ON HAMILTON CYCLES IN CUBIC (m,n)-METACIRCULANT GRAPHS

NGO DAC TAN *

Institute of Mathematics
P.O.Box 631 Bo Ho, 10 000 Hanoi, Vietnam

ABSTRACT. Connected cubic (m,n)-metacirculant graphs, other than the Petersen graph, have been previously proved to be hamiltonian for m odd, m divisible by 4 and m = 2. In this paper we give two sufficient conditions for connected cubic (m,n)-metacirculant graphs with m even, greater than 2 and not divisible by 4 to be hamiltonian. As corollaries, we show that every connected cubic (m,n)-metacirculant graph, other than the Petersen graph, has a Hamilton cycle if any one of the following conditions is met :

(i) Either m and n are positive integers such that n is even and every odd prime divisor of n is also a divisor of m; or

(ii) $n = 2^a p^b$, where p is an odd prime, $a > 0$ and $b \geq 0$.

1. INTRODUCTION

The class of (m,n)-metacirculant graphs was introduced in [1] as an interesting class of vertex-transitive graphs which included many non-Cayley graphs and which might contain further examples of non-hamiltonian graphs.

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It has been asked [2] whether or not every connected (m,n)-metacirculant graph, other than the Petersen graph, has a Hamilton cycle.

There are several papers that consider the above question. In [2, 3] it has been proved that if n is a prime, then every connected (m,n)-metacirculant graph, other than the Petersen graph, has a Hamilton cycle. Connected cubic (m,n)-metacirculant graphs, other than the Petersen graph, also have been proved to be hamiltonian for m odd [8], m divisible by 4 [11] (see also [9] for m = 4) and m = 2 [4, 8].

This paper is a sequel to [8, 11]. We consider here the above question for connected cubic (m,n)-metacirculant graphs with m even, greater than 2 and not divisible by 4. We will use techniques similar to ones used in [11]. In Section 3 we will give two sufficient conditions for connected cubic (m,n)-metacirculant graphs with m even, greater than 2 and not divisible by 4 to be hamiltonian. These conditions will be applied in Section 4 to prove the following Theorem 1.

THEOREM 1. Let G be a connected cubic (m,n)-metacirculant graph, other than the Petersen graph. Then G has a Hamilton cycle if any one of the following conditions is met:

(i) Either m and n are positive integers such that n is even and every odd prime divisor of n is also a divisor of m; or

(ii) $n = 2^a p^b$, where p is an odd prime, $a > 0$ and $b \geq 0$.

It is clear that this result is a partial answer to the above question for connected cubic (m,n)-metacirculant graphs with m even, greater than 2 and not divisible by 4. In addition to our result in Theorem 1 it is useful to mention that if $\gcd(m, \varphi(n)) = 1$ where φ is the Euler φ-function, then every (m,n)-metacirculant graph is a Cayley graph on an abelian group of order mn ([1], Corollary 5). Therefore, if $mn \geq 3$ and $\gcd(m,\varphi(n)) = 1$, then every connected (m,n)-metacirculant graph has a Hamilton cycle ([2], Corollary 3).

As a corollary of Theorem 1, the above mentioned result and ones obtained in [4, 8, 11] we will have immediately the following Theorem 2 which is a generalization of the main theorem in [4].

THEOREM 2. Let $\mathcal{G} = \langle \rho, \tau : \rho^n = \tau^m = 1, \tau^{-1}\rho\tau = \rho^\alpha$ with $\alpha^m \equiv 1 \pmod{n} \rangle$ be the semidirect product of a cyclic group of order n with a cyclic group of order m. Then every connected cubic Cayley graph on $\mathcal{G}$ has a Hamilton cycle if any one of the following conditions is met:

(i) Either m is odd;

(ii) m is divisible by 4;

(iii) $m = 2$;

(iv) $\gcd(m,\varphi(n)) = 1$, where φ is the Euler φ-function;

(v) n is even such that every odd prime divisor of n is also a divisor of m; or

(vi) $n = 2^a p^b$ with p an odd prime, $a > 0$ and $b \geq 0$.

2. PRELIMINARIES

(a) The reader is referred to [1] for basic properties of (m,n)-metacirculant graphs although their construction is now described.

We will denote the ring of integers modulo n by Z_n and the multiplicative group of units in Z_n by Z_n^*. Let m and n be two positive integers, $\alpha \in Z_n^*$, $\mu = \lfloor m/2 \rfloor$ and S_0, S_1, ..., S_μ be subsets of Z_n satisfying the following conditions: (1) $0 \notin S_0 = -S_0$; (2) $\alpha^m S_r = S_r$ for $0 \leqslant r \leqslant \mu$; (3) if m is even, then $\alpha^\mu S_\mu = -S_\mu$. Then we define the (m,n)-metacirculant graph $G = MC(m,n,\alpha,S_0,S_1,\ldots,S_\mu)$ to be the graph with vertex-set $V(G) = \{ v_j^i : i \in Z_m ; j \in Z_n \}$ and edge-set $E(G) = \{ v_j^i v_h^{i+r} : 0 \leqslant r \leqslant \mu ; i \in Z_m ; h, j \in Z_n$ and $(h-j) \in \alpha^i S_r \}$, where superscripts and subscripts are always reduced modulo m and modulo n, respectively.

The above construction is designed to allow the permutations ρ and τ on $V(G)$ defined by $\rho(v_j^i) = v_{j+1}^i$ and $\tau(v_j^i) = v_{\alpha j}^{i+1}$ to be automorphisms of G. Thus, (m,n)-metacirculant graphs are vertex-transitive. Some (m,n)-metacirculant graphs are Cayley graphs, but many of them are non-Cayley.

(b) A permutation β is said to be semiregular if all cycles in the disjoint cycle decomposition of β have the same length. If a graph G has a semiregular automorphism β, then the quotient graph G/β with respect to β is defined as follows. The vertices of G/β are the orbits of the subgroup $\langle\beta\rangle$ generated by β and two such vertices are adjacent if and only if there is an edge in G joining a vertex of one corresponding orbit to a vertex in the other orbit.

Let β be of order t and $G^0, G^1, \ldots, G^{\ell}$ be the subgraphs induced by G on the orbits of $\langle\beta\rangle$. Let $v_0^i, v_1^i, \ldots, v_{t-1}^i$ be a cyclic labelling of the vertices of G^i under the action of β and $C = G^0G^iG^j\ldots G^rG^0$ be a cycle of G/β. Consider a path P of G arising from a lifting of C, namely, start at v_0^0 and choose an edge from v_0^0 to a vertex v_a^i of G^i. Then take an edge from v_a^i to a vertex v_b^j of G^j following G^i in C. Continue in this way until returning to a vertex v_d^0 of G^0. The set of all paths that can be constructed in this way using C is called in [5] the coil of C and is denoted by coil(C).

It is not difficult to prove the following result.

LEMMA 1 ([9]). Let t be the order of a **semiregular** automorphism β of a graph G and G^0 be the subgraph induced by G on an orbit of $\langle \beta \rangle$. If there exists a Hamilton cycle C in G/β such that coil(C) contains a path P whose terminal vertices are distance d apart in G^o where P starts and terminates and $\gcd(d,t) = 1$, then G has a Hamilton cycle.

(c) The following results proved in [10] will be used.

LEMMA 2 ([10]). Let $G = MC(m,n,\alpha,S_o,S_1,\dots,S_\mu)$ be a cubic (m,n)-metacirculant graph such that m is even and greater than 2, $S_o = \emptyset$, $S_i = \{s\}$ with $0 \leqslant s < n$ for some $i \in \{1,2,\dots,\mu-1\}$, $S_j = \emptyset$ for all $i \neq j \in \{1,2,\dots,\mu-1\}$ and $S_\mu = \{k\}$ with $0 \leqslant k < n$. Then

(1) if G is connected, then either i is odd and $\gcd(i,m) = 1$ or i is even, μ is odd and $\gcd(i,m) = 2$;

(2) if i is odd and $\gcd(i,m) = 1$, then G is isomorphic to the cubic (m,n)-metacirculant graph $G' = MC(m,n,\alpha', S_o',S_1',\dots,S_\mu')$ with $\alpha' = \alpha^i$, $S_o' = \emptyset$, $S_1' = \{s\}$ $(0 \leqslant s < n)$, $S_2' = \dots = S_{\mu-1}' = \emptyset$ and $S_\mu' = \{k\}$ $(0 \leqslant k < n)$;

(3) if i is even, μ is odd, $\gcd(i,m) = 2$ and $i = 2^r i'$ with $r \geqslant 1$ and i' odd, then G is isomorphic to the cubic (m,n)-metacirculant graph $G'' = MC(m,n,\alpha'',S_o'',S_1'',\dots,S_\mu'')$ with $\alpha'' = \alpha^{i'}$, $S_o'' = S_1'' = \dots = S_{2^r-1}'' = \emptyset$, $S_{2^r}'' = \{s\}$ $(0 \leqslant s < n)$, $S_{2^r+1}'' = \dots = S_{\mu-1}'' = \emptyset$ and $S_\mu'' = \{k\}$ $(0 \leqslant k < n)$.

LEMMA 3 ([10]). Let $G = MC(m,n,\alpha,S_o,S_1,\dots,S_\mu)$ be a cubic (m,n)-metacirculant graph such that m is even, greater than 2 and not divisible by 4, $S_o = S_1 = \dots = S_{2^r-1} =$

$\emptyset$ with $r \geqslant 1$, $S_{2^r} = \{s\}$ with $0 \leqslant s < n$, $S_{2^r+1} = \dots = S_{\mu-1} = \emptyset$ and $S_\mu = \{k\}$ with $0 \leqslant k < n$. Then G is connected if and only if $\gcd(h,n) = 1$, where h is $[k(1+\alpha+\alpha^2+\dots+\alpha^{2^r-1}) - s(1+\alpha+\alpha^2+\dots+\alpha^{\mu-1})]$ reduced modulo n.

(d) For the next section we also need the following lemma.

LEMMA 4. Let $G = MC(m,n,\alpha,S_0,S_1,\dots,S_\mu)$ be a connected cubic (m,n)-metacirculant graph such that m is even, greater than 2 and not divisible by 4, $S_0 = S_1 = \dots = S_{2^r-1} = \emptyset$ with $r \geqslant 1$, $S_{2^r} = \{s\}$ with $0 \leqslant s < n$, $S_{2^r+1} = \dots = S_{\mu-1} = \emptyset$ and $S_\mu = \{k\}$ with $0 \leqslant k < n$. Let

$\bar{n} = \gcd(\alpha-1,n)$ and

$\bar{\bar{n}} = \gcd((1-\alpha+\alpha^2-\dots+\alpha^{\mu-1}),n)$.

Then $n/(\bar{n}\bar{\bar{n}})$ is a divisor of $(\alpha+1)$.

PROOF. By the definition of (m,n)-metacirculant graphs, we have

I. $\alpha^{2\mu}s \equiv s \pmod{n}$,

$$\Leftrightarrow (\alpha+1)(1-\alpha+\alpha^2-\dots+\alpha^{\mu-1})(\alpha-1)(1+\alpha+\alpha^2+\dots+\alpha^{\mu-1})s \equiv 0 \pmod{n}. \tag{2.1}$$

II. $\alpha^\mu k \equiv -k \pmod{n}$,

$$\Leftrightarrow (\alpha+1)(1-\alpha+\alpha^2-\dots+\alpha^{\mu-1})k \equiv 0 \pmod{n}. \tag{2.2}$$

Since G is connected, it follows from Lemma 3 that $\gcd(h,n) = 1$, where h is $[k(1+\alpha+\alpha^2+\dots+\alpha^{2^r-1}) - s(1+\alpha+\alpha^2+\dots+$

$\alpha^{\mu-1})]$ reduced modulo n. Hence,

$$\gcd(\gcd(k,n), \gcd(s(1+\alpha+\alpha^2+\ldots+\alpha^{\mu-1}), n)) = 1. \tag{2.3}$$

Assume first that $(\alpha+1)(1-\alpha+\alpha^2-\ldots+\alpha^{\mu-1}) \equiv 0 \pmod{n}$. Then we trivially have $(\alpha+1)(1-\alpha+\alpha^2-\ldots+\alpha^{\mu-1})(\alpha-1) \equiv 0 \pmod{n}$. Therefore, $n/(\bar{n}\bar{\bar{n}})$ is a divisor of $(\alpha+1)$.

Assume next that $(\alpha+1)(1-\alpha+\alpha^2-\ldots+\alpha^{\mu-1}) \not\equiv 0 \pmod{n}$ and let

$$z = n/\gcd([(\alpha+1)(1-\alpha+\alpha^2-\ldots+\alpha^{\mu-1})], n).$$

Then, by (2.2), z is a divisor of gcd(k,n). Since (2.1) and (2.3) hold, we see that z must be a divisor of $(\alpha-1)$. Thus, we again have $(\alpha+1)(1-\alpha+\alpha^2-\ldots+\alpha^{\mu-1})(\alpha-1) \equiv 0 \pmod{n}$. Therefore, $n/(\bar{n}\bar{\bar{n}})$ is a divisor of $(\alpha+1)$. Lemma 4 is proved.

3. SUFFICIENT CONDITIONS

In this section two sufficient conditions for connected cubic (m,n)-metacirculant graphs to be hamiltonian will be given. Since connected cubic (m,n)-metacirculant graphs, other than the Petersen graph, have been proved to be hamiltonian for m odd [8], m divisible by 4 [11] and m = 2 [4, 8], we may assume in the next lemmas that m is even, greater than 2 and not divisible by 4.

LEMMA 5. Let $G = MC(m,n,\alpha,S_0,S_1,\dots,S_\mu)$ be a connected cubic (m,n)-metacirculant graph such that m is even, greater than 2 and not divisible by 4, $S_0 = S_1 = \dots = S_{2^r-1} = \emptyset$ with $r \geq 1$, $S_{2^r} = \{s\}$ with $0 \leq s < n$, $S_{2^r+1} = \dots = S_{\mu-1} = \emptyset$ and $S_\mu = \{k\}$ with $0 \leq k < n$. Let $\bar{n} = \gcd(\alpha-1, n)$ and $\bar{\bar{n}} = \gcd((1-\alpha+\alpha^2-\dots+\alpha^{\mu-1}), n)$. Then G has a Hamilton cycle if any one of the following conditions is met:

(i) Either $\gcd(n/(\bar{n}\bar{\bar{n}}), \mu\bar{n}-1) = 1$; or

(ii) $\bar{\bar{n}} = 1$.

PROOF. Let G, $\bar{n}$ and $\bar{\bar{n}}$ be as in the formulation of Lemma 5.

(A) Assume first that assumption (i) is satisfied. Let ρ be the automorphism of G defined by $\rho(v_j^i) = v_{j+1}^i$ for every vertex $v_j^i \in V(G)$. Then $\rho^{\alpha-1}$ is semiregular. Thus, we can construct the quotient graph $G/\rho^{\alpha-1}$. It is not difficult to verify that $G/\rho^{\alpha-1}$ is isomorphic to the $(m,\bar{n})$-metacirculant graph $\bar{G} = MC(m,\bar{n},\bar{\alpha},\bar{S}_0,\bar{S}_1,\dots,\bar{S}_\mu)$, where $1 = \bar{\alpha} \equiv \alpha \pmod{\bar{n}}$, $\bar{S}_0 = \bar{S}_1 = \dots = \bar{S}_{2^r-1} = \emptyset$ with $r \geq 1$, $\bar{S}_{2^r} = \{\bar{s}\}$ with $\bar{s} \equiv s \pmod{\bar{n}}$ and $0 \leq \bar{s} < \bar{n}$, $\bar{S}_{2^r+1} = \dots = \bar{S}_{\mu-1} = \emptyset$ and $\bar{S}_\mu = \{\bar{k}\}$ with $\bar{k} \equiv k \pmod{\bar{n}}$ and $0 \leq \bar{k} < \bar{n}$. Therefore, from now on we can identify $G/\rho^{\alpha-1}$ with $\bar{G}$ and in order to avoid the confusion between vertices of G and $\bar{G}$ we assume that $V(\bar{G}) = \{w_j^i : i \in Z_m; j \in Z_{\bar{n}}\}$. Since G is connected, it follows that $\bar{G}$ is connected. Therefore, by Lemma 3,

$$\gcd(\bar{h}, \bar{n}) = 1, \qquad (3.1)$$

where $\bar{h}$ is

$$[\bar{k}(1+\bar{\alpha}+\bar{\alpha}^2+\ldots+\bar{\alpha}^{2^r-1})-\bar{s}(1+\bar{\alpha}+\bar{\alpha}^2+\ldots+\bar{\alpha}^{\mu-1})]$$

$$= [2^r\bar{k}-\mu\bar{s}] \qquad (3.2)$$

reduced modulo $\bar{n}$.

By definition, we have $\bar{\alpha}^{\mu}\bar{k} \equiv -\bar{k} \pmod{n} \iff 2\bar{k} \equiv 0 \pmod{\bar{n}}$. This means that

$$2\bar{k} = u\bar{n} \qquad (3.3)$$

for some integer u. If $\bar{n}$ is odd, then from (3.3) and $0 \leqslant \bar{k} < \bar{n}$ it follows that $\bar{k} = 0$. Therefore, from (3.1) and (3.2) we have $\gcd(\mu\bar{s}, \bar{n}) = 1$ in this subcase. If $\bar{n}$ is even but $\bar{k} = 0$, then we still have $\gcd(\mu\bar{s}, \bar{n}) = 1$ as before. If $\bar{n}$ is even but $\bar{k} \neq 0$, then from (3.3) and $0 \leqslant \bar{k} < \bar{n}$ it follows that $\bar{k} = \bar{n}/2$. Since $r \geqslant 1$, $2^r\bar{k} \equiv 0 \pmod{\bar{n}}$. Therefore, from (3.1) and (3.2) we again have $\gcd(\mu\bar{s}, \bar{n}) = \gcd([2^r\bar{k}-\mu\bar{s}], \bar{n}) = 1$. Thus, in all cases we have

$$\gcd(\mu\bar{s}, \bar{n}) = 1.$$

Denote $Q(w_j^i) = w_j^i w_{j+\bar{s}}^{i+2^r} w_{j+2\bar{s}}^{i+2\cdot 2^r} \ldots w_{j+(\mu-1)\bar{s}}^{i+(\mu-1)2^r}$. Then, since $\gcd(\mu\bar{s}, \bar{n}) = 1$,

$$C_1 = Q(w_0^0)Q(w_{\mu\bar{s}}^0)Q(w_{2\mu\bar{s}}^0)\ldots Q(w_{(\bar{n}-1)\mu\bar{s}}^0) \quad \text{and}$$

$$C_2 = Q(w_0^1)Q(w_{\mu\bar{s}}^1)Q(w_{2\mu\bar{s}}^1)\ldots Q(w_{(\bar{n}-1)\mu\bar{s}}^1)$$

are cycles of $\bar{G}$. Moreover, $V(C_1) \cap V(C_2) = \emptyset$ and $V(\bar{G}) = V(C_1) \cup V(C_2)$. Since $\bar{\alpha} = 1$, it follows that the vertex w_0^0

of C_1 is adjacent to the vertex $w^{\mu}_{\bar{k}}$ of C_2 and the vertex $w^{2^r}_{\bar{s}}$ of C_1 is adjacent to the vertex $w^{\mu+2^r}_{\bar{k}+\bar{s}}$ of C_2. So, we can construct the following Hamilton cycle C of $\bar{G}$ from C_1 and C_2 (see Figure 1). Start C with the edge $w^{o}_{o}w^{\mu}_{\bar{k}}$. Then go around C_2 from $w^{\mu}_{\bar{k}}$ in the direction of $w^{\mu-2^r}_{\bar{k}-\bar{s}}$ until reaching $w^{\mu+2^r}_{\bar{k}+\bar{s}}$. Proceed along it by taking the edge $w^{\mu+2^r}_{\bar{k}+\bar{s}}w^{2^r}_{\bar{s}}$ and go now around the cycle C_1 from $w^{2^r}_{\bar{s}}$ in the direction of $w^{2\cdot 2^r}_{2\bar{s}}$ until reaching w^{o}_{o}.

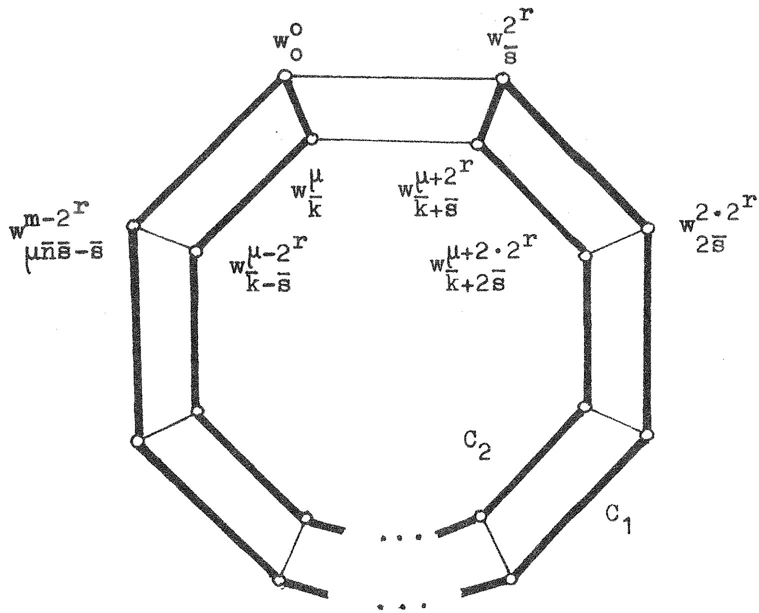

Figure 1

Let P be the path of coil(C) which starts at v_0^o. This path terminates at v_f^o with

$$f \equiv (k - \alpha^{\mu-2^r}s - \alpha^{\mu-2\cdot 2^r}s - \dots - \alpha^{\mu+2\cdot 2^r}s - \alpha^{\mu+2^r}s +$$

$$\alpha^{\mu+2^r}k + \alpha^{2^r}s + \alpha^{2\cdot 2^r}s + \dots + \alpha^{(\mu-1)2^r}s)(\text{mod } n),$$

where the numbers of $-\alpha^{\mu-2^r}s$, $-\alpha^{\mu-2\cdot 2^r}s$, ..., $-\alpha^{\mu+2^r}s$, $\alpha^{2^r}s$, $\alpha^{2\cdot 2^r}s$, ..., $\alpha^{(\mu-2)2^r}s$ and $\alpha^{(\mu-1)2^r}s$ terms are $\bar{n}$, whilst the numbers of s and $-\alpha^{\mu}s$ terms are $(\bar{n}-1)$ and the numbers of k and $\alpha^{\mu+2^r}k$ terms are 1. Therefore,

$$f \equiv (\alpha^{\mu}s - s + k + \alpha^{\mu+2^r}k) + \bar{n}(s + \alpha^{2^r}s + \alpha^{2\cdot 2^r}s +$$

$$\dots + \alpha^{(\mu-1)2^r}s) - \bar{n}(\alpha^{\mu}s + \alpha^{\mu+2^r}s + \alpha^{\mu+2\cdot 2^r}s +$$

$$\dots + \alpha^{\mu-2^r}s)(\text{mod } n). \tag{3.4}$$

Since $r \geqslant 1$ and $\mu = m/2$ is odd, we have $\gcd(2^r, \mu) = 1$. Therefore, 0, 2^r, $2\cdot 2^r$, ..., $(\mu-1)2^r$ are all even numbers modulo m and μ, $\mu+2^r$, $\mu+2\cdot 2^r$, ..., $\mu-2\cdot 2^r$, $\mu-2^r$ are all odd numbers modulo m. Therefore,

$$s + \alpha^{2^r}s + \alpha^{2\cdot 2^r}s + \dots + \alpha^{(\mu-1)2^r}s = s + \alpha^2 s +$$

$$\alpha^4 s + \dots + \alpha^{2\mu-2}s = s(1 + \alpha + \alpha^2 + \dots + \alpha^{\mu-1})(1 -$$

$$\alpha + \alpha^2 - \dots + \alpha^{\mu-1}) \tag{3.5}$$

and $\alpha^{\mu}s + \alpha^{\mu+2^r}s + \dots + \alpha^{\mu-2^r}s = \alpha s + \alpha^3 s + \dots$

$$+ \alpha^{2\mu-1}s = \alpha s(1 + \alpha + \alpha^2 + \dots + \alpha^{\mu-1})(1 - \alpha +$$

$$\alpha^2 - \dots + \alpha^{\mu-1}). \tag{3.6}$$

From (3.4), (3.5) and (3.6) we have

$$f \equiv (s(\alpha-1)(1+\alpha+\alpha^2+\dots+\alpha^{\mu-1})+k(1-\alpha)(1+\alpha+\alpha^2+\dots+\alpha^{2^r-1}))+(1-\alpha)\bar{n}s(1+\alpha+\alpha^2+\dots+\alpha^{\mu-1})(1-\alpha+\alpha^2-\dots+\alpha^{\mu-1}) \pmod{n}. \qquad (3.7)$$

By the definition of metacirculant graphs, we have $\alpha^{\mu}k \equiv -k \pmod{n} \Leftrightarrow (\alpha^{\mu}+1)k \equiv 0 \pmod{n}$. Therefore, we have

$$0 \equiv (\alpha-1)\bar{n}k(\alpha^{\mu}+1)(1+\alpha^2)(1+\alpha^{2^2})\dots(1+\alpha^{2^{r-1}})$$
$$\equiv (\alpha-1)\bar{n}k(1+\alpha+\alpha^2+\dots+\alpha^{2^r-1})(1-\alpha+\alpha^2-\dots+\alpha^{\mu-1}) \pmod{n}. \qquad (3.8)$$

From (3.7) and (3.8) it follows that

$$f \equiv f+0 \equiv \{-(\alpha-1)[k(1+\alpha+\alpha^2+\dots+\alpha^{2^r-1})-s(1+\alpha+\alpha^2+\dots+\alpha^{\mu-1})]\}+\{(\alpha-1)\bar{n}(1-\alpha+\alpha^2-\dots+\alpha^{\mu-1})[k(1+\alpha+\alpha^2+\dots+\alpha^{2^r-1})-s(1+\alpha+\alpha^2+\dots+\alpha^{\mu-1})]\} \equiv (\alpha-1)d \pmod{n},$$

where $d = [k(1+\alpha+\alpha^2+\dots+\alpha^{2^r-1})-s(1+\alpha+\alpha^2+\dots+\alpha^{\mu-1})][\bar{n}(1-\alpha+\alpha^2-\dots+\alpha^{\mu-1})-1]$.

It is not difficult to see that the automorphism $\rho^{\alpha-1}$ has order $t = n/\bar{n} = \bar{\bar{n}}(n/(\bar{n}\bar{\bar{n}}))$. Since G is conncted, by Lemma 3, $\gcd(h,n) = 1$, where h is $[k(1+\alpha+\alpha^2+\dots+\alpha^{2^r-1}) - s(1+\alpha+\alpha^2+\dots+\alpha^{\mu-1})]$ reduced modulo n. Hence,

$$\gcd(h,t) = 1. \qquad (3.9)$$

It is also clear that

$$\gcd([\bar{n}(1-\alpha+\alpha^2-\ldots+\alpha^{\mu-1})-1],\bar{\bar{n}}) = 1. \qquad (3.10)$$

Furthermore, we have $\alpha^{2i} = ((\alpha+1-1)^2)^i = (\alpha+1)x_i+1$ and $\alpha^{2i+1} = \alpha^{2i}\alpha = ((\alpha+1)x_i+1)((\alpha+1)-1) = (\alpha+1)y_i-1$, where x_i and y_i are integers. Consequently, $(1-\alpha+\alpha^2-\ldots+\alpha^{\mu-1}) = (\alpha+1)x+\mu$ for some integer x. Thus,

$$[\bar{n}(1-\alpha+\alpha^2-\ldots+\alpha^{\mu-1})-1] = \bar{n}(\alpha+1)x+(\mu\bar{n}-1). \qquad (3.11)$$

By Lemma 4, $n/(\bar{n}\bar{\bar{n}})$ is a divisor of $(\alpha+1)$. From this, (3.11) and assumption (i) of our lemma it is easy to see that

$$\gcd([\bar{n}(1-\alpha+\alpha^2-\ldots+\alpha^{\mu-1})-1],t) = 1. \qquad (3.13)$$

Thus, $\gcd(d,t) = 1$ because (3.9) and (3.13) hold. By Lemma 1, G has a Hamilton cycle in this case.

(B) Assume now that assumption (ii) is satisfied, i.e., $\bar{\bar{n}} = 1$. By the definition of metacirculant graphs, we have $\alpha^{\mu}k \equiv -k \pmod{n} \Leftrightarrow (\alpha^{\mu}+1)k \equiv 0 \pmod{n}$. Therefore,

$$\begin{aligned}0 &\equiv -k(\alpha^{\mu}+1)(1+\alpha^2)(1+\alpha^{2^2})\ldots(1+\alpha^{2^{r-1}})\\ &\equiv -k(1+\alpha+\alpha^2+\ldots+\alpha^{2^r-1})(1-\alpha+\alpha^2-\\ &\quad \ldots+\alpha^{\mu-1}) \pmod{n}. \qquad (3.14)\end{aligned}$$

On the other hand, since $\gcd(2^r,\mu) = 1$, the numbers $0, 2^r, 2\cdot 2^r, \ldots, (\mu-1)2^r$ are all even integers modulo m. Therefore, from (3.5) and (3.14), we have

$$s+\alpha^{2^r}s+\alpha^{2\cdot 2^r}s+\dots+\alpha^{(\mu-1)2^r}s \equiv -(1-\alpha+\alpha^2- \dots+\alpha^{\mu-1})[k(1+\alpha+\alpha^2+\dots+\alpha^{2^r-1})-s(1+\alpha+ \alpha^2+\dots+\alpha^{\mu-1})] \pmod n. \qquad (3.15)$$

Since G is connected, by Lemma 3, $\gcd(h,n) = 1$, where h is $[k(1+\alpha+\alpha^2+\dots+\alpha^{2^r-1})-s(1+\alpha+\alpha^2+\dots+\alpha^{\mu-1})]$ reduced modulo n. Furthermore, by assumption (ii), $\bar{\bar{n}} = 1$. Therefore, from (3.15) we have

$$\gcd((s+\alpha^{2^r}s+\alpha^{2\cdot 2^r}s+\dots+\alpha^{(\mu-1)2^r}s), n) = 1. \qquad (3.16)$$

Let $Q(v_j^i) = v_j^i v_{j+\alpha^i s}^{i+2^r} v_{j+\alpha^i(s+\alpha^{2^r}s)}^{i+2\cdot 2^r} \dots v_{j'}^{i+(\mu-1)2^r}$, where $j' = j+\alpha^i(s+\alpha^{2^r}s+\dots+\alpha^{(\mu-2)2^r}s)$. Let z be $(s+\alpha^{2^r}s+\alpha^{2\cdot 2^r}s+\dots+\alpha^{(\mu-1)2^r}s)$ reduced modulo n. Then, since (3.16) holds,

$$C_1 = Q(v_o^o)Q(v_z^o)Q(v_{2z}^o)\dots Q(v_{(n-1)z}^o) \text{ and}$$
$$C_2 = Q(v_o^1)Q(v_{\alpha z}^1)Q(v_{2\alpha z}^1)\dots Q(v_{(n-1)\alpha z}^1)$$

are cycles of G. Moreover, $V(C_1) \cap V(C_2) = \emptyset$ and $V(G) = V(C_1) \cup V(C_2)$.

Now we relabel the vertices of G as follows (see Figure 2). Choose a direction of C_1. Because the chosen direction, for every vertex v_j^i of C_1 we can talk about the vertex following v_j^i in C_1. The vertex v_o^o of C_1 is relabelled by u_o. The (unique) vertex of C_2 which is adjacent to v_o^o is relabelled by v_o. Suppose v_j^i of C_1 and the vertex of C_2 adjacent to v_j^i have been relabelled by u_x and v_x, respectively. Then the vertex $v_{j'}^{i'}$ following v_j^i in C_1 is relabelled by u_{x+1} and the vertex of C_2 adjacent to $v_{j'}^{i'}$ is relabelled by v_{x+1}.

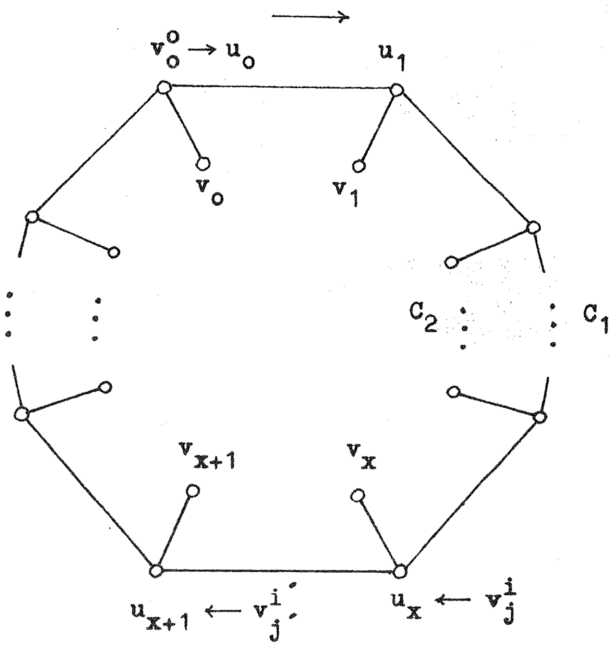

Figure 2

We show now that the relabelled graph G is a generalized Petersen graph. Let ρ and τ be the automorphisms of G defined by $\rho(v_j^i) = v_{j+1}^i$ and $\tau(v_j^i) = v_{\alpha j}^{i+1}$ for every $v_j^i \in V(G)$. Then $\gamma = \rho^s\tau^{2^r}$ is also an automorphism of G. For every $v_j^i \in V(G)$, we have

$$\gamma(v_j^i) = \rho^s \tau^{2^r}(v_j^i) = \rho^s(v^{i+2^r}_{(\alpha^{2^r} j)}) = v^{i+2^r}_{(s+\alpha^{2^r} j)} .$$

In particular,

$$\gamma(v_o^o) = v_s^{2^r},\ \gamma(v_s^{2^r}) = v^{2\cdot 2^r}_{(s+\alpha^{2^r} s)}, \ldots .$$

This means that depending on the chosen direction of C_1, either γ maps every vertex of C_1 to the vertex following it in C_1 or γ maps every vertex of C_1 to the vertex preceding it in C_1. Without loss of generality we may assume that γ maps every vertex of C_1 to the vertex following it in C_1. Therefore, in the relabelled graph G, $\gamma(u_i) = u_{i+1}$ and $\gamma(v_i) = v_{i+1}$. From this it follows immediately that the relabelled graph G is a generalized Petersen graph $GP(mn/2, \ell)$.

On the other hand, G is vertex-transitive. Therefore, either $\ell^2 \equiv \pm 1 \pmod{mn/2}$ or $mn/2 = 10$ and $\ell = 2$ [7]. In both cases, G has a Hamilton cycle [6]. Lemma 5 is completely proved.

LEMMA 6. Let $G = MC(m,n,\alpha,S_o,S_1,\ldots,S_\mu)$ be a connected cubic (m,n)-metacirculant graph such that m is even, greater than 2 and not divisible by 4, $S_o = \emptyset$, $S_1 = \{s\}$ with $0 \leqslant s < n$, $S_2 = S_3 = \ldots = S_{\mu-1} = \emptyset$ and $S_\mu = \{k\}$ with $0 \leqslant k < n$. Then G has a Hamilton cycle if n is even.

PROOF. The proof of the main result in [11] (Theorem 5) for the case of an even n can be repeated here to prove our Lemma 6 if some minor changes in this proof (in connection with the assumption on m which here is even, greater than 2 and not divisible by 4) are made. The reader is invited to do all these in details to complete the proof of Lemma 6.

4. PROOFS OF THEOREMS

PROOF OF THEOREM 1. Let $G = MC(m,n,\alpha,S_0,S_1,\dots,S_\mu)$ be a connected cubic (m,n)-metacirculant graph, other than the Petersen graph. If m is odd or m is divisible by 4 or $m = 2$, then G has a Hamilton cycle [8, 11, 4]. If m is even, greater than 2 and not divisible by 4 but $S_0 \neq \emptyset$, then by [8] G has a Hamilton cycle. Thus, we may assume from now on that m is even, greater than 2 and not divisible by 4 and $S_0 = \emptyset$. Since G is cubic, it is not difficult to see that in this case $S_i = \{s\}$ with $0 \leqslant s < n$ for some $i \in \{1,2,\dots,\mu-1\}$, $S_j = \emptyset$ for all $i \neq j \in \{1, 2, \dots, \mu-1\}$ and $S_\mu = \{k\}$ with $0 \leqslant k < n$. By Lemma 2, G is isomorphic to G' or G'', where G' and G'' are as in Lemma 2. Since G is connected, G' and G'' are also connected.

(A) Assume first that assumption (i) of Theorem 1 is satisfied. If G is isomorphic to G', then G has a Hamilton cycle because by Lemma 6 G' has a Hamilton cycle. If G is isomorphic to G'', then let $\bar{n}$ and $\bar{\bar{n}}$ be defined as in Lemma 5. Since n is even, the number $\bar{n}$ is also even. Therefore, $\mu\bar{n}-1$ is odd. Hence, $d = \gcd(n/(\bar{n}\bar{\bar{n}}), \mu\bar{n}-1)$ is odd. Suppose that $d > 1$ and let p be a prime divisor of d. Then p is odd. Since d is a divisor of $n/(\bar{n}\bar{\bar{n}})$, p is also a prime divisor of n. By assumption (i), p is also a divisor of m. Being odd, in fact, p is a divisor of μ. On the other hand, p is a divisor of $\mu\bar{n}-1$. Thus, p divides 1. This contradiction shows that $d = 1$. By Lemma 5(i), G'' has a Hamilton cycle. Therefore, G has a Hamilton cycle.

(B) Assume now that $n = 2^a p^b$, where p is an odd prime, $a > 0$ and $b \geq 0$. If G is isomorphic to G', then again by Lemma 6 G' has a Hamilton cycle. Therefore, G has a Hamilton cycle. If G is isomorphic to G'', then let $\bar{n}$ and $\bar{\bar{n}}$ be defined as in Lemma 5. Since n is even, α must be odd. Therefore, $\bar{n}$ is even and $\bar{\bar{n}}$ is odd. From this it follows that $\mu\bar{n}-1$ is odd and $\bar{\bar{n}} = p^c$ with $0 \leq c \leq b$. If $c = 0$, then G'' has a Hamilton cycle by Lemma 5(ii). If $c > 0$ and p is a divisor of $n/(\bar{n}\bar{\bar{n}})$, then p is also a divisor of $(\alpha+1)$ by Lemma 4. We have $(1-\alpha+\alpha^2-\ldots+\alpha^{\mu-1}) = (\alpha+1)x+\mu$ for some integer x. Therefore, p is also a divisor of μ. By Theorem 1(i) above, G'' has a Hamilton cycle in this subcase. If $c > 0$ and p is not a divisor of $n/(\bar{n}\bar{\bar{n}})$, then $n/(\bar{n}\bar{\bar{n}}) = 2^d$ with $0 \leq d \leq a$. Since $\mu\bar{n}-1$ is odd, we have in this subcase $\gcd(n/(\bar{n}\bar{\bar{n}}), \mu\bar{n}-1) = 1$ and G'' again has a Hamilton cycle by Lemma 5(i). Thus, in any cases, G'' has a

Hamilton cycle. Therefore, G has a Hamilton cycle.

Theorem 1 is completely proved.

PROOF OF THEOREM 2. It has been proved in [1] (Theorem 2) that every Cayley graph on $\mathcal{G}$ is an (m,n)-metacirculant graph. Therefore, the conclusions (i) - (iii) follow from the results obtained in [8, 11, 4], respectively. (iv) is the result mentioned after the formulation of Theorem 1. Finally, (v) and (vi) follow from Theorem 1. Theorem 2 is proved.

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