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Color-induced subgraphs of Grünbaum colorings of triangulations

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Abstract

We consider Grünbaum-colored triangulations that are dual to properly 3-edge-colored embedded cubic graphs. Previous studies of color-induced subgraphs (CISGs) of Grünbaum colorings of triangulations have focused on determining what properties of an embedding correspond to all CISGs being connected. Here we examine CISGs from multiple perspectives, including building on prior work to determine when these results do and do not generalize. We determine when CISGs can and cannot be trees or forests, and answer a question of Kasai, Matsumoto, and Nakamoto about whether there can be a Grünbaum-colored triangulation with every CISG a tree.

1 Background and Introduction

In a properly edge-colored cubic graph, the set of edges of each color forms a perfect matching. When such a graph is embedded on a surface without edges crossing, we may consider the topological dual; here, the set of edges of a given color has more interesting structure, and investigating this structure is the motivation for the present work. We begin by making these notions precise.

Recall that a *cubic* graph is 3-regular and a *proper* 3*-edge coloring* assigns three colors to its edges such that no two incident edges receive the same color. A *cellularly*

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embedded graph G on a surface S has the property that $S \setminus G$ is a collection of topological disks, each of which is called a *face* of the embedding. The *topological dual* of a cellularly embedded graph is another graph G° embedded on S such that each vertex of G° corresponds to a face of G embedded on S, and an edge joins two vertices of G° exactly when the two corresponding faces of G embedded on S share an edge. (If two faces of G embedded on S share k edges, then the corresponding vertices of G° are joined by k edges.) Note that $(G^\circ)^\circ = G$. A *triangulation* is an embedded graph where every face has three edges, and every triangulation is topologically dual to an embedded cubic graph. Dualizing a proper 3-edge coloring of an embedded cubic graph produces an edge coloring where each triangular face's edges use all three colors; this is called a *Grünbaum coloring*.

Not every cubic graph has a proper 3-edge coloring, and commensurately not every embedded triangulation has a Grünbaum coloring. However, most nondegenerate triangulations on a torus have Grünbaum colorings (see [1]) as do duals of all cubic graphs of $n \leq 30$ vertices (see [7]). In [6] the authors show that all even-degree triangulations of sufficiently low genus or high representativity have Grünbaum colorings, and [5] shows that on the projective plane, triangulations with all even degree except for two adjacent vertices of odd degree have Grünbaum colorings. Thus, the class of Grünbaum-colorable triangulations on surfaces is quite large. In this paper we only consider embedded cubic graphs for which there exists a proper 3-edge coloring, and their associated dual Grünbaum-colored triangulations.

We now establish some notation. We let C be a properly 3-edge colored cubic graph cellularly embedded on a surface S. Dual to C is a Grünbaum colored triangulation T, also embedded on S. Note that while both C and T are edge colored, those colorings need not be unique. The colors used will be denoted c_1, c_2, and c_3. We will say that an edge is a c_k edge if it is colored c_k, and that a cycle is a c_i-c_j cycle if its edges alternate in color between c_i and c_j. The subgraph of T induced by all edges of color c_i will be denoted G_i; this is called a *color-induced subgraph (CISG)*. Notice that this differs from inducing a subgraph from a subset of vertices. In particular, even though every vertex of C will be incident to edges of all three colors, not every vertex of T must be incident to a given color.

1.1 Prior work

The study of color-induced subgraphs is still in its infancy; there are only two papers extant on the topic. The first paper was by Gottleib and Shelton [4], in which they prove that a planar triangulation has a Grünbaum coloring such that each CISG is connected, and this coloring is unique, if and only if the degree of each vertex of the triangulation is even. Their proofs use planarity in an integral way (the Jordan Curve Theorem for one direction, and uniqueness of vertex 3-coloring for the other), so there is no straightforward generalization to any other surface.

More than a decade later, Kasai, Matsumoto, and Nakamoto in [5] proved that a projective-planar triangulation has a Grünbaum coloring such that each CISG is connected if and only if the degree of each vertex of the triangulation is even. Both papers note that there are even toroidal triangulations without CISGs connected, and [5] gives such examples for every orientable surface of genus $g \geq 1$.

1.2 Overview of results

The main results in this paper are about determining when there exists a Grünbaum coloring where all CISGs are connected, extending [4], and when CISGs can be trees or forests. In particular, we address the question raised in [5] as to whether it is possible for a triangulation on the sphere or projective plane to have a Grünbaum coloring with each CISG isomorphic to a tree. Here is a guide to the remainder of the paper.

In Section 2.1, we give examples to show that various generalizations of the main theorem in [4] do not hold. We contrast this in Section 2.2 with a surface-independent condition under which there exists a Grünbaum coloring where all CISGs are connected. This is accompanied by a family of Grünbaum-colored triangulations for each genus (orientable and nonorientable) with all connected CISGs. In Section 2.3, we characterize CISG components that are trees and prove that forest CISGs can only occur on the plane or projective plane. We follow this in Section 2.4 with our main theorem, that only on the projective plane can there be tree CISGs and that for any projective planar embedding there can be at most two tree CISGs.

2 Results about CISGs on Surfaces

2.1 First Observations

In [4], the authors show that there exists a Grünbaum coloring of a planar triangulation such that each color-induced subgraph is connected if and only if the degree of each vertex of the triangulation is even. Their proof cannot be generalized: Starting with a Grünbaum coloring with connected color-induced subgraphs, they use the Jordan Curve Theorem to show that the triangulation must have even degree, and many curves on surfaces are not Jordan curves. For the converse, the authors apply the Three Color Theorem [9] to an even-degree planar triangulation to produce a Grünbaum coloring with connected color-induced subgraphs; nonplanar even-degree triangulations are not necessarily 3-vertex colorable.

Moreover, neither implication direction of this theorem generalizes to other surfaces. In [4], the authors close with one example of an all-even-degree triangulation on the torus that does not have a Grünbaum coloring with connected color-induced subgraphs. We give in Example 2.1 an example of a not-all-even-degree triangulation on the torus that *does* have a Grünbaum coloring with connected color-induced subgraphs.

Example 2.1. The generalized Petersen graph $GP(9, 4)$ has an embedding on the torus with a proper 3-edge coloring such that each pair of colors forms a Hamilton

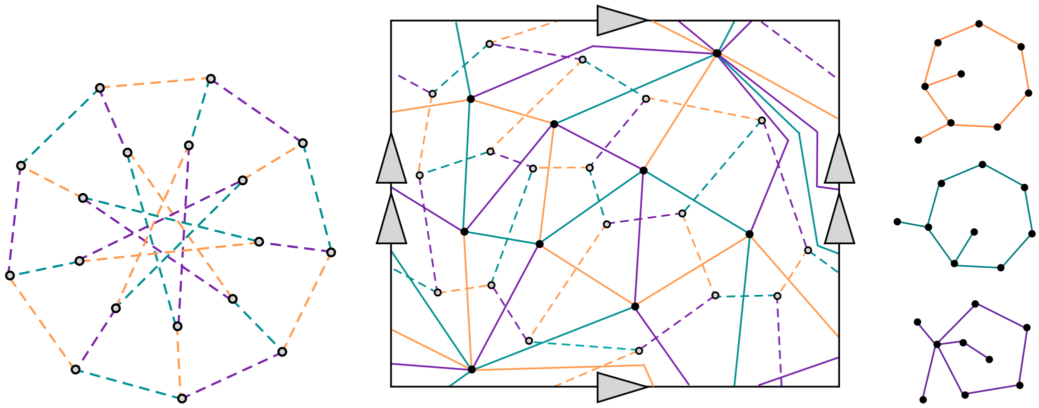

Figure 1: At left, $GP(9,4)$; center, a properly edge-colored embedding of $GP(9,4)$ together with its dual triangulation; at right, the CISGs of the triangulation.

cycle. The dual triangulation has all CISGs connected. See Figure 1 for $GP(9,4)$, the embedding and dual embedding, and CISGs.

In the planar case, there can be at most one Grünbaum coloring with connected color-induced subgraphs [4]. On some other surface, does there exist a triangulation with more than one Grünbaum coloring with connected color-induced subgraphs? The answer is *yes*.

Example 2.2. The $(5,4)$ embedding of K_6 on the torus (see [1] and [3]) is so called because all faces are triangles except for one pentagonal face and one quadrangular face. We transform this embedding into a triangulation L (for Lulu) by adding three edges. In Figures 2 and 3 we demonstrate two different Grünbaum colorings of L—these are completions of the fourth and sixth partial Grünbaum coloring of the $(5,4)$ embedding of K_6 on the torus (see [1])—and their associated connected CISGs. In each figure, isolated vertices that are not part of the CISGs are shown for convenience.

2.2 Connected-CISG Grünbaum colorings

In this section, we give a general condition under which we can assure that a Grünbaum coloring has all connected CISGs, and exhibit infinite families of examples with such Grünbaum colorings.

Proposition 2.3. *Given any surface S, if a triangulation T on S is 3-vertex colorable, then there exists a Grünbaum coloring where all CISGs are connected.*

Proof. We proceed by expanding on the argument in [4]: From a vertex 3-coloring, we produce an edge coloring by assigning the color c_k to every edge between vertices of colors c_i, c_j. Any triangle has three different vertex colors and thus three different edge colors, so this is a Grünbaum coloring.

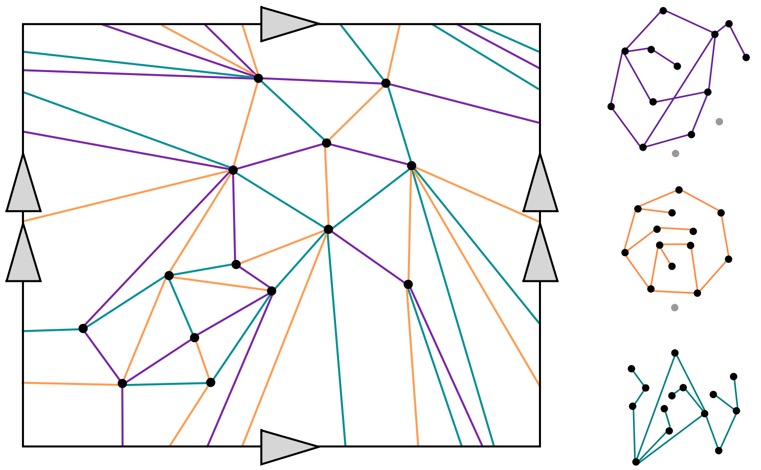

Figure 2: At left, one Grünbaum coloring of L; at right, the CISGs of L.

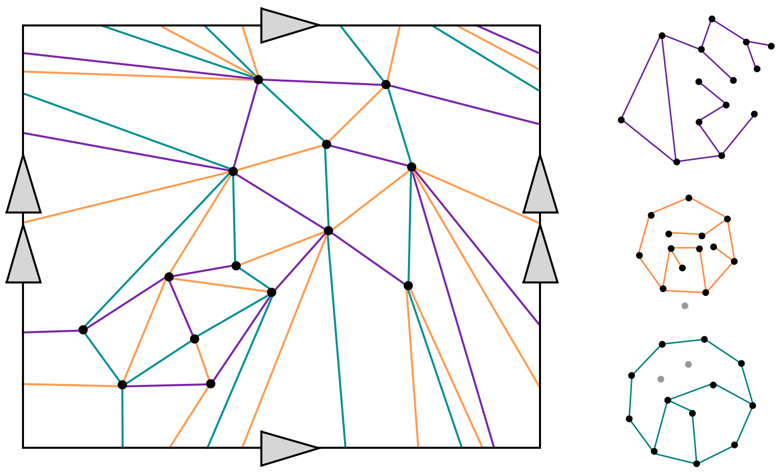

Figure 3: At left, a second Grünbaum coloring of L; at right, the CISGs of L.

Consider U_e, the portion of T formed by two faces that share an edge e. The vertices incident to e have colors c_i, c_j, so the remaining vertices of these faces are both color c_k. Therefore on U_e, the subgraph of edges of a single color forms either a single edge or a pair of edges joined by a vertex, and so is connected. An inductive argument shows that any sequence of faces that are pairwise adjacent has the same property that the subgraph in each color is connected.

Now, we claim that between any two same-color edges of T there exists such a sequence of faces. Consider the topological dual graph C; two same-color edges of T correspond to a pair of edges e_1, e_2 in C. Because C is connected, there is a path connecting a vertex incident to e_1 and a vertex incident to e_2. This path corresponds to a sequence of faces in T, where each edge in the path from C corresponds to an edge shared by two faces in T. This sequence of faces has the property desired. This shows that every CISG is connected. $\square$

Example 2.4. We will exhibit a family of vertex 3-colorable graphs on each surface of genus $g \geq 1$, and therefore a family of Grünbaum-colored triangulations with each CISG connected.

Consider first any $2m \times 2n$ grid graph on the torus. This is bipartite and thus vertex 2-colorable. Starring each face produces a vertex 3-colorable triangulation (by giving the new vertices a third color). The associated Grünbaum coloring has G_1 the original grid, and G_2, G_3 are two interlaced "diagonal" grids formed by the face-starring; each G_i is connected. See the left and center of Figure 4. In order to extend this construction to an n-holed torus, begin with an even grid graph on each of n tori and take the connected sum using disks contained in the interior of (quadrilateral) faces. The joined regions are cylindrical, with the boundaries of the two quadrilateral faces on the ends, and can be filled in with grid graphs as indicated in Figure 4(right). Then the face-starring procedure produces the desired example.

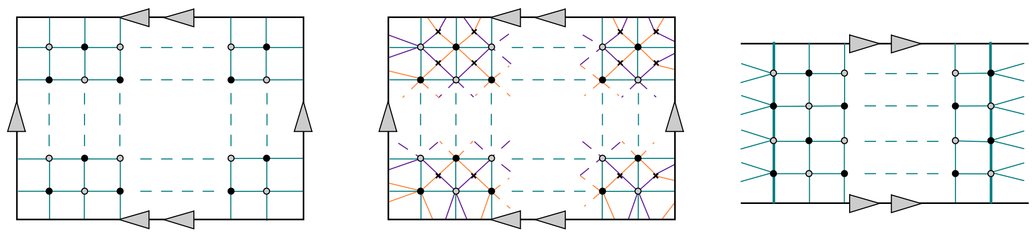

Figure 4: At left, an even grid graph on the torus; at center, a Grünbaum-colored triangulation on the torus; at right, a grid graph on a cylinder with original quadrilateral face boundaries bolded.

Now consider any $(2m+1)\times 2n$ grid graph on the Klein bottle (see Figure 5 (left)); this is bipartite and thus vertex 2-colorable. We can follow the procedure above to produce a Grünbaum-colored triangulation on a nonorientable surface of even genus with all CISGs connected. It remains to examine the projective plane, whence the procedure above gives a Grünbaum-colored triangulation on a nonorientable surface of odd genus with all CISGs connected. Figure 5 (right) shows a Grünbaum-colored face-starred bipartite grid graph on the projective plane.

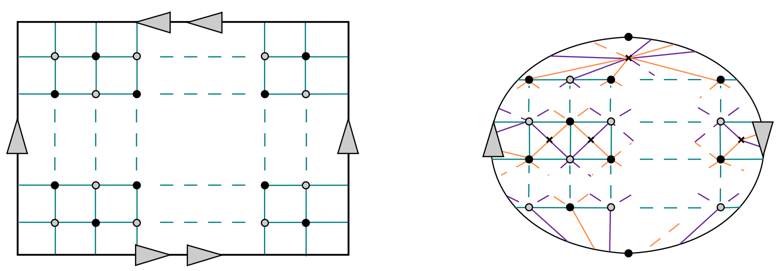

Figure 5: At left, a bipartite grid graph on the Klein bottle; at right, a Grünbaum-colored triangulation on the projective plane.

2.3 Forest CISGs

In [5] the authors note that aside from the sphere and projective plane, a counting argument shows that it is not possible for all three CISGs to be single trees. Is it ever possible on any surface for even one of the CISGs to be a single tree? Or a forest? Proposition 2.5 provides the answer.

Proposition 2.5. *If a CISG of a Grünbaum coloring of an embedded triangulation is a forest, then either*

- *the forest has two trees on a planar embedding, or*
- *the forest is a spanning tree on a projective-planar embedding.*

Proof. Suppose that we have a Grünbaum coloring of a triangulation T with v vertices embedded on S. Recall that $\chi(S)$ is the Euler characteristic of S, computed from a cellular embedding as $v - e + f = \chi(S)$. There are exactly $3(v - \chi(S))$ edges in T, and there are the same number of edges of each color (because a Grünbaum coloring is dual to a proper coloring of a cubic graph), namely $v - \chi(S)$. If $\chi(S) \leq 0$, then $v - \chi(S) > v - 1$, and there are too many edges of each color to form a tree.

This leaves the case $\chi(S) = 1$, which corresponds to a single (spanning) tree embedded on the projective plane, and the case $\chi(S) = 2$, which corresponds to two trees in a planar embedding. □

Recall that because CISGs are induced by edges, there may be isolated vertices in a CISG. Therefore, Proposition 2.5 leaves open the possibility that a CISG in a planar embedding may be a single non-spanning tree (i.e. two trees where one is the trivial tree). However, we now show that this is not possible.

Theorem 2.6. *No CISG of a Grünbaum coloring of a planar triangulation can consist of a single tree (spanning all but one vertex of T).*

Proof. Consider some Grünbaum coloring of a planar triangulation T and consider the c_k CISG. In the dual cubic graph C, the edges of any two colors c_i, c_j form a set of even cycles $\{Z_r\}$. At least one of the Z_r contains no other cycle in its interior; call this Z_0. The c_k edges of C contained in the interior of Z_0 form a matching M.

If M is empty, then all c_k edges incident to Z_0 are exterior to Z_0, so Z_0 is dual to a single vertex v in T with incident edges alternating color between c_i, c_j as in Figure 6. Each pair of adjacent edges at v form two edges of a triangle that must be completed by an edge colored c_k. The collection of these edges form a cycle in T surrounding Z_0 in C, as shown in Figure 6. Therefore at least one component of the c_k CISG is not a tree, and so the c_k CISG is not a single tree.

If M is nonempty, then it is dual to a c_k tree K in T. (If there were a c_k cycle of T inside Z_0, then there must also be additional c_i, c_j edges of C inside Z_0, contradicting the choice of Z_0.) It remains to show that K is not the entirety of the c_k CISG. If not all c_k edges incident to Z_0 are part of M (e.g. if some are also exterior to Z_0)

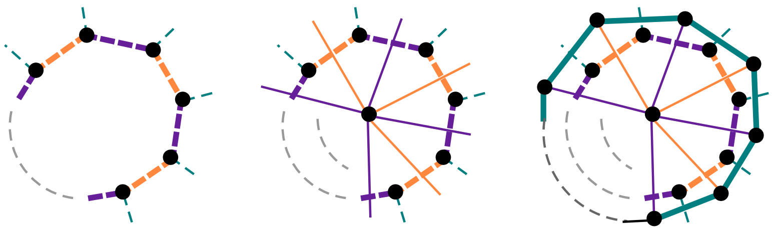

Figure 6: A 2-color cycle in C corresponds to a monochromatic cycle in T.

then certainly the c_k CISG has at least 2 components. If on the other hand all c_k edges incident to Z_0 are part of M, then because C is connected, there are no c_k edges of C exterior to Z_0. This means that Z_0 bounds a face (the exterior face) of the embedding of C, and that exterior face contains a single vertex of T. In turn, as in the case when M is empty, this implies that there is a c_k cycle of T, and thus of K, homotopic to Z_0. This contradicts that K is a tree. In neither case can the c_k CISG be a single tree. □

While there are few conditions under which a CISG can be a forest, there are no barriers to a CISG having a tree component. In this situation, independent of the embbeding surface, we understand the topology of the dual.

Proposition 2.7. *A component of a c_k CISG of T is a tree K if and only if a c_i-c_j cycle in C bounds a disk containing only the c_k edges dual to K.*

Proof. Suppose a c_i-c_j cycle in C bounds a disk containing only c_k edges. As in the proof of Theorem 2.6, these c_k edges form a matching M. If M is empty then there are no corresponding dual c_k edges in T. If M is nonempty then as in the proof of Theorem 2.6, M is dual to a c_k tree in T.

Now suppose that some component of a c_k CISG of T is a tree K. Every vertex of K corresponds to a face in C. That face is surrounded by edges; the c_k edges correspond to edges of K that separate adjacent faces in C. Consider the union of faces in C corresponding to vertices in K; it is a topological disk because there are no cycles in K. The boundary of this disk consists of edges colored c_i, c_j because every c_k edge is interior to (and a chord of) the disk. See Figure 7 for an example. □

2.4 Projective planar embeddings with tree CISGs

From Proposition 2.5, in the case of the projective plane it is possible to have a CISG that is a tree. Next, we show by example that it is possible for two of the CISGs to be trees.

Example 2.8. Consider $K_{3,3}$. It is cubic and has exactly one embedding on the projective plane. It also has a coloring in which every Kempe chain is a Hamiltonian

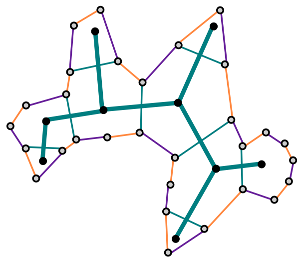

Figure 7: A c_k tree in T is surrounded by a c_i-c_j cycle in C.

circuit. The triangulation dual to this coloring of $K_{3,3}$ embedded on the projective plane has two CISGs that are trees, as shown in Figure 8. Note that this triangulation is not simple; we leave open whether a simple triangulation with this property exists.

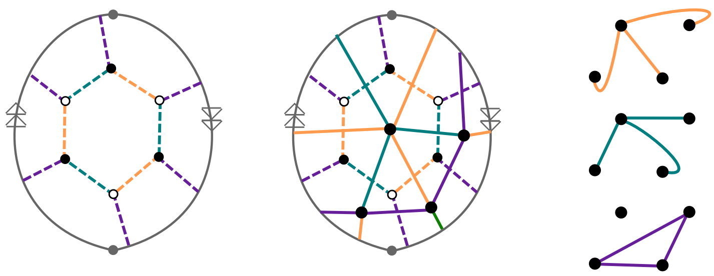

Figure 8: At left, the tri-Hamiltonian embedding of $K_{3,3}$ on the projective plane, with the dual triangulation shown at center and the three CISGs shown at right.

However, it is not possible for all three CISGs of a projective planar triangulation to be trees, as we will show in Theorem 2.11. This implies, by the arboricity theorem of Tutte [10] and Nash-Williams [8], that a decomposition of a projective planar triangulation into three edge-disjoint spanning trees cannot induce a Grünbaum coloring.

As preparation, we make some observations. Consider a Grünbaum-colored triangulation of the projective plane, T, with dual edge 3-colored embedded cubic graph C. If each CISG of T is a tree, then all three CISGs are connected, and by Theorem 5 of [5], each vertex of T has even degree and each face of C has an even number of sides. In Example 2.8, we noted that the tree CISGs of T happened to correspond to Hamilton cycles in C. It turns out that this is always true.

Theorem 2.9 ([2]). *If the c_k CISG of a Grünbaum coloring of a projective-planar triangulation T is a (spanning) tree, then C has a c_i-c_j Hamilton cycle.*

In fact, a slightly stronger statement holds. Examine Figure 8; careful checking of the embedding shows that the orange-purple and teal-purple Hamilton cycles are noncontractible. This also turns out to always be true. If the Hamilton cycle in question were contractible, then the corresponding CISG would have two components (one inside the contractible Hamilton cycle and one outside it) which contradicts the assumption that the CISG is a spanning tree. This proves the following.

Corollary 2.10. *A (spanning) tree c_k CISG of a projective-planar T corresponds to a noncontractible c_i-c_j Hamilton cycle of C.*

We are now ready to prove our main result. We have given an example where two of the CISGs of T are spanning trees. We will now show that there cannot be a Grünbaum-colored triangulation T on the projective plane P with all three CISGs trees.

Theorem 2.11. *At most two of the CISGs of a Grünbaum-colored projective-planar triangulation T can be (spanning) trees.*

Proof. Our proof proceeds by contradiction. Suppose that T is a Grünbaum-colored triangulation on the projective plane P with all three CISGs trees. From Theorem 2.9 and Corollary 2.10, we see that in such a case, each pair of colors in C forms a Hamilton cycle, and that Hamilton cycle must be homotopic to the generator of $\pi_1(P)$. (Recall that $\pi_1(P) \cong \mathbb{Z}_2$.)

Given a c_i-c_j Kempe cycle in C, contracting the c_i edges does not change the topological type (contractible or noncontractible) of the cycle. Now we have that a c_i-c_j Hamilton cycle and a c_i-c_k Hamilton cycle agree on the set of edges in the common color c_i, and this includes all c_i edges. Thus, consider the multigraph created by contracting the c_i edges of C. The c_j (resp. c_k) edges form a noncontractible Hamilton cycle. Without loss of generality, we may assume the c_j cycle is drawn as in Figure 9 by "straightening" it.

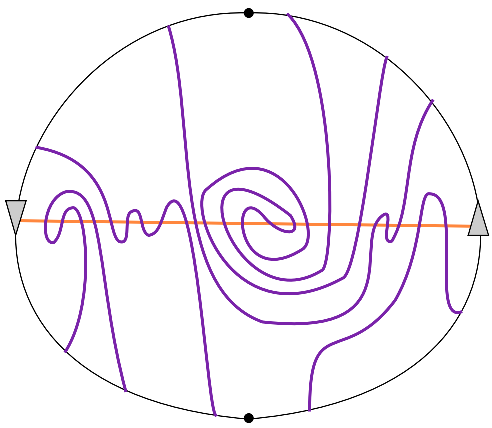

Figure 9: A cubic graph on the projective plane after contracting c_i edges. Here c_j is shown in orange.

Each intersection of the c_k cycle with the c_j cycle corresponds to a contraction of a c_i edge. Given noncontractible c_j and c_k cycles, there are three possible reversals of

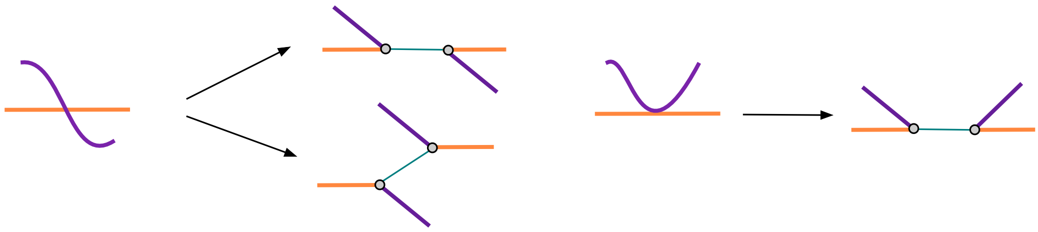

Figure 10: The possible reversals of c_j-c_k intersections to vertices of C.

a contraction of a c_i edge (henceforth called *reversals*): these are shown in Figure 10.

Notice that any particular pair of c_j and c_k cycles corresponds to an entire family of cubic graphs. That is, each set of choices of reversals at the intersections corresponds to a cubic graph and most of these will be distinct graphs. Our strategy will be to show that for any member of the family, the c_j, c_k edges cannot form a noncontractible c_j-c_k Hamiltonian cycle.

Consider a closed curve contained in the complement of the union of the c_j, c_k cycles and parallel to twice the homotopy generator equivalent to the c_j cycle. This curve bounds a topological disk; removing this disk leaves a Möbius band containing the union of the c_j, c_k cycles (see Figure 11). We call this a *ribbon graph neighborhood* of the union of the c_j, c_k cycles. The complement in the projective plane is a disc D,

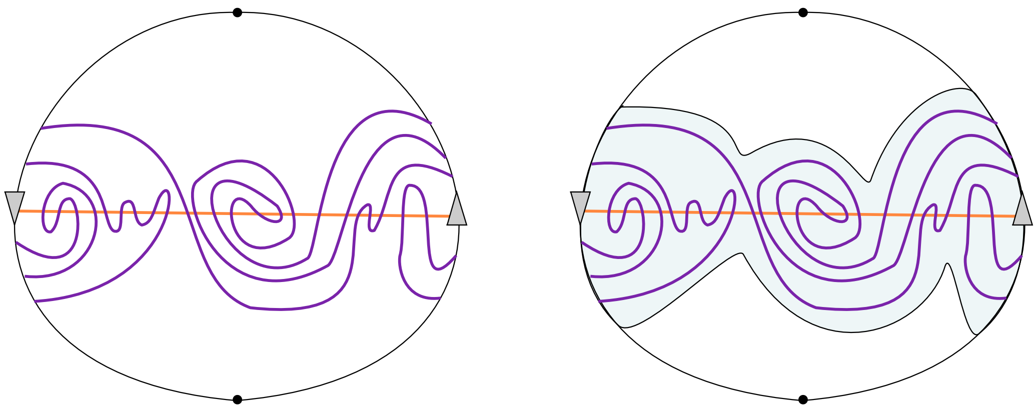

Figure 11: A redrawing of the c_j, c_k cycles from Figure 9 (left) and the ribbon graph neighborhood of the c_j, c_k cycles (right).

and the ribbon graph neighborhood is a Möbius band.

Each intersection of the c_j, c_k cycles may be incident to two boundary segments, one boundary segment, or no boundary segments of the ribbon graph neighborhood, as illustrated in Figure 12. We refer to these intersection types as 2-b, 1-b, and 0-b respectively, and now examine the effect of reversing each of them in turn. We look first at the effect of reversal of a 2-b intersection on the ribbon graph neighborhood. The possibilities are demonstrated in Figure 13.

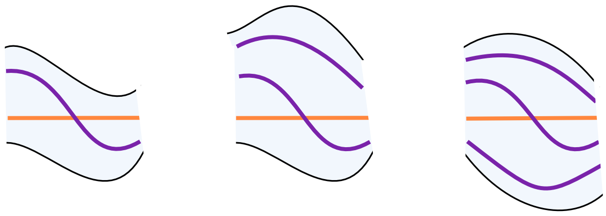

Figure 12: Examples of the three different ways in which an intersection between c_j (orange) and c_k (purple) cycles may be incident to the ribbon graph neighborhood boundary. At left, a 2-b intersection; at center, a 1-b intersection; at right, a 0-b intersection.

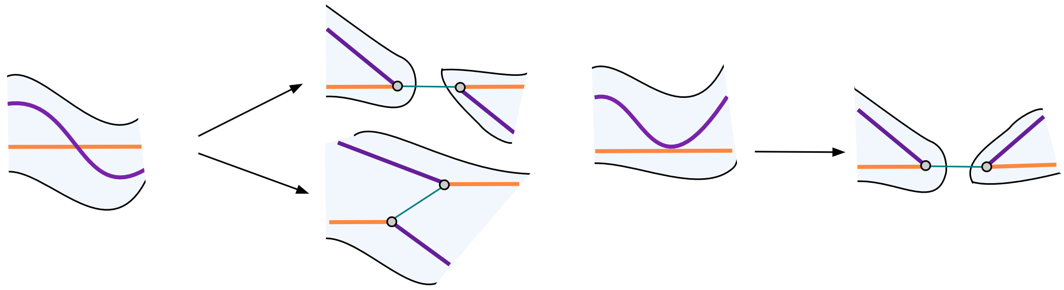

Figure 13: Ribbon graph neighborhoods of the possible reversals of certain c_j-c_k intersections to vertices of C. At left, the c_j and c_k cycles cross; at right, they are tangent.

Notice that in one of the three cases, the ribbon graph boundary is maintained, whereas in the other two cases the c_j-c_k boundary is changed so that instead of a Möbius band, the c_j, c_k ribbon graph neighborhood is a disk. In these latter cases, the c_j-c_k Hamilton cycle must be contractible, which is a contradiction.

If we reverse a 1-b intersection, then the c_j, c_k ribbon graph neighborhood remains a Möbius band; however, some of the remaining intersections may have additional incidences with the boundary. Figure 14 gives an example where this is the case. Of course, if we reverse a 0-b intersection then there is no effect on the c_j-c_k boundary.

We will now show that in any configuration of intersections, the c_j-c_k Hamilton cycle must be contractible. Recall that a bifacial edge is incident to two distinct faces of an embedding, whereas a monofacial edge is incident to only one face (twice). Our argument is as follows: reverse the intersections in any order; either the reversal of some intersection will change the ribbon graph boundary to that of a disk, or the reversal of no intersection changes the ribbon graph boundary, in which case the c_j, c_k edges are all bifacial. Here are the details.

Case 1: All intersections are 2-b intersections.
If any of the reversals changes the boundary of the c_j, c_k ribbon graph to that of a disk, then we have a contradiction; the c_j-c_k Hamilton cycle must be contractible. Otherwise, none of the reversals changes the boundary of the c_j, c_k ribbon graph.

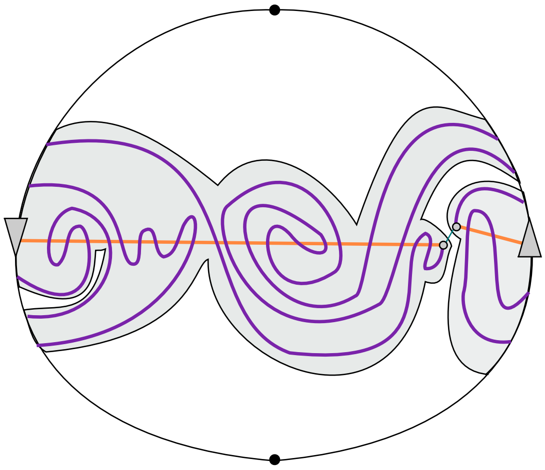

Figure 14: Reversal of a 1-b intersection in Figure 11 (third from right) changes two 1-b intersections to 2-b intersections and changes three 0-b intersections to 1-b intersections. The intersections that change types are the third and fourth from the left, as well as the fourth, fifth and sixth from the right.

In this case, consider the c_j-c_k Hamilton cycle, and retract the ribbon graph so that the Hamilton cycle is on the boundary of the Möbius band. This makes it evident that each edge is bifacial; one side of each edge is incident to the disc complement of the Möbius band in the projective plane, and the other side of each edge is incident to the interior of the Möbius band. This is a contradiction because a noncontractible Hamilton cycle must be homotopic to the generator of $\pi_1(P)$, which has only monofacial edges. Here the c_j-c_k Hamilton cycle is instead homotopic to twice the generator of $\pi_1(P)$, and thus contractible.

Case 2: All intersections are either 2-b intersections or 0-b intersections.
If any 2-b intersection reversal changes the boundary of the c_j, c_k ribbon graph to that of a disk, then we have a contradiction by Case 1. Thus, suppose no 2-b intersection reversal changes the boundary of the c_j, c_k ribbon graph. After all reversals of intersections, the boundary of the c_j, c_k ribbon graph is unchanged (because no reversal of a 0-b intersection can alter that boundary). Therefore we have bifacial c_j, c_k edges by the same argument as in Case 1, and thus the same contradiction.

Case 3: There is at least one 1-b intersection.
If any reversal of a 2-b intersection changes the boundary of the c_j, c_k ribbon graph to that of a disk, we have a contradiction by the Case 1 argument.

If no reversal of a 2-b intersection changes the boundary of the c_j, c_k ribbon graph to that of a disk, we have two subcases.

Subcase 3A: No reversal of any 1-b intersection affects the boundary incidence of any other intersection.
After all reversals, the boundary of the c_j, c_k ribbon graph is unchanged and therefore by the Case 1 argument we have bifacial c_j, c_k edges and a contradiction.

Subcase 3B: The reversal of some 1-b intersection affects the boundary incidence of some other intersection.
Note that after reversing this intersection, the types of some remaining intersections may have changed. Therefore we can consider whether the current intersection set falls in Case 1, Case 2, or Subcase 3A. In these three situations, we have a contradiction. Otherwise we find ourselves again in Subcase 3B, reverse a 1-b intersection that affects the boundary incidence of some other intersection, and re-consider the intersection set. In every case, we reach a contradiction by one of Case 1, Case 2, or Subcase 3A.

There are a finite number of intersections. Reversal of the last intersection cannot affect the boundary incidence of other intersections, so this reversal must be in Case 1, Case 2, or Subcase 3A. Thus, this algorithm only terminates in a contradiction to the noncontractability of the c_j-c_k Hamilton cycle. Therefore there exists at least one pair i, j such that a c_i-c_j Hamiltonian cycle must be contractible, and in turn we cannot have all three CISGs trees. □

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