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Gallai-Ramsey numbers of C_{10} and C_{12}

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Abstract

A Gallai coloring is a coloring of the edges of a complete graph without rainbow triangles, and a Gallai k-coloring is a Gallai coloring that uses at most k colors. Given an integer $k \geq 1$ and graphs $H_1, \ldots, H_k$, the Gallai-Ramsey number $GR(H_1, \ldots, H_k)$ is the least integer n such that every Gallai k-coloring of the complete graph K_n contains a monochromatic copy of H_i in color i for some $i \in \{1, \ldots, k\}$. When $H = H_1 = \cdots = H_k$, we simply write $GR_k(H)$. We continue to study Gallai-Ramsey numbers of even cycles and paths. For all $n \geq 3$ and $k \geq 1$, let $G_i = P_{2i+3}$ be a path on $2i+3$ vertices for all $i \in \{0, 1, \ldots, n-2\}$ and $G_{n-1} \in \{C_{2n}, P_{2n+1}\}$. Let $i_j \in \{0, 1, \ldots, n-1\}$ for all $j \in \{1, \ldots, k\}$ with $i_1 \geq i_2 \geq \cdots \geq i_k$. Song recently conjectured that $GR(G_{i_1}, \ldots, G_{i_k}) = |G_{i_1}| + \sum_{j=2}^{k} i_j$. This conjecture has been verified to be true for $n \in \{3, 4\}$ and all $k \geq 1$. In this paper, we prove that the aforementioned conjecture holds for $n \in \{5, 6\}$ and all $k \geq 1$. Our result implies that for all $k \geq 1$, $GR_k(C_{2n}) = GR_k(P_{2n}) = (n-1)k + n + 1$ for $n \in \{5, 6\}$ and $GR_k(P_{2n+1}) = (n-1)k + n + 2$ for $1 \leq n \leq 6$.

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1 Introduction

In this paper we consider graphs that are finite, simple and undirected. Given a graph G and a set $A \subseteq V(G)$, we use $|G|$ to denote the number of vertices of G, and $G[A]$ to denote the subgraph of G obtained from G by deleting all vertices in $V(G)\backslash A$. A graph H is an *induced subgraph* of G if $H = G[A]$ for some $A \subseteq V(G)$. We use P_n, C_n and K_n to denote the path, cycle and complete graph on n vertices, respectively. For any positive integer k, we write $[k]$ for the set $\{1, \ldots, k\}$.

Given an integer $k \geq 1$ and graphs $H_1, \ldots, H_k$, the classical Ramsey number $R(H_1, \ldots, H_k)$ is the least integer n such that every k-coloring of the edges of K_n contains a monochromatic copy of H_i in color i for some $i \in [k]$. Ramsey numbers are notoriously difficult to compute in general. In this paper, we study Ramsey numbers of graphs in Gallai colorings, where a *Gallai coloring* is a coloring of the edges of a complete graph without rainbow triangles (that is, a triangle with all its edges colored differently). Gallai colorings naturally arise in several areas including: information theory [17]; the study of partially ordered sets, as in Gallai's original paper [12] (his result was restated in [15] in the terminology of graphs); and the study of perfect graphs [5]. There are now a variety of papers which consider Ramsey-type problems in Gallai colorings (see, e.g., [2, 3, 4, 6, 10, 13, 14, 16, 21, 24]). These works mainly focus on finding various monochromatic subgraphs in such colorings. More information on this topic can be found in [9, 11].

A *Gallai k-coloring* is a Gallai coloring that uses at most k colors. Given an integer $k \geq 1$ and graphs $H_1, \ldots, H_k$, the *Gallai-Ramsey number* $GR(H_1, \ldots, H_k)$ is the least integer n such that every Gallai k-coloring of K_n contains a monochromatic copy of H_i in color i for some $i \in [k]$. When $H = H_1 = \cdots = H_k$, we simply write $GR_k(H)$ and $R_k(H)$. Clearly, $GR_k(H) \leq R_k(H)$ for all $k \geq 1$ and $GR(H_1, H_2) = R(H_1, H_2)$. In 2010, Gyárfás, Sárközy, Sebő and Selkow [14] proved the general behavior of $GR_k(H)$.

Theorem 1.1 ([14]) *Let H be a fixed graph with no isolated vertices and let $k \geq 1$ be an integer. Then $GR_k(H)$ is exponential in k if H is not bipartite, linear in k if H is bipartite but not a star, and constant (does not depend on k) when H is a star.*

It turns out that for some graphs H (e.g., when $H = C_3$), $GR_k(H)$ behaves nicely, while the order of magnitude of $R_k(H)$ seems hopelessly difficult to determine. It is worth noting that finding exact values of $GR_k(H)$ is far from trivial, even when $|H|$ is small. We will utilize the following important structural result of Gallai [12] on Gallai colorings of complete graphs.

Theorem 1.2 ([12]) *For any Gallai coloring c of a complete graph G with $|G| \geq 2$, $V(G)$ can be partitioned into nonempty sets $V_1, \ldots, V_p$ with $p \geq 2$ so that at most two colors are used on the edges in $E(G)\backslash(E(G[V_1]) \cup \cdots \cup E(G[V_p]))$ and only one color is used on the edges between any fixed pair (V_i, V_j) under c.*

The partition given in Theorem 1.2 is a *Gallai-partition* of the complete graph

G under c. Given a Gallai-partition $V_1, \ldots, V_p$ of the complete graph G under c, let $v_i \in V_i$ for all $i \in [p]$ and let $\mathcal{R} := G[\{v_1, \ldots, v_p\}]$. Then $\mathcal{R}$ is the *reduced graph* of G corresponding to the given Gallai-partition under c. Clearly, $\mathcal{R}$ is isomorphic to K_p. By Theorem 1.2, all edges in $\mathcal{R}$ are colored by at most two colors under c. One can see that any monochromatic H in $\mathcal{R}$ under c will result in a monochromatic H in G under c. It is not surprising that Gallai-Ramsey numbers $GR_k(H)$ are closely related to the classical Ramsey numbers $R_2(H)$. Recently, Fox, Grinshpun and Pach posed the following conjecture on $GR_k(H)$ when H is a complete graph.

Conjecture 1.3 ([9]) *For all integers $k \geq 1$ and $t \geq 3$,*

$$GR_k(K_t) = \begin{cases} (R_2(K_t) - 1)^{k/2} + 1 & \text{if } k \text{ is even} \\ (t-1)(R_2(K_t) - 1)^{(k-1)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

The first case of Conjecture 1.3 follows from a result of Chung and Graham [6] from 1983. A simpler proof of this case can be found in [14]. The case when $t = 4$ was recently settled in [18]. Conjecture 1.3 remains open for all $t \geq 5$. The next open case, when $t = 5$, involves $R_2(K_5)$. Angeltveit and McKay [1] recently proved that $R_2(K_5) \leq 48$. It is widely believed that $R_2(K_5) = 43$ (see [1]). It is worth noting that Schiermeyer [20] recently observed that if $R_2(K_5) = 43$, then Conjecture 1.3 fails for K_5 when $k = 3$. More recently, Gallai-Ramsey numbers of odd cycles on at most 15 vertices have been completely settled by Fujita and Magnant [10] for C_5, Bruce and Song [4] for C_7, Bosse and Song [2] for C_9 and C_{11}, and Bosse, Song and Zhang [3] for C_{13} and C_{15}. Very recently, the exact values of $GR_k(C_{2n+1})$ for $n \geq 8$ has been solved by Zhang, Song and Chen [23]. We summarize these results below.

Theorem 1.4 ([2, 3, 4, 23]) *For all $n \geq 3$ and $k \geq 1$, $GR_k(C_{2n+1}) = n \cdot 2^k + 1$.*

In this paper, we continue to study Gallai-Ramsey numbers of even cycles and paths. For all $n \geq 3$ and $k \geq 1$, let $G_{n-1} \in \{C_{2n}, P_{2n+1}\}$, $G_i := P_{2i+3}$ for all $i \in \{0, 1, \ldots, n-2\}$, and $i_j \in \{0, 1, \ldots, n-1\}$ for all $j \in [k]$. We want to determine the exact values of $GR(G_{i_1}, \ldots, G_{i_k})$. By reordering colors if necessary, we assume that $i_1 \geq \cdots \geq i_k$. Song and Zhang [22] recently proved that

Proposition 1.5 ([22]) *For all $n \geq 3$ and $k \geq 1$,*

$$GR(G_{i_1}, \ldots, G_{i_k}) \geq |G_{i_1}| + \sum_{j=2}^{k} i_j.$$

In the same paper, Song [22] further made the following conjecture.

Conjecture 1.6 ([22]) *For all $n \geq 3$ and $k \geq 1$,*

$$GR(G_{i_1}, \ldots, G_{i_k}) = |G_{i_1}| + \sum_{j=2}^{k} i_j.$$

To completely solve Conjecture 1.6, one only needs to consider the case $G_{n-1} = C_{2n}$.

Proposition 1.7 ([22]) *For all $n \geq 3$ and $k \geq 1$, if Conjecture 1.6 holds for $G_{n-1} = C_{2n}$, then it also holds for $G_{n-1} = P_{2n+1}$.*

Let M_n denote a matching of size n on $2n$ vertices. As observed in [22], the truth of Conjecture 1.6 implies that $GR_k(C_{2n}) = GR_k(P_{2n}) = GR_k(M_n) = (n-1)k+n+1$ for all $n \geq 3$ and $k \geq 1$, and $GR_k(P_{2n+1}) = (n-1)k + n + 2$ for all $n \geq 1$ and $k \geq 1$. It is worth noting that Dzido, Nowik and Szuca [7] proved that $R_3(C_{2n}) \geq 4n$ for all $n \geq 3$. The truth of Conjecture 1.6 implies that $GR_3(C_{2n}) = 4n - 2 < R_3(C_{2n})$ for all $n \geq 3$. Conjecture 1.6 has recently been verified to be true for $n \in \{3, 4\}$ and all $k \geq 1$.

Theorem 1.8 ([22]) *For $n \in \{3, 4\}$ and all $k \geq 1$, let $G_i = P_{2i+3}$ for all $i \in \{0, 1, \ldots, n-2\}$, $G_{n-1} = C_{2n}$, and $i_j \in \{0, 1, \ldots, n-1\}$ for all $j \in [k]$ with $i_1 \geq \cdots \geq i_k$. Then*

$$GR(G_{i_1}, \ldots, G_{i_k}) = |G_{i_1}| + \sum_{j=2}^{k} i_j.$$

In this paper, we continue to establish more evidence for Conjecture 1.6. We prove that Conjecture 1.6 holds for $n \in \{5, 6\}$ and all $k \geq 1$.

Theorem 1.9 *For $n \in \{5, 6\}$ and all $k \geq 1$, let $G_i = P_{2i+3}$ for all $i \in \{0, 1, \ldots, n-2\}$, $G_{n-1} = C_{2n}$, and $i_j \in \{0, 1, \ldots, n-1\}$ for all $j \in [k]$ with $i_1 \geq \cdots \geq i_k$. Then*

$$GR(G_{i_1}, \ldots, G_{i_k}) = |G_{i_1}| + \sum_{j=2}^{k} i_j.$$

We prove Theorem 1.9 in Section 2. Applying Theorem 1.9 and Proposition 1.7, we obtain the following.

Corollary 1.10 *Let $G_i = P_{2i+3}$ for all $i \in \{0, 1, 2, 3, 4, 5\}$. For every integer $k \geq 1$, let $i_j \in \{0, 1, 2, 3, 4, 5\}$ for all $j \in [k]$ with $i_1 \geq \cdots \geq i_k$. Then*

$$GR(G_{i_1}, \ldots, G_{i_k}) = |G_{i_1}| + \sum_{j=2}^{k} i_j.$$

Corollary 1.11 *For all $k \geq 1$,*

(a) $GR_k(P_{2n+1}) = (n-1)k + n + 2$ *for all* $n \in [6]$.

(b) $GR_k(C_{2n}) = GR_k(P_{2n}) = (n-1)k + n + 1$ *for* $n \in \{5, 6\}$.

Finally, we shall make use of the following results on 2-colored Ramsey numbers of cycles and paths in the proof of Theorem 1.9.

Theorem 1.12 ([19]) *For all $n \geq 3$, $R_2(C_{2n}) = 3n - 1$.*

Theorem 1.13 ([8]) *For all integers n, m satisfying $2n \geq m \geq 3$, $R(P_m, C_{2n}) = 2n + \lfloor \frac{m}{2} \rfloor - 1$.*

2 Proof of Theorem 1.9

We are ready to prove Theorem 1.9. Let $n \in \{5, 6\}$. By Proposition 1.5, it suffices to show that $GR(G_{i_1}, \ldots, G_{i_k}) \leq |G_{i_1}| + \sum_{j=2}^{k} i_j$.

By Theorem 1.8 and Proposition 1.7, we may assume that $i_1 = n-1$. Then $|G_{i_1}| = 2n$. By Theorem 1.12 and Theorem 1.13, we have $GR(G_{i_1}, G_{i_2}) = R(G_{i_1}, G_{i_2}) = 2n + i_2$. So we may assume $k \geq 3$. Let $N := |G_{i_1}| + \sum_{j=2}^{k} i_j$. Then $N \geq 2n$. Let G be a complete graph on N vertices and let $c : E(G) \to [k]$ be any Gallai coloring of G using at least three colors. We next show that G contains a monochromatic copy of G_{i_j} in color j for some $j \in [k]$. Suppose G contains no monochromatic copy of G_{i_j} in color j for any $j \in [k]$ under c. Such a Gallai k-coloring c is called a *bad coloring*. Among all complete graphs on N vertices with a bad coloring, we choose G with N minimum, taken over all $n - 1 \geq i_1 \geq \cdots \geq i_k \geq 0$.

By Theorem 1.2, we may consider a Gallai-partition of G with parts $A_1, \ldots, A_p$, where $p \geq 2$. We may assume that $|A_1| \geq \cdots \geq |A_p| \geq 1$. Let $\mathcal{R}$ be the reduced graph of G with vertices $a_1, \ldots, a_p$, where $a_i \in A_i$ for all $i \in [p]$. By Theorem 1.2, assume that the edges of $\mathcal{R}$ are colored either red or blue. Since c uses at least three colors, we see that $\mathcal{R} \neq G$ and so $|A_1| \geq 2$. By abusing the notation, we use i_b to denote i_j when the color j is blue. Similarly, we use i_r (respectively, i_g) to denote i_j when the color j is red (respectively, green). Let

$$A_b := \{a_i \in \{a_2, \ldots, a_p\} \mid a_i a_1 \text{ is colored blue in } \mathcal{R}\},$$
$$A_r := \{a_j \in \{a_2, \ldots, a_p\} \mid a_j a_1 \text{ is colored red in } \mathcal{R}\}.$$

Then $|A_b| + |A_r| = p - 1$. Let $B := \bigcup_{a_i \in A_b} A_i$ and $R := \bigcup_{a_j \in A_r} A_j$. Then $|A_1| + |R| + |B| = N$ and $\max\{|B|, |R|\} \neq 0$ because $p \geq 2$. Thus G contains a blue P_3 between B and A_1, or a red P_3 between R and A_1, and so $\max\{i_b, i_r\} \geq 1$. We next prove several claims.

Claim 1. Let $r \in [k]$ and let $s_1, \ldots, s_r$ be nonnegative integers with $s_1 + \cdots + s_r \geq 1$. If $i_{j_1} \geq s_1, \ldots, i_{j_r} \geq s_r$ for colors $j_1, \ldots, j_r \in [k]$, then for any $S \subseteq V(G)$ with $|S| \geq |G| - (s_1 + \cdots + s_r)$, $G[S]$ must contain a monochromatic copy of $G_{i^*_{j_q}}$ in color j_q for some $j_q \in \{j_1, \ldots, j_r\}$, where $i^*_{j_q} = i_{j_q} - s_q$.

Proof. Let $i^*_{j_1} := i_{j_1} - s_1, \ldots, i^*_{j_r} := i_{j_r} - s_r$, and $i^*_j := i_j$ for all $j \in [k] \backslash \{j_1, \ldots, j_r\}$. Let $i^*_\ell := \max\{i^*_j \mid j \in [k]\}$. Then $i^*_\ell \leq i_1$. Let $N^* := |G_{i^*_\ell}| + [(\sum_{j=1}^{k} i^*_j) - i^*_\ell]$. Then $N^* \geq 3$ and $N^* \leq N - (s_1 + \cdots + s_r) < N$ because $s_1 + \cdots + s_r \geq 1$. Since $|S| \geq N - (s_1 + \cdots + s_r) \geq N^*$ and $G[S]$ does not have a monochromatic copy of G_{i_j}

in color j for all $j \in [k]\backslash\{j_1, \ldots, j_r\}$ under c, by minimality of N, $G[S]$ must contain a monochromatic copy of $G_{i^*_{j_q}}$ in color j_q for some $j_q \in \{j_1, \ldots, j_r\}$. ■

Claim 2. $|A_1| \le n-1$, and so G does not contain a monochromatic copy of a graph on $|A_1|+1 \le n$ vertices in color m, where $m \in [k]$ is a color that is neither red nor blue.

Proof. Suppose $|A_1| \ge n$. We first claim that $i_b \ge |B|$ and $i_r \ge |R|$. Suppose $i_b \le |B|-1$ or $i_r \le |R|-1$. Then we obtain a blue G_{i_b} using the edges between B and A_1, or a red G_{i_r} using the edges between R and A_1, a contradiction. Thus $i_b \ge |B|$ and $i_r \ge |R|$, as claimed. Let $i^*_b := i_b - |B|$ and $i^*_r := i_r - |R|$. Since $|A_1| = N - |B| - |R|$, by Claim 1 applied to $i_b \ge |B|$, $i_r \ge |R|$ and A_1, $G[A_1]$ must have a blue $G_{i^*_b}$ or a red $G_{i^*_r}$, say the latter. Then $i_r > i^*_r$. Thus $|R| > 0$ and $G_{i^*_r}$ is a red path on $2i^*_r + 3$ vertices. Note that

$$\begin{aligned}
|A_1| &= |G_{i_1}| + \sum_{j=2}^{k} i_j - |B| - |R| \\
&\ge \begin{cases} |G_{i_r}| + i_b - |B| - |R| & \text{if } \ i_r \ge i_b \\ |G_{i_b}| + i_r - |B| - |R| & \text{if } \ i_r < i_b, \end{cases} \\
&\ge \begin{cases} |G_{i_r}| + i^*_b - |R| & \text{if } \ i_r \ge i_b \\ 2i_b + 2 + i_r - |B| - |R| \ge i^*_b + (2i_r+3) - |R| & \text{if } \ i_r < i_b, \end{cases} \\
&\ge |G_{i_r}| - |R|.
\end{aligned}$$

Then

$$\begin{aligned}
|A_1| - |G_{i^*_r}| &\ge |G_{i_r}| - |G_{i^*_r}| - |R| \\
&= \begin{cases} (3+2i_r) - (3+2i^*_r) - |R| = |R| & \text{if } i_r \le n-2 \\ (2+2i_r) - (3+2i^*_r) - |R| = |R| - 1 & \text{if } i_r = n-1. \end{cases}
\end{aligned}$$

But then $G[A_1 \cup R]$ contains a red G_{i_r} using the edges of the $G_{i^*_r}$ and the edges between $A_1 \backslash V(G_{i^*_r})$ and R, a contradiction. This proves that $|A_1| \le n-1$. Next, let $m \in [k]$ be any color that is neither red nor blue. Suppose G contains a monochromatic copy of a graph, say J, on $|A_1|+1$ vertices in color m. Then $V(J) \subseteq A_\ell$ for some $\ell \in [p]$. But then $|A_\ell| \ge |A_1| + 1$, contrary to $|A_1| \ge |A_\ell|$. ■

For two disjoint sets $U, W \subseteq V(G)$, we say U is *blue-complete* (respectively, *red-complete*) to W if all the edges between U and W are colored blue (respectively, red) under c. For convenience, we say u is *blue-complete* (respectively, *red-complete*) to W when $U = \{u\}$.

Claim 3. $\min\{|B|, |R|\} \ge 1$, $p \ge 3$, and B is neither red- nor blue-complete to R under c.

Proof. Suppose $B = \emptyset$ or $R = \emptyset$. By symmetry, we may assume that $R = \emptyset$. Then $B \ne \emptyset$ and so $i_b \ge 1$. By Claim 2, $|A_1| \le n-1 \le 5$ because $n \in \{5, 6\}$. Then

$|A_1| \le i_b + 4$. If $i_b \le |A_1| - 1$, then $i_b \le n-2$ by Claim 2. But then we obtain a blue G_{i_b} using the edges between B and A_1. Thus $i_b \ge |A_1|$. Let $i_b^* = i_b - |A_1|$. By Claim 1 applied to $i_b \ge |A_1|$ and B, $G[B]$ must have a blue $G_{i_b^*}$. Since $|B| \ge n+1+i_b^*$, we see that G contains a blue G_{i_b}, a contradiction. Hence $R \neq \emptyset$, and similarly $B \neq \emptyset$, and so $p \ge 3$ for any Gallai-partition of G. It follows that B is neither red- nor blue-complete to R, otherwise $\{B \cup A_1, R\}$ or $\{B, R \cup A_1\}$ yields a Gallai-partition of G with only two parts. ∎

Claim 4. Let $m \in [k]$ be a color that is neither red nor blue. Then $i_m \le n-4$. In particular, if $i_m \ge 1$, then G contains a monochromatic copy of P_{2i_m+1} in color m under c.

Proof. Note that $i_m \le n-4$ is is trivially true when $i_m = 0$ because $n \in \{5,6\}$ and $n-4 \ge 1$. Suppose $i_m \ge 1$. By Claim 2, $|A_1| \le n-1$ and G contains no monochromatic copy of $P_{|A_1|+1}$ in color m under c. Let $i_m^* := i_m - 1$. By Claim 1 applied to $i_m \ge 1$ and $V(G)$, G must have a monochromatic copy of $G_{i_m^*}$ in color m under c. Since $n \in \{5,6\}$, $|A_1| \le n-1$ and G contains no monochromatic copy of $P_{|A_1|+1}$ in color m, we see that $i_m^* \le n-5$. Thus $i_m \le n-4$ and G contains a monochromatic copy of P_{2i_m+1} in color m under c if $i_m \ge 1$. ∎

By Claim 3 and the fact that $|A_1| \ge 2$, G has a red P_3 and a blue P_3. Thus $\min\{i_b, i_r\} \ge 1$. By Claim 4, $\max\{i_b, i_r\} = i_1 = n-1$. Then $|G| = |G_{i_1}| + \sum_{j=2}^k i_j \ge 2n+1$. For the remainder of the proof of Theorem 1.9, we choose $p \ge 3$ to be as large as possible.

Claim 5. $\min\{|B|, |R|\} \le n-1$ if $|A_1| \ge n-3$.

Proof. Suppose $|A_1| \ge n-3$ but $\min\{|B|, |R|\} \ge n$. By symmetry, we may assume that $|B| \ge |R| \ge n$. Let $B := \{x_1, x_2, \dots, x_{|B|}\}$ and $R := \{y_1, y_2, \dots, y_{|R|}\}$. Let $H := (B, R)$ be the complete bipartite graph obtained from $G[B \cup R]$ by deleting all the edges with both ends in B or in R. Then H has no blue P_7 with both ends in B and no red P_7 with both ends in R, else we obtain a blue C_{2n} or a red C_{2n} because $|A_1| \ge n-3$. We next show that H has no red $K_{3,3}$.

Suppose H has a red $K_{3,3}$. We may assume that $H[\{x_1, x_2, x_3, y_1, y_2, y_3\}]$ is a red $K_{3,3}$ under c. Since H has no red P_7 with both ends in R, $\{y_4, \dots, y_{|R|}\}$ must be blue-complete to $\{x_1, x_2, x_3\}$. Thus $H[\{x_1, x_2, x_3, y_4, y_5\}]$ has a blue P_5 with both ends in $\{x_1, x_2, x_3\}$ and $H[\{x_1, x_2, x_3, y_1, y_2, y_3\}]$ has a red P_5 with both ends in $\{y_1, y_2, y_3\}$. If $|A_1| \ge n-2$ or $\min\{i_b, i_r\} \le n-2$, then we obtain a blue G_{i_b} or a red G_{i_r}, a contradiction. It follows that $|A_1| = n-3$ and $i_b = i_r = n-1$. Then $|G| = |G_{i_1}| + \sum_{j=2}^k i_j \ge 2n + (n-1) = 3n-1$. Thus $|B \cup R| = |G| - |A_1| \ge 2n+2$. If $|R| \ge 6$, then $\{y_4, y_5, y_6\}$ must be red-complete to $\{x_4, x_5, x_6\}$, else H has a blue P_7 with both ends in B. But then we obtain a red C_{2n} in G. Thus $|R| = 5$, $n = 5$, and so $|B| \ge 7$. Let $A_1 = \{a_1, a_1^*\}$. For each $j \in \{4,5,6,7\}$ and every $W \subseteq \{x_1, x_2, x_3\}$ with $|W| = 2$, no x_j is red-complete to W under c, else, say, x_4 is red-complete to $\{x_1, x_2\}$, then we obtain a red C_{10} with vertices $a_1, y_1, x_1, x_4, x_2, y_2, x_3, y_3, a_1^*, y_4$

in order, a contradiction. We may assume that x_4x_1, x_5x_2 are colored blue. But then we obtain a blue C_{10} with vertices $a_1, x_4, x_1, y_4, x_3, y_5, x_2, x_5, a_1^*, x_6$ in order, a contradiction. This proves that H has no red $K_{3,3}$.

Let $X := \{x_1, x_2, \dots, x_5\}$ and $Y := \{y_1, y_2, \dots, y_5\}$. Let H_b and H_r be the spanning subgraphs of $H[X \cup Y]$ induced by all the blue edges and red edges of $H[X \cup Y]$ under c, respectively. By the Pigeonhole Principle, there exist at least three vertices, say x_1, x_2, x_3, in X such that either $d_{H_b}(x_i) \geq 3$ for all $i \in [3]$ or $d_{H_r}(x_i) \geq 3$ for all $i \in [3]$. Suppose $d_{H_r}(x_i) \geq 3$ for all $i \in [3]$. We may assume that x_1 is red-complete to $\{y_1, y_2, y_3\}$. Since $|Y| = 5$ and H has no red P_7 with both ends in R, we see that $N_{H_r}(x_1) = N_{H_r}(x_2) = N_{H_r}(x_3) = \{y_1, y_2, y_3\}$. But then $H[\{x_1, x_2, x_3, y_1, y_2, y_3\}]$ is a red $K_{3,3}$, contrary to H has no red $K_{3,3}$. Thus $d_{H_b}(x_i) \geq 3$ for all $i \in [3]$. Since $|Y| = 5$, we see that any two of x_1, x_2, x_3 have a common neighbor in H_b. Furthermore, two of x_1, x_2, x_3, say x_1, x_2, have at least two common neighbors in H_b. It can be easily checked that H has a blue P_5 with ends in $\{x_1, x_2, x_3\}$, and there exist three vertices, say y_1, y_2, y_3, in Y such that y_ix_i is blue for all $i \in [3]$ and $\{x_4, \dots, x_{|B|}\}$ is red-complete to $\{y_1, y_2, y_3\}$. Then H has a blue P_5 with both ends in $\{x_1, x_2, x_3\}$ and a red P_5 with both ends in $\{y_1, y_2, y_3\}$. If $|A_1| \geq n-2$ or $\min\{i_b, i_r\} \leq n-2$, then we obtain a blue G_{i_b} or a red G_{i_r}, a contradiction. It follows that $|A_1| = n-3$ and $i_b = i_r = n-1$. Thus $|B \cup R| \geq 1+n+i_b+i_r-|A_1| = 2n+2$. Then $|B| \geq n+1$ and so $H[\{x_4, x_5, x_6, y_1, y_2, y_3\}]$ is a red $K_{3,3}$, contrary to the fact that H has no red $K_{3,3}$. ■

Claim 6. $|A_1| \geq 3$.

Proof. Suppose $|A_1| = 2$. Then G has no monochromatic copy of P_3 in color j for any $j \in \{3, \dots, k\}$ under c. By Claim 4, $i_3 = \cdots = i_k = 0$ and so $N = 1+n+i_b+i_r$. We may assume that $|A_1| = \cdots = |A_t| = 2$ and $|A_{t+1}| = \cdots = |A_p| = 1$ for some integer t satisfying $p \geq t \geq 1$. Let $A_i = \{a_i, b_i\}$ for all $i \in [t]$. By reordering if necessary, each of $A_1, \dots, A_t$ can be chosen as the largest part in the Gallai-partition $A_1, A_2, \dots, A_p$ of G. For all $i \in [t]$, let

$$A_b^i := \{a_j \in V(\mathcal{R}) \mid a_ja_i \text{ is colored blue in } \mathcal{R}\},$$
$$A_r^i := \{a_j \in V(\mathcal{R}) \mid a_ja_i \text{ is colored red in } \mathcal{R}\}.$$

Let $B^i := \bigcup_{a_j \in A_b^i} A_j$ and $R^i := \bigcup_{a_j \in A_r^i} A_j$. Then $|B^i| + |R^i| = 2n-2+\min\{i_b, i_r\} = n-1+i_b+i_r$. Let

$$E_B := \{a_ib_i \mid i \in [t] \text{ and } |R^i| < |B^i|\},$$
$$E_R := \{a_ib_i \mid i \in [t] \text{ and } |B^i| < |R^i|\},$$
$$E_Q := \{a_ib_i \mid i \in [t] \text{ and } |B^i| = |R^i|\}.$$

Let c^* be obtained from c by recoloring all the edges in E_B blue, all the edges in E_R red, and all the edges in E_Q either red or blue. Then all the edges of G are colored red or blue under c^*. Note that $|G| = n+1+i_b+i_r = R(G_{i_b}, G_{i_r})$. By

Theorem 1.12 and Theorem 1.13, we see that G must contain a blue G_{i_b} or a red G_{i_r} under c^*. By symmetry, we may assume that G has a blue $H := G_{i_b}$ under c^*. Then H contains no edges of E_R but must contain at least one edge of $E_B \cup E_Q$, else we obtain a blue H in G under c. We choose H so that $|E(H) \cap (E_B \cup E_Q)|$ is minimal. We may further assume that $a_1b_1 \in E(H) \cap (E_B \cup E_Q)$, so that $|B^1| \geq |R^1|$. Since $|B^1| + |R^1| = 2n - 2 + \min\{i_b, i_r\} \geq 2n - 2 + 1$, we see that $|B^1| \geq n \geq 5$ and $|R^1| \leq n - 1 + \lfloor \frac{\min\{i_b, i_r\}}{2} \rfloor \leq 7$. So $i_b \geq 2$. By Claim 5, $|R^1| \leq 4$ when $n = 5$. Let $W := V(G) \backslash V(H)$.

We next claim that $i_b = n-1$. Suppose $i_b \leq n-2$. Then $H = P_{2i_b+3}$, $i_r = n-1$, $|G| = 2n+i_b$ and $|W| = 2n-3-i_b \geq n-1$. Let $x_1, x_2, \ldots, x_{2i_b+3}$ be the vertices of H in order. We may assume that $x_\ell x_{\ell+1} = a_1b_1$ for some $\ell \in [2i_b+2]$. If a vertex $w \in W$ is blue-complete to $\{a_1, b_1\}$, then we obtain a blue $H' := G_{i_b}$ under c^* with vertices $x_1, \ldots, x_\ell, w, x_{\ell+1}, \ldots, x_{2i_b+2}$ in order (when $\ell \neq 2i_b + 2$) or $x_1, x_2, \ldots, x_{2i_b+2}, w$ in order (when $\ell = 2i_b + 2$) such that $|E(H') \cap (E_B \cup E_Q)| < |E(H) \cap (E_B \cup E_Q)|$, contrary to the choice of H. Thus no vertex in W is blue-complete to $\{a_1, b_1\}$ under c and so W must be red-complete to $\{a_1, b_1\}$ under c. This proves that $W \subseteq R^1$. We next claim that $\ell = 1$ or $\ell = 2i_b + 2$. Suppose $\ell \in \{2, \ldots, 2i_b + 1\}$. Then $\{x_1, x_{2i_b+3}\}$ must be red-complete to $\{a_1, b_1\}$, else, we obtain a blue $H' := G_{i_b}$ with vertices $x_\ell, \ldots, x_1, x_{\ell+1}, \ldots, x_{2i_b+3}$ or $x_1, \ldots, x_\ell, x_{2i_b+3}, x_{\ell+1}, \ldots, x_{2i_b+2}$ in order under c^* such that $|E(H') \cap (E_B \cup E_Q)| < |E(H) \cap (E_B \cup E_Q)|$. Thus $\{x_1, x_{2i_b+3}\} \subseteq R^1$ and so $W \cup \{x_1, x_{2i_b+3}\}$ is red-complete to $\{a_1, b_1\}$. If $n = 5$, then $4 \geq |R^1| \geq |W \cup \{x_1, x_{2i_b+3}\}| \geq 6$, a contradiction. Thus $n = 6$ and $7 \geq |R^1| \geq |W \cup \{x_1, x_{2i_b+3}\}| \geq 7$. It follows that $R^1 \cap V(H) = \{x_1, x_{2i_b+3}\}$ and thus either $\{x_{\ell-2}, x_{\ell-1}\}$ or $\{x_{\ell+2}, x_{\ell+3}\}$ is blue-complete to $\{a_1, b_1\}$. In either case, we obtain a blue $H' := G_{i_b}$ under c^* such that $|E(H') \cap (E_B \cup E_Q)| < |E(H) \cap (E_B \cup E_Q)|$, a contradiction. This proves that $\ell = 1$ or $\ell = 2i_b + 2$. By symmetry, we may assume that $\ell = 1$. Then x_1x_3 is colored blue under c because $A_1 = \{a_1, b_1\}$. Similarly, for all $j \in \{3, \ldots, 2i_b + 2\}$, $\{x_j, x_{j+1}\}$ is not blue-complete to $\{a_1, b_1\}$, else we obtain a blue $H' := G_{i_b}$ with vertices $x_1, x_j, \ldots, x_2, x_{j+1}, \ldots, x_{2i_b+3}$ in order under c^* such that $|E(H') \cap (E_B \cup E_Q)| < |E(H) \cap (E_B \cup E_Q)|$. It follows that $x_4 \in R^1$ and so $|R^1 \cap \{x_4, \ldots, x_{2i_b+3}\}| \geq i_b$. Then $|R^1| \geq |W| + |R^1 \cap \{x_4, \ldots, x_{2i_b+3}\}| \geq 2n - 3$, so $4 \geq |R^1| \geq 7$ (when $n = 5$) or $7 \geq |R^1| \geq 9$ (when $n = 6$), a contradiction. This proves that $i_b = n - 1$.

Since $i_b = n - 1$, we see that $H = C_{2n}$. Then $|G| = 2n + i_r$ and so $|W| = i_r$. Let $a_1, x_1, \ldots, x_{2n-2}, b_1$ be the vertices of H in order and let $W := \{w_1, \ldots, w_{i_r}\}$. Then x_1b_1 and a_1x_{2n-2} are colored blue under c because $A_1 = \{a_1, b_1\}$. Suppose $\{x_j, x_{j+1}\}$ is blue-complete to $\{a_1, b_1\}$ for some $j \in [2n - 3]$. We then obtain a blue $H' := C_{2n}$ with vertices $a_1, x_1, \ldots, x_j, b_1, x_{2n-2}, \ldots, x_{j+1}$ in order under c^* such that $|E(H') \cap (E_B \cup E_Q)| < |E(H) \cap (E_B \cup E_Q)|$, contrary to the choice of H. Thus, for all $j \in [2n - 3]$, $\{x_j, x_{j+1}\}$ is not blue-complete to $\{a_1, b_1\}$. Since $\{x_1, x_{2n-2}\}$ is blue-complete to $\{a_1, b_1\}$ under c, we see that $x_2, x_{2n-3} \in R^1$, and so $4 \geq |R^1| \geq |R^1 \cap V(H)| \geq 4$ (when $n = 5$) and $7 \geq 5 + \lfloor \frac{i_r}{2} \rfloor \geq |R^1| \geq |R^1 \cap V(H)| \geq 5$ (when $n = 6$). Thus, when $n = 5$, the distinct cases are $R^1 = \{x_2, x_4, x_5, x_7\}$ or $R^1 = \{x_2, x_4, x_6, x_7\}$, as depicted in Figure 1(a) and Figure 1(b); when $n = 6$, we have $R^1 \cap V(H) = \{x_2, x_9\} \cup \{x_j \mid j \in J\}$, where $J \in \{\{4,6,8\}, \{4,6,7\}, \{3,4,6,7\}, \{3,5,6,7\}, \{4,5,6,7\}, \{4,6,7,8\}, \{3,5,7,8\}, \{3,5,6,8\}, \{3,4,5,6,7\}, \{3,4,5,6,8\}, \{3,4,5,7,8\}\}$.

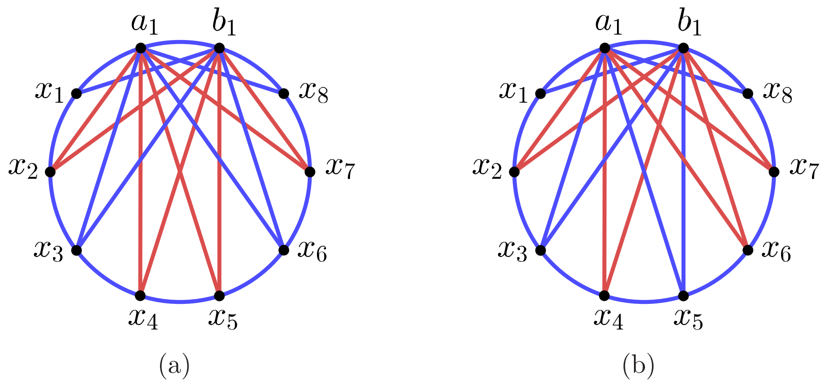

Figure 1: Two cases of R^1 when $i_b = 4$ and $n = 5$.

Since $|R^1| \geq n-1$ and R^1 is red-complete to $\{a_1, b_1\}$ under c, we see that $i_r \geq 2$. Let $W' := W \backslash R^1$. Then $W' \subseteq B^1$. Since $|B^1| \geq |R^1|$, it follows that $|W'| \geq \lceil \frac{i_r}{2} \rceil \geq 1$. We may assume $W' = \{w_1, \ldots, w_{|W'|}\}$. We claim that $E(H) \cap (E_B \cup E_Q) = \{a_1b_1\}$. Suppose, say $a_2b_2 \in E(H) \cap (E_B \cup E_Q)$. Since $\{x_1, x_2\} \neq A_i$ and $\{x_{2n-3}, x_{2n-2}\} \neq A_i$ for all $i \in [t]$, we may assume that $a_2 = x_j$ and $b_2 = x_{j+1}$ for some $j \in \{2, \ldots, 2n-4\}$. Then $x_{j-1}x_{j+1}$ and x_jx_{j+2} are colored blue under c. But then we obtain a blue $H' := C_{2n}$ under c^* with vertices $a_1, x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{2n-2}, b_1, w_1$ in order such that $|E(H') \cap (E_B \cup E_Q)| < |E(H) \cap (E_B \cup E_Q)|$, contrary to the choice of H. Thus $E(H) \cap (E_B \cup E_Q) = \{a_1b_1\}$, as claimed.

$(*)$ Let $w \in W'$. For $j \in \{1, 2n-2\}$, if $\{x_j, w\} \neq A_i$ for all $i \in [t]$, then x_jw is colored red. For $j \in \{2, \ldots, 2n-3\}$, if $\{x_j, w\} \neq A_i$ for all $i \in [t]$ and x_{j-2} or $x_{j+2} \in B^1$, then x_jw is colored red.

Proof. Suppose there is some $j \in [2n-2]$ such that $\{x_j, w\} \neq A_i$ for all $i \in [t]$, and x_{j-2} or $x_{j+2} \in B^1$ if $j \in \{2, \ldots, 2n-3\}$, but x_jw is colored blue. Then we obtain a blue C_{2n} under c with vertices $a_1, w, x_1, \ldots, x_{2n-2}$ (when $j = 1$) or $a_1, x_1, \ldots, x_{2n-2}, w$ (when $j = 2n-2$) in order if $j \in \{1, 2n-2\}$, and with vertices $b_1, x_{2n-2}, x_{2n-3}, \ldots, x_{j+2}, a_1, w, x_j, \ldots, x_1$ in order (when $x_{j+2} \in B^1$) or $a_1, x_1, \ldots, x_{j-2}, b_1, w, x_j, \ldots, x_{2n-2}$ in order (when $x_{j-2} \in B^1$) if $j \in \{2, \ldots, 2n-3\}$, a contradiction. ■

$(**)$ For $j \in [2n-4]$, x_jx_{j+2} is colored red if $\{x_j, x_{j+2}\} \neq A_i$ for all $i \in [t]$.

Proof. Suppose x_jx_{j+2} is colored blue for some $j \in [2n-4]$. Then we obtain a blue C_{2n} under c with vertices $a_1, x_1, \ldots, x_j, x_{j+2}, \ldots, x_{2n-2}, b_1, w_1$ in order, a contradiction. ■

We claim that $n = 6$. Suppose $n = 5$. Then $R^1 = \{x_2, x_4, x_\alpha, x_\beta\}$, where $(\alpha, \beta) \in \{(5,7), (7,6)\}$. Thus $W' = W$ and $x_{\alpha+1}, x_{\alpha-2} \in B^1$. Since $\{x_{\alpha-1}, w_j\} \neq A_i$

and $\{x_\alpha, w_j\} \neq A_i$ for all $w_j \in W$ and $i \in [t]$, it follows from $(*)$ that $\{x_{\alpha-1}, x_\alpha\}$ must be red-complete to W under c. Then for any $w_j \in W$, $\{x_{\alpha-2}, w_j\} \neq A_i$ and $\{x_{\alpha+1}, w_j\} \neq A_i$ for all $i \in [t]$ since $x_{\alpha-1}x_{\alpha-2}$ and $x_\alpha x_{\alpha+1}$ are colored blue under c. Thus $\{x_{\alpha-2}, x_{\alpha+1}\}$ is red-complete to W by $(*)$. So $\{x_{\alpha-2}, x_{\alpha-1}, x_\alpha, x_{\alpha+1}\}$ is red-complete to W under c. But then we obtain a red P_9 under c (when $i_r \leq 3$) with vertices $x_2, a_1, x_{\alpha-1}, b_1, x_\alpha, w_1, x_{\alpha-2}, w_2, x_{\alpha+1}$ in order, or a red C_{10} under c (when $i_r = 4$) with vertices $a_1, x_2, b_1, x_{\alpha-1}, w_1, x_{\alpha-2}, w_2, x_{\alpha+1}, w_3, x_\alpha$ in order, a contradiction. This proves that $n = 6$, as claimed. By $(*)$, we may assume x_1 is red-complete to $W' \backslash w_1$ and x_{10} is red-complete to $W' \backslash w_{|W'|}$ because $|A_1| = 2$. Recall that $5 \leq |R^1 \cap V(H)| \leq 7$ when $n = 6$. We next consider three cases based on the value of $|R^1 \cap V(H)|$.

Case 1. $|R^1 \cap V(H)| = 5$. Then $R^1 \cap V(H) = \{x_2, x_4, x_6, x_\alpha, x_\beta\}$, where $(\alpha, \beta) \in \{(9, 8), (7, 9)\}$. Then $x_{\alpha+1}, x_{\alpha-2} \in B^1$. Since $\{x_{\alpha-1}, w_j\} \neq A_i$ and $\{x_\alpha, w_j\} \neq A_i$ for all $w_j \in W'$ and $i \in [t]$, $\{x_{\alpha-1}, x_\alpha\}$ must be red-complete to W' under c by $(*)$. Then for any $w_j \in W'$, $\{x_{\alpha-2}, w_j\} \neq A_i$ and $\{x_{\alpha+1}, w_j\} \neq A_i$ for all $i \in [t]$ since $x_{\alpha-1}x_{\alpha-2}$ and $x_\alpha x_{\alpha+1}$ are colored blue under c. Thus $\{x_{\alpha-2}, x_{\alpha+1}\}$ is red-complete to W' by $(*)$. So $\{x_{\alpha-2}, x_{\alpha-1}, x_\alpha, x_{\alpha+1}\}$ is red-complete to W' under c. We see that G has a red P_7 with vertices $x_{\alpha-1}, w_1, x_\alpha, a_1, x_2, b_1, x_4$ in order, and so $i_r \geq 3$ and $|W'| \geq \lceil \frac{i_r}{2} \rceil \geq 2$. Moreover, $x_{\alpha-1}x_{\alpha+1}$ and $x_{\alpha-2}x_\alpha$ are colored red by $(**)$. Then G has a red P_{11} with vertices $x_1, w_2, x_{\alpha-1}, x_{\alpha+1}, w_1, x_{\alpha-2}, x_\alpha, a_1, x_2, b_1, x_4$ in order under c. Thus $i_r = 5$ and so $|W'| \geq \lceil \frac{i_r}{2} \rceil \geq 3$. Since $|A_1| = 2$ and $x_{\alpha-6} \in B^1$, by $(*)$, we may assume $x_{\alpha-4}$ is red-complete to $W' \backslash w_2$. But then we obtain a red C_{12} with vertices $a_1, x_\alpha, x_{\alpha-2}, w_1, x_{\alpha-4}, w_3, x_1, w_2, x_{\alpha+1}, x_{\alpha-1}, b_1, x_2$ in order under c, a contradiction.

Case 2. $|R^1 \cap V(H)| = 6$. We claim that $i_r \geq 3$. Suppose $i_r = 2$. Then $|B^1| = |R^1| = 6$ and $G[B^1 \cup R^1]$ contains no red P_3 with at least one end in R^1, else we obtain a red P_7. By Claim 3, B^1 is not blue-complete to R^1. Let $x \in B^1$ and $y \in R^1$ such that xy is colored red. Then x is blue-complete to $R^1 \backslash y$ and there exists at most one vertex $w \in B^1$ such that x is blue-complete to $B^1 \backslash \{x, w\}$ because $G[B^1 \cup R^1]$ contains no red P_3 with at least one end in R^1. Let $i_b^* := 1$, $i_r^* := 0$, $i_j^* := 0$ for all colors j other than red and blue. Let $N^* := |G_{i_b^*}| + [(\sum_{j=1}^k i_j^*) - i_b^*] = 5$. Observe that $|R^1 \backslash y| = 5 = N^*$, by minimality of N, $G[R^1 \backslash y]$ contains a blue P_5. Let $y_1, y_2, \ldots, y_5$ be the vertices of the P_5 in order. Then y is blue-complete to $\{y_j, y_{j+1}\}$ for some $j \in [4]$ and $x_1 \in B^1 \backslash x$ is not red-complete to $\{y_1, y_5\}$ because $G[B^1 \cup R^1]$ contains no red P_3 with at least one end in R^1 and $|A_1| = 2$. So we may assume x_1y_1 is colored blue. But then we obtain a blue C_{12} under c with vertices $a_1, x_1, y_1, \ldots, y_j, y, y_{j+1}, \ldots, y_5, x, x_2, b_1, x_3$ in order, where $x_2, x_3 \in B^1 \backslash \{x, x_1, w\}$, a contradiction. Thus $i_r \geq 3$, as claimed. Note that $|B^1 \cap V(H)| = 4$, so $|W'| \geq 3$. We may further assume that $\{x_1, w_2\} \neq A_i$ and $\{x_1, w_3\} \neq A_i$ for all $i \in [t]$; and $\{x_{10}, w_1\} \neq A_i$ and $\{x_{10}, w_2\} \neq A_i$ for all $i \in [t]$. By $(*)$, x_1 is red-complete to $\{w_2, w_3\}$ under c; and x_{10} is red-complete to $\{w_1, w_2\}$ under c. Let $(\alpha, \beta, \gamma) \in \{(5, 2, 4), (4, 7, 5)\}$. Suppose $R^1 \cap V(H) = \{x_2, x_3, x_\alpha, x_6, x_7, x_9\}$. Since $\{x_\beta, w_j\} \neq A_i$, $\{x_3, w_j\} \neq A_i$ and $\{x_6, w_j\} \neq A_i$ for all $w_j \in W'$ and $i \in [t]$, by $(*)$, $\{x_\beta, x_3, x_6\}$ must be red-complete to

W' under c. By $(**)$, x_γ is red-complete to $\{x_{\gamma-2}, x_{\gamma+2}\}$. But then we obtain a red C_{12} under c with vertices $a_1, x_2, x_4, x_6, w_1, x_{10}, w_2, x_1, w_3, x_3, b_1, x_5$ (when $\alpha = 5$) or $a_1, x_3, x_5, x_7, w_1, x_{10}, w_2, x_1, w_3, x_6, b_1, x_4$ (when $\alpha = 4$) in order, a contradiction. Let $(\alpha, \beta, \gamma, \delta) \in \{(3, 8, 5, 6), (3, 5, 7, 8), (4, 6, 8, 2)\}$. Suppose $R^1 \cap V(H) = V(H)\backslash\{a_1, b_1, x_1, x_{10}, x_\alpha, x_\beta\}$. Since $\{x_\gamma, w_j\} \neq A_i$ and $\{x_\delta, w_j\} \neq A_i$ for all $w_j \in W'$ and $i \in [t]$, $\{x_\gamma, x_\delta\}$ must be red-complete to W' under c by $(*)$. Moreover, $x_\gamma x_{\gamma-2}$ and $x_\delta x_{\delta+2}$ are colored red by $(**)$. Since $|A_1| = 2$, at least one of $x_1, x_{10}, x_\alpha, x_\beta$ is red-complete to $\{w_1, w_2, w_3\}$ by $(*)$. So we may assume x_α is red-complete to $W'\backslash w_2$ and x_β is red-complete to $\{w_1, w_2, w_3\}$. But then we obtain a red C_{12} with vertices $a_1, x_\gamma, x_{\gamma-2}, w_1, x_{10}, w_2, x_1, w_3, x_{\delta+2}, x_\delta, b_1, x_7$ in order if $(\alpha, \beta, \gamma, \delta) \in \{(3, 8, 5, 6), (4, 6, 8, 2)\}$ and $a_1, x_7, x_5, w_1, x_3, w_3, x_1, w_2, x_{10}, x_8, b_1, x_6$ in order if $(\alpha, \beta, \gamma, \delta) = (3, 5, 7, 8)$, a contradiction. Finally if $R^1 \cap V(H) = \{x_2, x_3, x_5, x_6, x_8, x_9\}$. By $(*)$, $R^1 \cap V(H)$ is red-complete to W'. Then G has a red P_{11} with vertices $x_2, a_1, x_3, b_1, x_5, w_1, x_6, w_2, x_8, w_3, x_9$ in order. Thus $i_r = 5$ and so $|W'| \geq 4$. But then we obtain a red C_{12} with vertices $a_1, x_2, w_1, x_3, w_2, x_5, w_3, x_6, w_4, x_8, b_1, x_9$ in order, a contradiction.

Case 3. $R^1 = |R^1 \cap V(H)| = 7$, then $i_r \geq 4$ and $|W'| = |W| = i_r$. Let $(\alpha, \beta) \in \{(6, 5), (7, 4)\}$. Suppose $R^1 = \{x_2, x_3, x_4, x_5, x_\alpha, x_8, x_9\}$. Since $\{x_3, w_j\} \neq A_i$, $\{x_\beta, w_j\} \neq A_i$ and $\{x_8, w_j\} \neq A_i$ for all $i \in [t]$ and any $w_j \in W'$, $\{x_3, x_\beta, x_8\}$ must be red-complete to W' under c by $(*)$. But then we obtain a red C_{12} with vertices $a_1, x_3, w_1, x_{10}, w_2, x_1, w_3, x_\beta, w_4, x_8, b_1, x_2$ in order, a contradiction. Finally if $R^1 = \{x_2, x_3, x_4, x_5, x_6, x_7, x_9\}$. Since $\{x_3, w_j\} \neq A_i$ and $\{x_6, w_j\} \neq A_i$ for all $i \in [t]$ and any $w_j \in W'$, $\{x_3, x_6\}$ must be red-complete to W' under c by $(*)$. We may assume x_8 is red-complete to $W'\backslash w_2$ by $(*)$. But then we obtain a red C_{12} with vertices $a_1, x_3, w_1, x_{10}, w_2, x_1, w_3, x_8, w_4, x_6, b_1, x_2$ in order, a contradiction. This proves that $|A_1| \geq 3$. ■

Claim 7. For any A_i with $3 \leq |A_i| \leq 4$, $G[A_i]$ has a monochromatic copy of P_3 in some color $m \in [k]$ other than red and blue.

Proof. Suppose there exists a part A_i with $3 \leq |A_i| \leq 4$ but $G[A_i]$ has no monochromatic copy of P_3 in any color $m \in [k]$ other than red and blue. We may assume $i = 1$. Since $GR_k(P_3) = 3$, we see that $G[A_1]$ must contain a red or blue P_3, say blue. We may assume a_1, b_1, c_1 are the vertices of the blue P_3 in order. Then $|A_1| = 4$, else $\{b_1\}, \{a_1, c_1\}, A_2, \ldots, A_p$ is a Gallai partition of G with $p + 1$ parts. Let $z_1 \in A_1\backslash\{a_1, b_1, c_1\}$. Then z_1 is not blue-complete to $\{a_1, c_1\}$, else $\{a_1, c_1\}, \{b_1, z_1\}, A_2, \ldots, A_p$ is a Gallai partition of G with $p + 1$ parts. Moreover, $b_1 z_1$ is not colored blue, else $\{b_1\}, \{a_1, c_1, z_1\}, A_2, \ldots, A_p$ is a Gallai partition of G with $p + 1$ parts. If $b_1 z_1$ is colored red, then $a_1 z_1$ and $c_1 z_1$ are colored either red or blue because G has no rainbow triangle. Similarly, z_1 is not red-complete to $\{a_1, c_1\}$, else $\{z_1\}, \{a_1, b_1, c_1\}, A_2, \ldots, A_p$ is a Gallai partition of G with $p + 1$ parts. Thus, by symmetry, we may assume $a_1 z_1$ is colored blue and $c_1 z_1$ is colored red, and so $a_1 c_1$ is colored blue or red because G has no rainbow triangle. But then $\{a_1\}, \{b_1\}, \{c_1\}, \{z_1\}, A_2, \ldots, A_p$ is a Gallai partition of G with $p + 3$ parts, a contradiction. Thus $b_1 z_1$ is colored neither red nor blue. But then $a_1 z_1$ and $c_1 z_1$ must be colored blue because $G[A_1]$ has neither rainbow triangle nor monochromatic P_3 in any color $m \in [k]$ other than red and blue, a contradiction. ■

For the remainder of the proof of Theorem 1.9, we assume that $|B| \geq |R|$. By Claim 5, $|R| \leq n-1$. Let $\{a_i, b_i, c_i\} \subseteq A_i$ if $|A_i| \geq 3$ for any $i \in [p]$. Let $B := \{x_1, \ldots, x_{|B|}\}$ and $R := \{y_1, \ldots, y_{|R|}\}$. We next show that

Claim 8. $i_r \geq |R|$.

Proof. Suppose $i_r \leq |R| - 1 \leq n-2$. Then $i_b = n-1$, $i_r \geq 3$, $|A_1| \leq 4$, else we obtain a red G_{i_r} because R is not blue-complete to B and $|A_1| \geq 3$. By Claim 7, $G[A_1]$ has a monochromatic, say green, copy of P_3. By Claim 4, $i_g = 1$. We have $|G| \geq n + 1 + i_b + i_r + i_g \geq 2n + 4$. This implies that there exist two independent edges between B and R, say x_1y_1, x_2y_2, that are colored red, else we obtain a blue C_{2n}. Then $G[A_1 \cup R \cup \{x_1, x_2\}]$ has a red P_9, it follows that $n = 6$, $i_r = 4$ and $|R| = 5$. Then $|A_1 \cup B| = |G| - |R| \geq 7 + i_b + i_r + i_g - |R| = 12$, and so $G[B]$ has no blue $G_{i_b - |A_1|}$, else we obtain a blue C_{12}. Let $i_b^* := i_b - |A_1| \leq 2$, $i_r^* := i_r - |R| + 2 = 1$, $i_j^* := i_j \leq 2$ for all color $j \in [k]$ other than red and blue. Let $i_\ell^* := \max\{i_j^* \mid j \in [k]\}$. Then $i_\ell^* \leq i_1$. Let $N^* := |G_{i_\ell^*}| + [(\sum_{j=1}^{k} i_j^*) - i_\ell^*]$. Observe that $|B| \geq N^*$. By minimality of N, $G[B]$ has a red $G_{i_r^*} = P_5$ with vertices, say $x_1, \ldots, x_5$, in order. Because there is a red P_7 with both ends in R by using edges between A_1 and R, we see that R is blue-complete to $\{x_1, x_2, x_4, x_5\}$, else $G[A_1 \cup R \cup \{x_1, \ldots, x_5\}]$ has a red P_{11}. But then we obtain a blue C_{12} under c with vertices $a_1, x_1, y_1, x_2, y_2, x_4, y_3, x_5, b_1, x_3, c_1, x_6$ in order, a contradiction. ■

Claim 9. $i_b > |A_1|$ and so $|A_1| \leq n-2$.

Proof. Suppose $i_b \leq |A_1|$. If $i_b \leq |A_1| - 1$, then $i_b \leq n-2$ by Claim 2 and so $i_r = n-1$. Thus $|B| \geq 2 + i_b$ because $|B| + |R| = |G| - |A_1| \geq n + 1 + i_b + (i_r - |A_1|) \geq 3 + 2i_b$. But then G has a blue G_{i_b} using edges between A_1 and B, a contradiction. Thus $i_b = |A_1|$. By Claims 5 and 8, $|R| \leq n-1$ and $i_r \geq |R|$. Observe that $|B| \geq 1 + n + i_r - |R| \geq 1 + n$. Then $G[B \cup R]$ has no blue P_3 with both ends in B, else we obtain a blue G_{i_b} in G. Let $i_b^* := i_b - |A_1| = 0$, $i_r^* := i_r - |R|$, and $i_j^* := i_j \leq n-4$ for all colors $j \in [k]$ other than blue and red. Let $i_\ell^* := \max\{i_j^* \mid j \in [k]\}$. Then $i_\ell^* \leq i_1$. Let $N^* := |G_{i_\ell^*}| + [(\sum_{j=1}^{k} i_j^*) - i_\ell^*]$. Then $3 < N^* < N$. Suppose first that $|R| \geq 2$. Since B is not red-complete to R, we may assume that y_1x is colored blue for some $x \in B$. Note that $i_r^* \leq n-3$ and $|B \backslash x| = N - |A_1| - |R| - 1 \geq N^*$. By minimality of N, $G[B \backslash x]$ must have a red $G_{i_r^*} = P_{2i_r^*+3}$ with vertices, say $x_1, \ldots, x_q$, in order, where $q = 2i_r^* + 3$. Since $G[B \cup R]$ contains no blue P_3 with both ends in B and xy_1 is colored blue, we see that y_1 must be red-complete to $B \backslash x$ and y_2 is not blue-complete to $\{x_1, x_q\}$. We may assume that x_qy_2 is colored red in G. Then $n = 6$, $i_r = |R| = 5$ and $i_b = |A_1| = 3$, else we obtain a red G_{i_r} using vertices in $V(P_{2i_r^*+3}) \cup R \cup A_1$. Let $x' \in B \backslash \{x, x_1, x_2, x_3\}$. Then $\{x, x'\} \not\subseteq A_i$ and $\{x, x_1\} \not\subseteq A_i$ for all $i \in [p]$ because y_1x is colored blue and y_1x', y_1x_1 are colored red, and so xx' and xx_1 are colored red, else $G[A_1 \cup B \cup \{y_1\}]$ has a blue P_9. But then we obtain a red

C_{12} with vertices $a_1, y_1, x', x, x_1, x_2, x_3, y_2, b_1, y_3, c_1, y_4$ in order, a contradiction. Thus $|R| = 1$. By Claim 1 applied to $i_b = |A_1|$, $i_r \geq |R|$ and B, $G[B]$ must have a red P_{2i_r+1} with vertices, say $x_1, x_2, \ldots, x_{2i_r+1}$, in order. Since $G[B \cup R]$ contains no blue P_3 with both ends in B, we may assume that y_1x_1 is colored red under c. Then $i_r = n - 1$, else we obtain a red G_{i_r}, a contradiction. Moreover, y_1x_{2n-1} must be colored blue, else G has a red C_{2n} with vertices $y_1, x_1, \ldots, x_{2n-1}$ in order. Thus y_1 is red-complete to $\{x_1, \ldots, x_{2n-2}\}$, and so $\{x_j, x_{2n-1}\} \not\subseteq A_i$ for all $i \in [p]$ and $j \in [2n-2]$. So $x_{2n-1}x_i$ must be colored red for some $i \in [2n-3]$ because $G[B]$ has no blue P_3. But then we obtain a red C_{2n} with vertices $y_1, x_1, \ldots, x_i, x_{2n-1}, x_{2n-2}, \ldots, x_{i+1}$ in order, a contradiction. This proves that $i_b > |A_1|$, and so $|A_1| \leq n - 2$. ■

By Claims 6 and 9, we have $3 \leq |A_1| \leq n - 2$. By Claim 7, $G[A_1]$ has a monochromatic, say green, copy of P_3. By Claim 4, $i_g = 1$.

Claim 10. If $|A_1| = 3$, then $|A_2| = 3$, $|A_3| \leq 2$, and $i_j = 0$ for all colors $j \in [k]$ other than red, blue and green.

Proof. We may assume that the first three colors in $[k]$ are red, blue, and green. Assume $|A_1| = 3$. To prove $|A_2| = 3$, we show that $G[B \cup R]$ has a green P_3. Suppose $G[B \cup R]$ has no green P_3. By Claim 9, $i_b \geq |A_1| + 1 = 4$. Let $i_g^* := 0$ and $i_j^* := i_j$ for all $j \in [k]$ other than green. Let $i_\ell^* := \max\{i_j^* \mid j \in [k]\}$ and $N^* := |G_{i_\ell^*}| + [(\sum_{j=1}^{k} i_j^*) - i_\ell^*]$. Then $N^* = N - 1$ and $|G \backslash a_1| = N - 1 = N^*$. But then $G \backslash a_1$ has no monochromatic copy of $G_{i_j^*}$ in color j for all $j \in [k]$, contrary to the minimality of N. Thus $G[B \cup R]$ has a green P_3 and so $|A_2| = 3$. For the rest of the proof of Claim 10, we do not use the condition $|B| \geq |R|$ because we make no use of Claim 8 and Claim 9.

Suppose $|A_3| = 3$. For all $i \in [3]$, let

$$A_b^i := \{a_j \in V(\mathcal{R}) \mid a_ja_i \text{ is colored blue in } \mathcal{R}\},$$
$$A_r^i := \{a_j \in V(\mathcal{R}) \mid a_ja_i \text{ is colored red in } \mathcal{R}\}.$$

Let $B^i := \bigcup_{a_j \in A_b^i} A_j$ and $R^i := \bigcup_{a_j \in A_r^i} A_j$. Since each of A_1, A_2, A_3 can be chosen as the largest part in the Gallai-partition $A_1, A_2, \ldots, A_p$ of G, by Claim 5, either $|B^i| \leq 5$ or $|R^i| \leq 5$ for all $i \in [3]$. Without loss of generality, we may assume that A_2 is blue-complete to $A_1 \cup A_3$. Let $X := V(G) \backslash (A_1 \cup A_2 \cup A_3) = \{v_1, \ldots, v_{|X|}\}$. Then $|X| \geq 1 + n + i_b + i_r + i_g - 9 = 2n - 8 + \min\{i_b, i_r\}$. Suppose $|X \cap B^1| \geq 2$. We may assume $v_1, v_2 \in X \cap B^1$. Then G has a blue C_{10} with vertices $a_1, v_1, b_1, v_2, c_1, a_2, a_3, b_2, b_3, c_2$ in order and a blue P_{11} with vertices $a_1, v_1, b_1, v_2, c_1, a_2, a_3, b_2, b_3, c_2, c_3$ in order, and so $n = 6$ and $i_b = 5$. Moreover, $X \backslash \{v_1, v_2\} \subseteq R^3$, else, say v_3 is blue-complete to A_3, then we obtain a blue C_{12} under c with vertices $a_1, v_1, b_1, v_2, c_1, a_2, a_3, v_3, b_3, b_2, c_3, c_2$ in order. Thus $|R^3| \geq |X \backslash \{v_1, v_2\}| \geq 2 + i_r$, and so $i_r \geq 3$, else G has a red G_{i_r} using the edges between A_3 and R^3. Then there exist at least two vertices in $X \backslash \{v_1, v_2\}$, say v_3, v_4, such that $\{v_3, v_4\}$ is blue-complete to A_1, else $G[A_1 \cup A_3 \cup (X \backslash \{v_1, v_2\})]$ contains a red G_{i_r}. Thus $|B^1| \geq |A_2 \cup \{v_1, \ldots, v_4\}| = 7$ and so

$|R^1| \leq 5$. Moreover, $\{v_1, v_2\} \subset R^3$, else, say v_1 is blue-complete to A_3, we then obtain a blue C_{12} under c with vertices $a_1, v_3, b_1, v_4, c_1, a_2, a_3, v_1, b_3, b_2, c_3, c_2$ in order. Then $X \subseteq R^3$ and $|R^3| \geq |X| \geq 4 + i_r \geq 7$, and so $|B^3| \leq 5$ and A_1 is red-complete to A_3. Furthermore, $G[B^1 \backslash A_2]$ has no blue P_3, else, say v_1, v_2, v_3 is such a blue P_3 in order, we obtain a blue C_{12} with vertices $a_1, v_1, v_2, v_3, b_1, v_4, c_1, a_2, a_3, b_2, b_3, c_2$ in order. Therefore for any $U \subseteq B^1 \backslash A_2$ with $|U| \geq 4$, $G[U]$ contains a red P_3 because $|A_1| = 3$ and $GR_k(P_3) = 3$. Since $|R^1| \leq 5$ and $A_3 \subseteq R^1$, we may assume $v_1, \ldots, v_{|X|-2} \in B^1 \backslash A_2$. Then $G[\{v_1, \ldots, v_4\}]$ must contain a red P_3 with vertices, say v_1, v_2, v_3, in order. We claim that $X \subset B^1$. Suppose $v_{|X|} \in R^1$. Then $v_{|X|}$ is red-complete to A_1 and so G has a red P_{11} with vertices $c_1, v_{|X|}, a_1, a_3, b_1, b_3, v_1, v_2, v_3, c_3, v_4$ in order, it follows that $i_r = 5$. Thus $|X| \geq 9$, and $G[\{v_4, \ldots, v_7\}]$ has a red P_3 with vertices, say v_4, v_5, v_6, in order. But then we obtain a red C_{12} with vertices $a_1, v_{|X|}, b_1, a_3, v_1, v_2, v_3, b_3, v_4, v_5, v_6, c_3$ in order, a contradiction. Thus $X \subset B^1$ as claimed. Since $|X| \geq 7$, $G[\{v_4, \ldots, v_7\}]$ contains a red P_3 with vertices, say v_4, v_5, v_6, in order. Then G has a red P_{11} with vertices $a_1, a_3, b_1, b_3, v_1, v_2, v_3, c_3, v_4, v_5, v_6$ in order, and so $i_r = 5$, $|X| \geq 9$. Suppose $G[\{v_4, \ldots, v_9\}]$ has no red P_5. Then $G[\{v_4, \ldots, v_9\}]$ contains at most one part of the Gallai-partition with order three, say A_4, and we may assume $G[A_4]$ has a monochromatic P_3 in some color m other than red and blue if $|A_4| = 3$ by Claim 7. Let $i_r^* := 1$, $i_m^* := 1$, $i_j^* := 0$ for all color $j \in [k] \backslash \{m\}$ other than red. Let $N^* := |G_{i_r^*}| + [(\sum_{j=1}^{k} i_j^*) - i_r^*] = 6 < N$. Then $G[\{v_4, \ldots, v_9\}]$ has no monochromatic copy of $G_{i_j^*}$ in any color $j \in [k]$, which contradicts the minimality of N. Thus $G[\{v_4, \ldots, v_9\}]$ has a red P_5 with vertices, say $v_4, \ldots, v_8$, in order. But then we obtain a red C_{12} with vertices $a_3, v_1, v_2, v_3, b_3, v_4, \ldots, v_8, c_3, v_9$ in order, a contradiction. Therefore, $|X \cap B^1| \leq 1$. By symmetry, $|X \cap B^3| \leq 1$. Let $w \in X \cap B^1$ when $X \cap B^1 \neq \emptyset$ and $w' \in X \cap B^3$ when $X \cap B^3 \neq \emptyset$. Then $A_1 \cup A_3$ is red-complete to $X \backslash \{w, w'\}$. It follows that $n = 5$ and $|X \cap B^1| = |X \cap B^3| = 1$, else $G[A_1 \cup A_3 \cup (X \backslash \{w, w'\})]$ has a red G_{i_r} because $|X| \geq 2n - 8 + \min\{i_b, i_r\}$, a contradiction. But then we obtain a blue C_{10} with vertices $a_2, a_1, w, b_1, b_2, a_3, w', b_3, c_2, c_3$ in order, a contradiction. This proves that $|A_3| \leq 2$ and so $G[A_i]$ has no monochromatic copy of P_3 for all $i \in [p]$ with $i \geq 3$. Since $G[R \cup B]$ has a green P_3, it follows that $G[A_2]$ has a green P_3, so $i_j = 0$ for all color $j \in [k]$ other than red, blue and green by Claim 4. ∎

Claim 11. If $i_b = |A_1| + 1$, then $|R| \leq 2$.

Proof. Suppose $i_b = |A_1| + 1$ but $|R| \geq 3$. By Claim 8, $i_r \geq |R|$, it follows that $|B| \geq 1 + n + i_b + i_r + i_g - |A_1| - |R| \geq 3 + n$. Thus $G[B \cup R]$ has no blue P_5 with both ends in B, else we obtain a blue G_{i_b}. Let $i_b^* := i_b - |A_1| = 1$, $i_r^* := i_r - |R| + 1$ (when $n = 5$) or $i_r^* := \max\{i_r - |R| + 1, 2\}$ (when $n = 6$), $i_j^* := i_j$ for all $j \in [k]$ other than red and blue. Let $i_\ell^* := \max\{i_j^* \mid j \in [k]\}$ and $N^* := |G_{i_\ell^*}| + [(\sum_{j=1}^{k} i_j^*) - i_\ell^*]$. Then $3 < N^* < N$. Observe that $|B| \geq N^*$. By minimality of N, $G[B]$ has a red $G_{i_r^*} = P_{2i_r^*+3}$ with vertices, say $x_1, \ldots, x_q$, in order, where $q = 2i_r^* + 3$. If R is blue-complete to $\{x_1, x_q\}$, then R is red-complete to $B \backslash \{x_1, x_q\}$ because $G[B \cup R]$ has no blue P_5 with both ends in B. But then $G[A_1 \cup R \cup \{x_2, \ldots, x_{q-1}\}]$ has a red G_{i_r}, a contradiction. Thus R is not blue-complete to $\{x_1, x_q\}$, and so we may assume y_1x_1 is colored red. Then $i_r = n-1$ and $R\backslash\{y_1\}$ is blue-complete to $\{x_{q-2}, x_q\}$, else $G[A_1 \cup R \cup \{x_1, \dots, x_q\}]$ has a red G_{i_r}. So $R\backslash\{y_1\}$ is red-complete to $B\backslash\{x_{q-2}, x_q\}$ because $G[B \cup R]$ has no blue P_5 with both ends in B. But then $G[A_1 \cup R \cup \{x_2, \dots, x_{q-1}\}]$ has a red $G_{i_r} = C_{2n}$, a contradiction. ■

Claim 12. $i_b = n-1$.

Proof. Suppose $i_b \leq n-2$. By Claim 6 and Claim 9, $|A_1| \geq 3$ and $i_b > |A_1|$, it follows that $n = 6$, $i_r = n-1 = 5$, $i_b = 4$, and $|A_1| = 3$. By Claim 10, $|A_2| = 3$, $|A_3| \leq 2$, $i_j = 0$ for all colors $j \in [k]\backslash[3]$. By Claim 11, $|R| \leq 2$ and so $A_2 \subset B$. It follows that $|B| = 7+i_b+i_r+i_g-|A_1 \cup R| = 14-|R| \geq 12$. Then $G[B \cup R]$ has no blue P_5 with both ends in B, else G has a blue P_{11} because $|A_1| = 3$. Thus there exists a set W such that $(B \cup R)\backslash(A_2 \cup W)$ is red-complete to A_2, where $W \subset (B \cup R)\backslash A_2$ with $|W| \leq 1$. Let $i_b^* := i_b - |A_1| = 1$, $i_r^* := 2$, $i_j^* := 0$ for all $j \in [k]$ other than red and blue. Let $N^* := |G_{i_r^*}| + [(\sum_{j=1}^{k} i_j^*) - i_r^*] = 8$. Then $N^* < N$. Observe that $|B\backslash(A_2 \cup W)| = |B| - |A_2| - |W| \geq 8 = N^*$. By minimality of N, $G[B\backslash(A_2 \cup W)]$ must contain a red $G_{i_r^*} = P_7$. But then $G[(B \cup R)\backslash W]$ has a red C_{12}, a contradiction. Thus $i_b = n-1$. ■

Claim 13. $|A_1| = n-2$.

Proof. By Claim 9, $|A_1| \leq n-2$. Suppose $|A_1| \leq n-3$. By Claim 6, $n = 6$ and $|A_1| = 3$. By Claim 12, $i_b = 5$. By Claim 10, $|A_2| = 3$, $|A_3| \leq 2$ and $i_j = 0$ for all colors $j \in [k]\backslash[3]$. By Claim 8, $i_r \geq |R|$. Then $|B| = 7+i_b+i_r+i_g-|A_1|-|R| \geq 10$, and so $G[B \cup R]$ has neither blue P_7 nor blue $P_5 \cup P_3$ with all ends in B else we obtain a blue C_{12}.

Suppose $|R| \leq 2$. Then $A_2 \subset B$ and there exists a set $W \subset (B \cup R)\backslash A_2$ with $|W| \leq 3$ such that W is blue-complete to A_2 and $(B \cup R)\backslash(A_2 \cup W)$ is red-complete to A_2. Since $|B\backslash(A_2 \cup W)| \geq 4$, we see that there is a red P_7 using edges between A_2 and $B\backslash(A_2 \cup W)$, so $i_r \geq 3$ and $i_r - |R| \geq 1$. Let $i_b^* := 2$ (when $|B \cap W| \leq 1$) or $i_b^* := 0$ (when $|B \cap W| \geq 2$), $i_r^* := \min\{i_r - |R| - 1, 2\}$, $i_j^* := 0$ for all colors $j \in [k]$ other than red and blue. Let $i_\ell^* := \max\{i_j^* \mid j \in [k]\}$ and $N^* := |G_{i_\ell^*}| + [(\sum_{j=1}^{k} i_j^*) - i_\ell^*] = 3 + \max\{i_b^*, i_r^*\} + i_b^* + i_r^*$. Observe that $|B\backslash(A_2 \cup W)| = 7 + i_r - |R \cup W| \geq N^*$. By minimality of N, $G[B\backslash(A_2 \cup W)]$ has a red $G_{i_r^*} = P_{2i_r^*+3}$ because $G[B]$ has neither blue P_7 nor blue $P_5 \cup P_3$ and $|A_3| \leq 2$. But then $G[(B \cup R)\backslash W]$ has a red G_{i_r} because $|(B \cup R)\backslash W| \geq 7 + i_r \geq |G_{i_r}|$ and A_2 is red-complete to $(B \cup R)\backslash(A_2 \cup W)$, a contradiction. Therefore, $3 \leq |R| \leq 5$ and so $i_r \geq 3$.

We claim that $i_r = 5$. Suppose $3 \leq i_r \leq 4$. Let $i_b^* := 2$, $i_r^* := 2$, $i_j^* := i_j$ for all colors $j \in [k]$ other than red and blue, and $N^* := |G_{i_r^*}| + [(\sum_{j=1}^{k} i_j^*) - i_r^*] = 10$. Observe that $|B| \geq 10 = N^*$. Since $G[B]$ has no blue P_7, by minimality of N, $G[B]$ has a red P_7 with vertices, say $x_1, \dots, x_7$, in order. Then R is blue-complete to $\{x_1, \dots, x_7\}\backslash x_4$, else $G[A_1 \cup R \cup \{x_1, \dots, x_7\}]$ has a red $G_{i_r} = P_{2i_r+3}$. But then $G[B \cup R]$ has a blue P_7 with vertices $x_1, y_1, x_2, y_2, x_3, y_3, x_5$ in order, a contradiction. Thus $i_r = 5$ and so $|G| = 18$, $|B| = 15 - |R|$.

We next consider the case $|R| = 3$. Suppose first $A_2 = R$. Since R is not red-complete to B, we may assume that A_2 is blue-complete to x_1. Let $i_b^* := 2$, $i_r^* := 3$, $i_j^* := 0$ for all colors $j \in [k]$ other than red and blue, and $N^* := |G_{i_r^*}| + [(\sum_{j=1}^k i_j^*) - i_r^*] = 11$. Observe that $|B\backslash x_1| = 11 = N^*$. By minimality of N, $G[B\backslash x_1]$ has a red P_9 with vertices, say $x_2, \ldots, x_{10}$, in order. We claim that A_2 is blue-complete to $\{x_2, x_{10}\}$, else, say x_2 is red-complete to A_2. Then A_2 is blue-complete to $\{x_8, x_{10}\}$, else $G[A_1 \cup A_2 \cup \{x_2, \ldots, x_{10}\}]$ has a red C_{12}. Thus A_2 is red-complete to $B\backslash\{x_1, x_8, x_{10}\}$ because $G[B \cup R]$ has no blue P_7 with both ends in B. But then we obtain a red C_{12} with vertices $a_1, a_2, x_3, \ldots, x_9, b_2, b_1, c_2$ in order, a contradiction. Thus, A_2 is blue-complete to $\{x_1, x_2, x_{10}\}$, and so A_2 is red-complete to $B\backslash\{x_1, x_2, x_{10}\}$ because $G[B \cup R]$ has no blue P_7 with both ends in B. But then we obtain a red C_{12} with vertices $a_1, a_2, x_3, \ldots, x_9, b_2, b_1, c_2$ in order, a contradiction. This proves that $A_2 \subset B$. Then there exists a set $W \subset (B \cup R)\backslash A_2$ with $|W \cap B| \le 3$ such that W is blue-complete to A_2 and $(B \cup R)\backslash(A_2 \cup W)$ is red-complete to A_2. Then $|W| \le 3$ and $|W \cap B| \le 3$ or $|W| = 4$ and $|W \cap B| = 1$ because $G[B \cup R]$ has no blue P_7 with both ends in B. Let

$$\begin{aligned}
&i_b^* := 2 - |W|,\ i_r^* := 2 \ \text{when}\ |W| \in \{0, 1\},\\
&i_b^* := 0,\ i_r^* := 2 \ \text{when}\ |W| \ge 2 \ \text{and}\ \ |W \cap B| \le 2,\\
&i_b^* := 0,\ i_r^* := 1 \ \text{when}\ |W| = |W \cap B| = 3,
\end{aligned}$$

$i_j^* := 0$ for all colors $j \in [k]$ other than red and blue, and $N^* := |G_{i_r^*}| + [(\sum_{j=1}^k i_j^*) - i_r^*] = 3 + 2i_r^* + i_b^*$. Observe that $|B\backslash(A_2 \cup W)| \ge N^*$. By minimality of N, $G[B\backslash(A_2 \cup W)]$ has a red $G_{i_r^*} = P_{2i_r^*+3}$ because $G[B \cup R]$ has neither blue P_7 nor blue $P_5 \cup P_3$ with all ends in B and $|A_3| \le 2$. If $|W| \le 3$ and $|W \cap B| \le 2$, then $G[(B \cup R)\backslash W]$ has a red C_{12} because $|(B \cup R)\backslash W| \ge 12$ and A_2 is red-complete to $(B \cup R)\backslash(A_2 \cup W)$. Thus $|W| = |W \cap B| = 3$ or $|W| = 4$ and $|W \cap B| = 1$. For the former case, $G[B\backslash(A_2 \cup W)]$ has a red P_5 with vertices, say $x_1, \ldots, x_5$, in order. Let $W := \{w_1, w_2, w_3\} \subset B$. Then A_2 is blue-complete to W and red-complete to $\{x_1, \ldots, x_5\}$, and so W is red-complete to $\{x_1, \ldots, x_5\}$ because $G[B]$ has no blue P_7. But then we obtain a red C_{12} with vertices $a_2, x_1, w_1, x_2, w_2, x_3, w_3, x_4, b_2, x_5, c_2, x_6$ in order, where $x_6 \in B\backslash(A_2 \cup W \cup \{x_1, \ldots, x_5\})$, a contradiction. For the latter case, $G[B\backslash(A_2 \cup W)]$ has a red P_7 with vertices, say $x_1, \ldots, x_7$, in order. Let $W \cap B := \{w\}$. Then w is red-complete to $\{x_1, \ldots, x_7\}$ because $G[B]$ has no blue P_7. But then we obtain a red C_{12} with vertices $a_2, x_1, w, x_2, \ldots, x_6, b_2, x_7, c_2, x_8$ in order, where $x_8 \in B\backslash(A_2 \cup W \cup \{x_1, \ldots, x_7\})$, a contradiction. This proves that $|R| \in \{4, 5\}$.

We claim that $G[E(B, R)]$ has no blue P_5 with both ends in B. Suppose there is a blue $H := P_5$ with vertices, say x_1, y_1, x_2, y_2, x_3, in order. Then $G[(B \cup R)\backslash V(H)]$ has no blue P_3 with both ends in B. Let $i_b^* := 0$, $i_r^* := i_r - |R| + 1 = 6 - |R|$, $i_j^* := i_j$ for all colors $j \in [k]$ other than red and blue, and $N^* := |G_{i_r^*}| + [(\sum_{j=1}^k i_j^*) - i_r^*] = 3 + 2(6 - |R|) + 1 = 16 - 2|R|$. Observe that $|B\backslash\{x_1, x_2, x_3\}| = 12 - |R| \ge N^*$ since $|R| \in \{4, 5\}$. By minimality of N, $G[B\backslash\{x_1, x_2, x_3\}]$ has a red $G_{i_r^*}$ with vertices, say $x_4, \ldots, x_q$, in order, where $q = 2i_r^* + 6$. Then y_3 is not blue-complete to $\{x_4, x_q\}$ because $G[(B \cup R)\backslash V(H)]$ has no blue P_3 with both ends in B. We may assume x_4y_3 is colored red.

Then $R\backslash\{y_1, y_2, y_3\}$ is blue-complete to x_8, else say if x_8y_4 is colored red, we obtain a red C_{12} with vertices $a_1, y_3, x_4, \ldots, x_8, y_4, b_1, y_1, c_1, y_2$ in order, a contradiction. Since $G[(B \cup R)\backslash V(H)]$ has no blue P_3 with both ends in B, we see that $R\backslash\{y_1, y_2, y_3\}$ is red-complete to $\{x_4, \ldots, x_q\}\backslash x_8$. But then we obtain a red C_{12} with vertices $a_1, y_3, x_4, \ldots, x_{10}, y_4, b_1, y_1$ (when $|R| = 4$), or $a_1, y_3, x_4, x_5, x_6, y_4, x_7, y_5, b_1, y_1, c_1, y_2$ (when $|R| = 5$) in order, a contradiction. Thus, $G[E(B, R)]$ has no blue P_5 with both ends in B. Let $i_b^* := 2$, $i_r^* := 2$, $i_j^* := i_j$ for all colors $j \in [k]$ other than red and blue, and $N^* := |G_{i_r^*}| + [(\sum_{j=1}^k i_j^*) - i_r^*] = 10$. Observe that $|B| \geq 10 = N^*$. By minimality of N, $G[B]$ has a red P_7 with vertices, say $x_1, \ldots, x_7$, in order. We claim that x_1 is blue-complete to R. Suppose x_1y_1 is colored red. Then $R\backslash y_1$ is blue-complete to $\{x_5, x_7\}$, else $G[A_1 \cup R \cup \{x_1, \ldots, x_7\}]$ has a red C_{12}. Thus $R\backslash y_1$ is red-complete to $B\backslash\{x_5, x_7\}$ because $G[E(B, R)]$ has no blue P_5 with both ends in B. But then we obtain a red C_{12} with vertices $a_1, y_2, x_2, \ldots, x_6, y_3, b_1, y_4, c_1, y_1$ in order, a contradiction. Therefore, x_1 is blue-complete to R. By symmetry, x_7 is blue-complete to R. Then R is red-complete to $B\backslash\{x_1, x_7\}$ because $G[E(B, R)]$ has no blue P_5 with both ends in B. But then we obtain a red C_{12} with vertices $a_1, y_2, x_2, \ldots, x_6, y_3, b_1, y_4, c_1, y_1$ in order, a contradiction. This proves that $|A_1| = n - 2$. ■

By Claims 12, 13 and 8, $i_b = n - 1$, $|A_1| = n - 2$, $i_r \geq |R|$. By Claim 11, $|R| \leq 2$. Then $|B| \geq 3 + n + i_r - |R| \geq 3 + n$, and so $G[B \cup R]$ has no blue P_5 with both ends in B, else there is a blue C_{2n}.

Claim 14. $i_r = n - 1$.

Proof. Suppose $i_r \leq n-2$. By Claim 3, B is not blue-complete to R. Let $x \in B$ and $y \in R$ such that xy is colored red. Let $i_b^* := i_b - |A_1| = 1$ and $i_r^* := i_r - |R| \leq n - 3$, $i_j^* := i_j \leq n - 4$ for all colors $j \in [k]$ other than red and blue. Let $N^* := |G_{i_r^*}| + [(\sum_{j=1}^k i_j^*) - i_r^*]$. Then $3 < N^* < N$ and $|B\backslash x| = N - |A_1| - |R| - 1 \geq N^*$. By minimality of N, $G[B\backslash x]$ must have a red $P_{2i_r^*+3}$ with vertices, say $x_1, x_2, \ldots, x_{2i_r^*+3}$, in order. Then $\{x_1, x_{2i_r^*+3}\}$ must be blue-complete to $\{x, y\}$ and xx_2 must be colored blue under c, else we obtain a red P_{2i_r+3} using vertices in $V(P_{2i_r^*+3}) \cup \{x, y\} \cup A_1$. But then $G[B \cup R]$ has a blue P_5 with vertices $x_2, x, x_1, y, x_{2i_r^*+3}$ in order, a contradiction. ■

Recall that $|A_1| = n - 2$, $G[A_1]$ has a green P_3, and $i_g = 1$. We next show that $|A_2| \geq 3$. Suppose $|A_2| \leq 2$. Then by Claim 10, $|A_1| = 4$ and so $n = 6$. Let $A_1 := \{a_1, b_1, c_1, z_1\}$. Let $i_b^* := i_b - |A_1| = 1$, $i_r^* := i_r - |R| + 1 = 6 - |R| \geq 4$, $i_g^* := i_g - 1 = 0$ and $i_j^* := i_j$ for all $j \in [k]$ other than red, blue and green. Let $i_\ell^* := \max\{i_j^* \mid j \in [k]\}$ and $N^* := |G_{i_\ell^*}| + [(\sum_{j=1}^k i_j^*) - i_\ell^*]$. Then $3 < N^* < N$ and $|B| = |G| - |A_1| - |R| = N^*$. By minimality of N, $G[B]$ must contain a red $G_{i_r^*}$. It follows that $|R| = 2$ and $G_{i_r^*} = P_{11}$. Let $x_1, x_2, \ldots, x_{11}$ be the vertices of the red P_{11} in order. If R is blue-complete to $\{x_1, x_{11}\}$, then R is red-complete to $B\backslash\{x_1, x_{11}\}$ because $G[B \cup R]$ has no blue P_5 with both ends in B. But then G has a red C_{12} with vertices $a_1, y_1, x_2, \ldots, x_{10}, y_2$ in order, a contradiction. Thus, R is not blue-complete to $\{x_1, x_{11}\}$ and we may assume x_1y_1 is colored red. Then $x_{11}y_1$ and x_9y_2 are colored blue, else $G[\{x_1, \ldots, x_{11}\} \cup R \cup A_1]$ has a red C_{12}. If $x_{11}y_2$ is colored red, then x_1y_2 and x_3y_1 are colored blue by the same reasoning. But then we obtain a blue C_{12} with vertices $a_1, x_1, y_2, x_9, b_1, x_3, y_1, x_{11}, c_1, x_2, z_1, x_4$ in order, a contradiction. Thus $x_{11}y_2$ is colored blue. Then y_1 is red-complete to $B\backslash\{x_9, x_{11}\}$, else, say y_1w is colored blue with $w \in B\backslash\{x_9, x_{11}\}$, then $G[B \cup R]$ has a blue P_5 with vertices w, y_1, x_{11}, y_2, x_9 in order. It follows that $\{x_{11}, w\} \nsubseteq A_j$ for all $j \in [p]$, where $w \in B\backslash\{x_9, x_{11}\}$. Moreover, x_2y_2 is colored blue, else G has a red C_{12} with vertices $a_1, y_2, x_2, \ldots, x_{10}, y_1$ in order, a contradiction. Thus, $G[B\backslash\{x_2, x_9\}]$ has no blue P_3, else $G[A_1 \cup B \cup \{y_2\}]$ has a blue C_{12}. Therefore, x_ix_{11} is colored red for some $i \in \{3, \ldots, 7\}$. But then we obtain a red C_{12} with vertices $y_1, x_1, \ldots, x_i, x_{11}, x_{10}, \ldots, x_{i+1}$ in order, a contradiction. Thus $3 \leq |A_2| \leq n-2$ and $A_2 \subset B$ because $|R| \leq 2$.

Since $G[B \cup R]$ has no blue P_5 with both ends in B, there exists at most one vertex, say $w \in (B \cup R)\backslash A_2$, such that $(B \cup R)\backslash(A_2 \cup \{w\})$ is red-complete to A_2, and w is blue-complete to A_2. Suppose $3 \leq |A_3| \leq n-2$. Then $n = 6$ and $|A_1| = 4$ by Claim 10, $A_3 \subseteq B$ and A_3 must be red-complete to A_2, so $w \notin A_3$. Since $G[B \cup R]$ has no blue P_5 with both ends in B, there exists at most one vertex, say $w' \in (B \cup R)\backslash(A_2 \cup A_3)$, such that $(B \cup R)\backslash(A_2 \cup A_3 \cup \{w'\})$ is red-complete to A_3. Note that we may have $w' = w$. Since $|(B \cup R)\backslash\{w, w'\}| \geq |G| - |A_1| - 2 = 18 - 4 - 2 = 12$, we see that $G[(B \cup R)\backslash\{w, w'\}]$ has a red C_{12}, a contradiction. Thus $|A_3| \leq 2$ and so $G[B\backslash A_2]$ has no monochromatic copy of P_3 in color j for all $j \in [k]$ other than red and blue. Let $i_b^* := 1$, $i_r^* := n - 1 - |A_2|$, and $i_j^* := 0$ for all colors $j \in [k]$ other than red and blue. Let $N^* := |G_{i_r^*}| + [(\sum_{j=1}^{k} i_j^*) - i_r^*] = 2i_r^* + 1 = 2n - 1 - 2|A_2|$. Then $3 < N^* < N$ and $|B\backslash(A_2 \cup \{w\})| \geq 2n + 1 - |R| - |A_2| \geq N^*$. By minimality of N, $G[B\backslash(A_2 \cup \{w\})]$ has a red $G_{i_r^*} = P_{2i_r^*+3}$. But then $G[(B \cup R)\backslash\{w\}]$ has a red C_{2n}, a contradiction.

This completes the proof of Theorem 1.9. ■

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