^{-a}	p^a	qp^{b+2}	qp^{-a-b}
p^a	qp^{-a-b}	p^2	qp^{a-b}
p^2	qp^{2-b}	qp^b	p^2
qp^{2-b}	p^{-a}	qp^{a-b}	qp^{b+2}
qp^{-a-b}	p^2	p^{-a}	p^{-a}
qp^{a-b}	qp^{-b}	qp^{-a-b}	p^a
qp^{-b}	qp^{a-b}	p^a	qp^{-b}

The structure of $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ is shown in Figure 1: the 2-orthogonalities are represented by horizontal lines, and the 1-orthogonalities are represented by vertical lines.

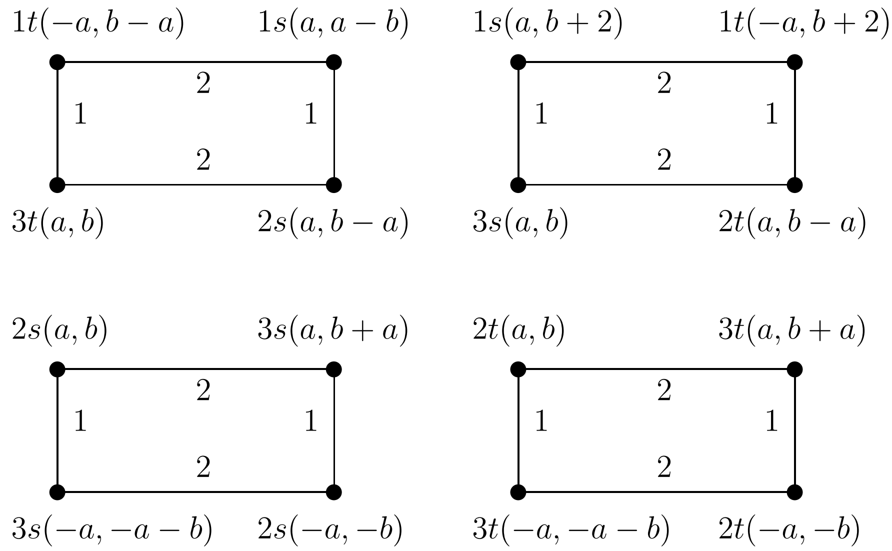

Figure 1: The structure of $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$, $a = \pm 1$, $b = 0, 2$

From the structure of $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$, shown in Figure 1, we can easily determine $\omega(\mathbb{Z}_2 \times \mathbb{Z}_4)$ and the maximal cliques in $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$.

Corollary 3.2 *The maximal cliques in* $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ *are all* 2*-cliques, and* $\omega(\mathbb{Z}_2 \times \mathbb{Z}_4) = 2$.

4 The structure of $\mathrm{Orth}(D_8)$

In this section we will construct and classify the orthomorphisms in $\mathrm{Orth}(D_8)$ and use this classification to prove that $\omega(D_8) = 1$.

Analogous to the classification of orthomorphisms in $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$, we will give a complete classification of orthomorphisms in $\mathrm{Orth}(D_8)$. In this classification, just as in the classification of orthomorphisms in $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$, every orthomorphism will belong to one of three classes, class 1, class 2, or class 3; will be of one of two types, type s, or type t; and will have parameters (a, b), where $a = \pm 1$ and $b \in \mathbb{Z}_4$. We will write $\theta = xy(a, b)$, whenever θ is in class x, of type y, and has parameters (a, b).

Theorem 4.1 (The classification of orthomorphisms in $\mathrm{Orth}(D_8)$**)**
Let $\theta \in \mathrm{Orth}(D_8)$ *have parameters* (a, b), $a = \pm 1$, $b \in \mathbb{Z}_4$. *Then* θ *is one of the following.*

Class 1, type s

$$1s(a,b) \sim \left[\begin{array}{cc} \begin{array}{rr} 0 & \mathbf{2} \\ 0 & \mathbf{a} \\ \hline 0 & -a \end{array} & \begin{array}{rr} \mathbf{a} & -a \\ \mathbf{b} & \mathbf{b+a} \\ \hline b+a & b \end{array} \\ & \\ \begin{array}{rr} b-a & b+a \\ -a & 2 \\ \hline b+2 & b-a \end{array} & \begin{array}{rr} b & b+2 \\ b+2 & b-a \\ \hline 2 & a \end{array} \end{array}\right]$$

Class 1, type t

$$1t(a,b) \sim \left[\begin{array}{cc} \begin{array}{rr} 0 & \mathbf{2} \\ 0 & \mathbf{a} \\ \hline 0 & -a \end{array} & \begin{array}{rr} \mathbf{a} & -a \\ \mathbf{b} & \mathbf{b-a} \\ \hline b+a & b+2 \end{array} \\ & \\ \begin{array}{rr} b+2 & b \\ 2 & -a \\ \hline b & b-a \end{array} & \begin{array}{rr} b-a & b+a \\ b+a & b+2 \\ \hline 2 & a \end{array} \end{array}\right].$$

Class 2, type s

$$2s(a,b) \sim \left[\begin{array}{cc} \begin{array}{rr} 0 & \mathbf{a} \\ 0 & \mathbf{2} \\ \hline 0 & a \end{array} & \begin{array}{rr} \mathbf{2} & -a \\ \mathbf{b} & b+2 \\ \hline b+2 & b+a \end{array} \\ & \\ \begin{array}{rr} b-a & \mathbf{b} \\ a & -a \\ \hline b & b-a \end{array} & \begin{array}{rr} b+a & b+2 \\ b-a & b+a \\ \hline 2 & -a \end{array} \end{array}\right].$$

Class 2, type t

$$2t(a,b) \sim \left[\begin{array}{cc} \begin{array}{rr} 0 & \mathbf{a} \\ 0 & \mathbf{2} \\ \hline 0 & a \end{array} & \begin{array}{rr} \mathbf{2} & -a \\ \mathbf{b} & b+2 \\ \hline b+2 & b+a \end{array} \\ & \\ \begin{array}{rr} b+a & b+2 \\ -a & a \\ \hline b & b-a \end{array} & \begin{array}{rr} \mathbf{b} & b-a \\ b-a & b+a \\ \hline -a & 2 \end{array} \end{array}\right].$$

Class 3, type s

$$3s(a,b) \sim \left[\begin{array}{cc} \begin{array}{rr} 0 & \mathbf{a} \\ 0 & -\mathbf{a} \\ \hline 0 & 2 \end{array} & \begin{array}{rr} -\mathbf{a} & 2 \\ \mathbf{b} & b-a \\ \hline b-a & b+a \end{array} \\ & \\ \begin{array}{rr} b-a & \mathbf{b} \\ a & 2 \\ \hline b & b+2 \end{array} & \begin{array}{rr} b+a & b+2 \\ b+2 & b+a \\ \hline a & -a \end{array} \end{array}\right].$$

Class 3, type t

$$3t(a,b) \sim \left[\begin{array}{cc} \begin{array}{rr} 0 & \mathbf{a} \\ 0 & -\mathbf{a} \\ \hline 0 & 2 \end{array} & \begin{array}{rr} -\mathbf{a} & 2 \\ \mathbf{b} & b-a \\ \hline b-a & b+a \end{array} \\ & \\ \begin{array}{rr} b+a & b+2 \\ a & 2 \\ \hline b+2 & b \end{array} & \begin{array}{rr} b-a & \mathbf{b} \\ b+2 & b+a \\ \hline -a & a \end{array} \end{array}\right].$$

Proof: Similar to the proof of Theorems 3.1 and 3.2. □

As in the classification of orthomorphisms in $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$, in each of the arrays of tables in Theorem 3.2, five entries are shown in bold. In each array, two entries in the second column of the α-table, and two entries in the first column of the β-table are shown in bold: these entries determine the class and parameters of an orthomorphism. The fifth entry shown in bold distinguishes between types.

We obtain the following as corollaries.

Corollary 4.1 $|\mathrm{Orth}(D_8)| = 48$.

Proof: This is the same as the proof of Corollary 3.1. □

Corollary 4.2 *Let S_{xy} denote the set of orthomorphisms in $\mathrm{Orth}(D_8)$, in class x and of type y. For $y = s, t$, I is a bijection $S_{1y} \to S_{2y}$ and R is a bijection $S_{2y} \to S_{3y}$.*

Proof: Routine. □

Using the classification in Theorem 4.1 we can determine $\omega(D_8)$.

Theorem 4.2 $\omega(D_8) = 1$.

Proof: We will first show that there are no 2-orthogonalities in $\mathrm{Orth}(D_8)$.

Let θ and ϕ be in class 1, and let $\theta \perp \phi$. Then $\theta \perp_2 \phi$, and $\theta = 1x(a,b)$ and $\phi = 1y(-a,c)$ for some $x, y \in \{s,t\}$, some $a = \pm 1$, and some $b, c \in \mathbb{Z}_4$. If $x = y = s$, then

$$\beta_\phi(a) - \beta_\theta(a) = c - a - b = \beta_\phi(-a) - \beta_\theta(-a)$$

and so θ and ϕ are not 2-orthogonal by Lemma 2.2. If $x = y = t$, then

$$\beta_\phi(a) - \beta_\theta(a) = c + a - b = \beta_\phi(-a) - \beta_\theta(-a)$$

and so θ and ϕ are not 2-orthogonal by Lemma 2.2. Hence, $x \neq y$. Without loss of generality $\theta = 1s(a,b)$ and $\phi = 1t(-a,c)$. Now $B'_\theta = \{b, b+2\} = \{c, c+2\} = B_\phi$. Therefore $c = b, b+2$. If $c = b$, then

$$\delta_\phi(b+a) + \gamma_\theta(b+a) = b + a = \delta_\phi(b-a) + \gamma_\theta(b-a)$$

and so θ and ϕ are not 2-orthogonal. If $c = b+2$, then

$$\gamma_\phi(b) + \delta_\theta(b) = b = \gamma_\phi(b+2) + \delta_\theta(b+2)$$

and so θ and ϕ are not 2-orthogonal by Lemma 2.2. It follows from Lemma 2.4 that there are no 2-orthogonalities in $\mathrm{Orth}(D_8)$.

We will next show that there are no 1-orthogonalities in $\mathrm{Orth}(D_8)$ within the union of class 2 and class 3.

Let θ, ϕ be in the union of class 2 and class 3, $\theta \perp_1 \phi$. Then we may assume that θ has parameters (a,b) and ϕ has parameters $(-a,c)$, $a = \pm 1$, $b, c \in \mathbb{Z}_4$. Now $|\{b, b-a\} \cap \{c, c+a\}| = 1$. As $c = b - a$ if and only if $c + a = b$, it must be that $c = b$ or $b + 2$.

If θ, ϕ are both in class 3, then $\beta_\phi(2) - \beta_\theta(2) = c - b + 2 \neq 0$ and so $c = b$. If $x = y = s$, then

$$\gamma_\phi(b) - \gamma_\theta(b) = 0 = \alpha_\phi(0) - \alpha_\theta(0)$$

and so, by Lemma 2.2, θ and ϕ are not 1-orthogonal. If $x = y = t$, then

$$\gamma_\phi(b+2) - \gamma_\theta(b+2) = 0 = \alpha_\phi(0) - \alpha_\theta(0)$$

and so, by Lemma 2.2, θ and ϕ are not 1-orthogonal. Hence $x \neq y$. Without loss of generality $\theta = 3s(a,b)$ and $\phi = 3t(-a,b)$. Then

$$\gamma_\phi(b-a) - \gamma_\theta(b-a) = 2 = \beta_\phi(2) - \beta_\theta(2)$$

and so, by Lemma 2.2, θ and ϕ are not 1-orthogonal.

If θ, ϕ are both in class 2, then $R[\theta]$ and $R[\phi]$ are both in class 3, and so $R[\theta]$ and $R[\phi]$ are not 1-orthogonal and, hence, by Lemma 2.2, θ and ϕ are not 1-orthogonal.

Thus we may assume without loss of generality that $\theta = 2x(a,b)$ and $\phi = 3y(-a,c)$ for some $x, y = s, t$, some $a = \pm 1$, and some $b, c \in \mathbb{Z}_4$. If $x = t$, $y = s$ and $c = b$, then

$$\gamma_\phi(b+a) - \gamma_\theta(b+a) = 0 = \alpha_\phi(0) - \alpha_\theta(0)$$

and so, by Lemma 2.2, θ and ϕ are not 1-orthogonal. If $x = t$, $y = s$ and $c = b + 2$, then

$$\beta_\phi(a) + \alpha_\theta(a) = b = \gamma_\phi(b - a) + \delta_\theta(b - a)$$

and so, by Lemma 2.2, θ and ϕ are not 1-orthogonal. If $x = s$ and $y = t$, then $R[\theta]$ and $R[\phi]$ are not 1-orthogonal, and so θ and ϕ are not 1-orthogonal by Lemma 2.4.

Hence $x = y$. If $x = y = s$ and $c = b$, then

$$\gamma_\phi(b + a) + \delta_\theta(b + a) = b + 2 = \beta_\phi(a) + \alpha_\theta(a)$$

and so, by Lemma 2.2, θ and ϕ are not 1-orthogonal.

If $x = y = s$ and $c = b + 2$, then

$$\gamma_\phi(b + 2) + \delta_\theta(b + 2) = b - a = \alpha_\phi(-a) + \beta_\theta(-a)$$

and so, by Lemma 2.2, θ and ϕ are not 1-orthogonal.

If $x = y = t$ and $c = b$, then

$$\delta_\phi(b) - \delta_\theta(b) = 0 = \alpha_\phi(0) - \alpha_\theta(0)$$

and so, by Lemma 2.2, θ and ϕ are not 1-orthogonal.

If $x = y = t$ and $c = b + 2$, then

$$\gamma_\phi(b + a) - \gamma_\theta(b + a) = 0 = \alpha_\phi(0) - \alpha_\theta(0)$$

and so, by Lemma 2.2, θ and ϕ are not 1-orthogonal.

The proof that there are no other 1-orthogonalities in $\mathrm{Orth}(D_8)$ is the same as the proof in Theorem 3.2 that there are no excess 1-orthogonalities between orthomorphisms in the union of class 2 and class 3, and orthomorphisms in class 1. The result follows. □

5 The structure of $\mathrm{Orth}(Q_8)$

In this section we will construct and classify the orthomorphisms in $\mathrm{Orth}(Q_8)$ and use this classification to prove that $\omega(Q_8) = 1$.

Analogous to the classification of orthomorphisms in $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ and the classification of orthomorphisms in $\mathrm{Orth}(D_8)$, we will give a complete classification of orthomorphisms in $\mathrm{Orth}(Q_8)$. In this classification, just as in the classification of orthomorphisms in $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$, every orthomorphism will belong to one of three classes, class 1, class 2, or class 3; will be of one of two types, type s, or type t; and will have parameters (a, b), where $a = \pm 1$ and $b \in \mathbb{Z}_4$. We will write $\theta = xy(a, b)$, whenever θ is in class x, of type y, and has parameters (a, b).

Theorem 5.1 (The classification of orthomorphisms in $\mathrm{Orth}(Q_8)$) *Let $\theta \in \mathrm{Orth}(Q_8)$ have parameters (a, b), $a = \pm 1$, $b \in \mathbb{Z}_4$. Then θ is one of the following.*

Class 1, type s

$$1s(a,b) \sim \left[\begin{array}{rr@{\qquad}rr}
0 & \mathbf{2} & \mathbf{a} & -a \\
0 & \mathbf{a} & \mathbf{b} & \mathbf{b+a} \\
\cline{1-2}\cline{3-4}
0 & -a & b+a & b \\
 & & & \\
b+a & b-a & b & b+2 \\
-a & 2 & b+2 & b-a \\
\cline{1-2}\cline{3-4}
b+2 & b-a & 2 & a
\end{array}\right]$$

Class 1, type t

$$1t(a,b) \sim \left[\begin{array}{rr@{\qquad}rr}
0 & \mathbf{2} & \mathbf{a} & -a \\
0 & \mathbf{a} & \mathbf{b} & \mathbf{b-a} \\
\cline{1-2}\cline{3-4}
0 & -a & b+a & b+2 \\
 & & & \\
b & b+2 & b+a & b-a \\
2 & -a & b+2 & b+a \\
\cline{1-2}\cline{3-4}
b & b-a & a & 2
\end{array}\right].$$

Class 2, type s

$$2s(a,b) \sim \left[\begin{array}{rr@{\qquad}rr}
0 & \mathbf{a} & \mathbf{2} & -a \\
0 & \mathbf{2} & \mathbf{b} & b+2 \\
\cline{1-2}\cline{3-4}
0 & a & b+2 & b+a \\
 & & & \\
b+a & b+2 & b-a & \mathbf{b} \\
a & -a & b+a & b-a \\
\cline{1-2}\cline{3-4}
b & b-a & 2 & -a
\end{array}\right].$$

Class 2, type t

$$2t(a,b) \sim \left[\begin{array}{rr@{\qquad}rr}
0 & \mathbf{a} & \mathbf{2} & -a \\
0 & \mathbf{2} & \mathbf{b} & b+2 \\
\cline{1-2}\cline{3-4}
0 & a & b+2 & b+a \\
 & & & \\
\mathbf{b} & b-a & b+2 & b+a \\
a & -a & b+a & b-a \\
\cline{1-2}\cline{3-4}
b-a & b & -a & 2
\end{array}\right].$$

Class 3, type s

$$3s(a,b) \sim \left[\begin{array}{rrcrr} 0 & \mathbf{a} & & -\mathbf{a} & 2 \\ 0 & -\mathbf{a} & & \mathbf{b} & b-a \\ \hline 0 & 2 & & b-a & b+a \\ & & & & \\ \mathbf{b} & b-a & & b+2 & b+a \\ 2 & a & & b+a & b+2 \\ \hline b & b+2 & & -a & a \end{array}\right].$$

Class 3, type t

$$3t(a,b) \sim \left[\begin{array}{rrcrr} 0 & \mathbf{a} & & -\mathbf{a} & 2 \\ 0 & -\mathbf{a} & & \mathbf{b} & b-a \\ \hline 0 & 2 & & b-a & b+a \\ & & & & \\ b+a & b+2 & & \mathbf{b} & b-a \\ a & 2 & & b+a & b+2 \\ \hline b & b+2 & & a & -a \end{array}\right].$$

Proof: Similar to the proofs of Theorems 3.1 and 3.2. □

As in the classification of orthomorphisms in $\mathrm{Orth}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ and $\mathrm{Orth}(D_8)$, in each of the arrays of tables in Theorem 3.2, five entries are shown in bold. In each array, two entries in the second column of the α-table, and two entries in the first column of the β-table are shown in bold: these entries determine the class and parameters of an orthomorphism. The fifth entry shown in bold distinguishes between types.

We obtain the following as corollaries.

Corollary 5.1 $|\mathrm{Orth}(Q_8)| = 48$.

Proof: This is the same as the proof of Corollary 3.1. □

Corollary 5.2 *Let S_{xy} denote the set of orthomorphisms in* $\mathrm{Orth}(D_8)$*, in class x and of type y. For $y = s, t$, I is a bijection $S_{1y} \to S_{2y}$ and R is a bijection $S_{2y} \to S_{3y}$.*

Proof: Routine. □

Using the classification in Theorem 5.1 we can determine $\omega(Q_8)$.

Theorem 5.2 $\omega(Q_8) = 1$.

Proof: We will first show that there are no 2-orthogonalities in $\mathrm{Orth}(Q_8)$.

Let θ and ϕ be in class 1, and let $\theta \perp \phi$. Then $\theta \perp_2 \phi$, and $\theta = 1x(a,b)$ and $\phi = 1y(-a,c)$ for some $x, y \in \{s,t\}$, some $a = \pm 1$, and some $b, c \in \mathbb{Z}_4$. If $x = y = s$, then

$$\beta_\phi(a) - \beta_\theta(a) = c - a - b = \beta_\phi(-a) - \beta_\theta(-a)$$

and so θ and ϕ are not 2-orthogonal by Lemma 2.2. If $x = y = t$, then

$$\beta_\phi(a) - \beta_\theta(a) = c + a - b = \beta_\phi(-a) - \beta_\theta(-a)$$

and so θ and ϕ are not 2-orthogonal by Lemma 2.2. Hence, $x \neq y$. Without loss of generality $\theta = 1s(a,b)$ and $\phi = 1t(-a,c)$. Now $B'_\theta = \{b, b+2\} = \{c, c+2\} = B_\phi$. Therefore $c = b, b+2$. If $c = b$, then

$$\delta_\phi(b+a) + \gamma_\theta(b+a) = b + 2 = \gamma_\phi(b) + \delta_\theta(b) + 2$$

and so θ and ϕ are not 2-orthogonal. If $c = b+2$, then

$$\beta_\phi(a) - \beta_\theta(a) = -a = \beta_\phi(-a) - \beta_\theta(-a)$$

and so θ and ϕ are not 2-orthogonal by Lemma 2.2. It follows from Lemma 2.4 that there are no 2-orthogonalities in $\mathrm{Orth}(Q_8)$.

We will next show that there are no 1-orthogonalities in $\mathrm{Orth}(Q_8)$ within the union of class 2 and class 3.

Let θ, ϕ be in the union of class 2 and class 3, $\theta \perp_1 \phi$. Then we may assume that θ has parameters (a,b) and ϕ has parameters $(-a,c)$, $a = \pm 1$, $b, c \in \mathbb{Z}_4$. Now $|\{b, b-a\} \cap \{c, c+a\}| = 1$. As $c = b - a$ if and only if $c + a = b$, it must be that $c = b$ or $b + 2$.

If θ, ϕ are both in class 2, then $\beta_\phi(2) - \beta_\theta(2) = c - b \neq 0$ and so $c = b + 2$. If $x = y = s$, then

$$\delta_\phi(b-a) - \delta_\theta(b-a) = 0 = \alpha_\phi(0) - \alpha_\theta(0)$$

and so, by Lemma 2.2, θ and ϕ are not 1-orthogonal. If $x = y = t$, then

$$\gamma_\phi(b-a) - \gamma_\theta(b-a) = 2 = \beta_\phi(2) - \beta_\theta(2)$$

and so, by Lemma 2.2, θ and ϕ are not 1-orthogonal. If $x \neq y$, then, without loss of generality, $\theta = 2s(a,b)$ and $\phi = 2t(-a, b+2)$. As

$$\gamma_\phi(b+2) - \gamma_\theta(b+2) = 0 = \alpha_\phi(0) - \alpha_\theta(0)$$

and so, by Lemma 2.2, θ and ϕ are not 1-orthogonal.

If θ, ϕ are both in class 3, then $R[\theta]$ and $R[\phi]$, then $R[\theta]$ and $R[\phi]$ are both in class 2, and so $R[\theta]$ and $R[\phi]$ are not 1-orthogonal and, hence, by Lemma 2.4, θ and ϕ are not 1-orthogonal.

Thus we may assume without loss of generality that $\theta = 2x(a, b)$ and $\phi = 3y(-a, c)$ for some $x, y = s, t$, some $a = \pm 1$, and some $b, c \in \mathbb{Z}_4$. If $x = t$, $y = s$ and $c = b$, then

$$\beta_\phi(2) - \beta_\theta(2) = a = \gamma_\phi(b) - \gamma_\theta(b)$$

and so, by Lemma 2.2, θ and ϕ are not 1-orthogonal. If $x = t$, $y = s$ and $c = b + 2$, then

$$\gamma_\phi(b - a) - \gamma_\theta(b - a) = 0 = \alpha_\phi(0) - \alpha_\theta(0)$$

and so, by Lemma 2.2, θ and ϕ are not 1-orthogonal. If $x = s$ and $y = t$, then $R[\theta]$ and $R[\phi]$ are not 1-orthogonal, and so θ and ϕ are not 1-orthogonal by Lemma 2.4.

Hence $x = y$. If $x = y = s$ and $c = b$, then

$$\delta_\phi(b - a) - \delta_\theta(b - a) = a = \beta_\phi(2) - \beta_\theta(2)$$

and so, by Lemma 2.2, θ and ϕ are not 1-orthogonal.

If $x = y = s$ and $c = b + 2$, then

$$\gamma_\phi(b + 2) - \gamma_\theta(b + 2) = -a = \beta_\phi(2) - \beta_\theta(2)$$

and so, by Lemma 2.2, θ and ϕ are not 1-orthogonal.

If $x = y = t$ and $c = b$, then

$$\gamma_\phi(b - a) - \gamma_\theta(b - a) = 0 = \alpha_\phi(0) - \alpha_\theta(0)$$

and so, by Lemma 2.2, θ and ϕ are not 1-orthogonal.

If $x = y = t$ and $c = b + 2$, then

$$\delta_\phi(b + 2) - \delta_\theta(b + 2) = 0 = \alpha_\phi(0) - \alpha_\theta(0)$$

and so, by Lemma 2.2, θ and ϕ are not 1-orthogonal.

The proof that there are no other 1-orthogonalities in $\mathrm{Orth}(Q_8)$ is the same as the proof in Theorem 3.2 that there are no excess 1-orthogonalities between orthomorphisms in the union of class 2 and class 3, and orthomorphisms in class 1. The result follows. □

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