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Star edge coloring of the Cartesian product of graphs

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Abstract

The star chromatic index of a graph G is the smallest integer k for which G admits a proper edge coloring with k colors such that there is no bi-colored path nor cycle of length four. In this paper, we first obtain an upper bound for the star chromatic index of the Cartesian product of two graphs. We then determine the exact value of the star chromatic index of 2-dimensional grids. We also obtain upper bounds on the star chromatic index of the Cartesian product of a path with a cycle, d-dimensional grids, d-dimensional hypercubes and d-dimensional toroidal grids, for every natural number $d \geq 2$.

1 Introduction

Here we briefly introduce the graph theory terminology and notations that we need throughout the paper. All graphs considered in this paper are finite, simple and undirected. We use P_n and C_n to denote a path and a cycle of order n, respectively. A path (cycle) of length k (i.e. with k edges) is referred to as a *k-path* (*k-cycle*). Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The number of edges that are incident to a specific vertex in G is called the *degree* of that vertex. The *maximum degree* of G, denoted by $\Delta(G)$ or simply Δ, is the maximum degree among all the vertices in G. The *distance* between two edges in G is the minimum length of the paths between every two end-points of these edges. A subset M of edges in G is called a *matching* if every two edges in M have no common end-point. A *perfect matching* is a matching which covers all vertices of G.

The *Cartesian product* of two graphs G and H, denoted by $G \square H$, is a graph with vertex set $V(G) \times V(H)$, and $(a,x)(b,y) \in E(G \square H)$ if either $ab \in E(G)$ and

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$x = y$, or $xy \in E(H)$ and $a = b$. For a natural number d, we denote by G^d the d-th *Cartesian power*, that is $G^d = G$ when $d = 1$, and $G^d = G^{d-1} \Box G$ when $d > 1$. A *d-dimensional hypercube* Q_d is the d-th Cartesian power of P_2. A *d-dimensional grid* $G_{\ell_1,\ell_2,\ldots,\ell_d} = P_{\ell_1} \Box P_{\ell_2} \Box \ldots \Box P_{\ell_d}$ is the Cartesian product of d paths. A *d-dimensional toroidal grid* $T_{\ell_1,\ell_2,\ldots,\ell_d} = C_{\ell_1} \Box C_{\ell_2} \Box \ldots \Box C_{\ell_d}$ is the Cartesian product of d cycles.

A *proper vertex* (respectively *edge*) *coloring* of graph G is an assignment of colors to the vertices (respectively edges) of G such that no two adjacent vertices (respectively two incident edges) receive the same color. The minimum number of colors that is needed to properly color the vertices (respectively edges) of G is called the *chromatic number* (respectively *chromatic index*) of G, and is denoted by $\chi(G)$ (respectively $\chi'(G)$). A *star vertex coloring* of G, is a proper vertex coloring of G such that no path or cycle on four vertices in G is bi-colored (see [3, 5]).

In 2008, Liu and Deng [7] introduced the edge version of the star vertex coloring that is defined as follows. A *star edge coloring* of G is a proper edge coloring of G such that no path or cycle of length four in G is bi-colored. We call a star edge coloring of G with k colors, a *k-star edge coloring* of G. The smallest integer k for which G admits a k-star edge coloring is called the *star chromatic index* of G and is denoted by $\chi'_s(G)$. Liu and Deng [7] presented an upper bound of $\lceil 16(\Delta - 1)^{\frac{3}{2}} \rceil$ for the star chromatic index of graphs with $\Delta \geq 7$. In [2], Dvořák et al. obtained the lower bound $2\Delta(1 + o(1))$ and the near-linear upper bound $\Delta.2^{O(1)\sqrt{\log \Delta}}$ for the star chromatic index of graphs. They also presented some upper bounds and lower bounds for the star chromatic index of complete graphs and subcubic graphs (graphs with maximum degree at most 3). In [1], Bezegová et al. obtained some bounds on the star chromatic index of subcubic outerplanar graphs, trees and outerplanar graphs. Some other results on the star chromatic index of graphs can be found in [6, 8, 10–12].

This paper is organized as follows. In Section 2, we give a tight upper bound for the star chromatic index of the Cartesian product of two arbitrary graphs G and H. In Section 3, we determine the exact value of $\chi'_s(P_m \Box P_n)$ for all natural numbers $m, n \geq 2$. Then, we give some upper bounds for the star chromatic index of the Cartesian product of a path and a cycle and the Cartesian product of two cycles. Moreover, applying the upper bounds obtained in Section 2, we give upper bounds on the star chromatic index of grids, hypercubes, and toroidal grids.

2 General upper bounds

In this section, we define the concept of star compatibility and use this concept to obtain an upper bound for the star chromatic index of the Cartesian product of two graphs. Naturally, these bounds imply some upper bounds on the star chromatic index of $G \Box P_n$ and $G \Box C_n$, for every graph G.

Let f be a star edge coloring of a graph G. For every vertex v of G, we denote the set of colors of the edges incident to v by $\mathcal{C}_f(v)$. Two star edge colorings f_1 and

f_2 of G are *star compatible* if for every vertex v, $\mathcal{C}_{f_1}(v) \cap \mathcal{C}_{f_2}(v) = \emptyset$. We say that graph G is (k,t)-*star colorable* if G has t pairwise star compatible colorings with k colors.

Theorem 2.1. *If G and H are two graphs such that G is (k_G, t_G)-star colorable and $t_G \geq \chi(H)$, then*

$$\chi'_s(G \square H) \leq k_G + \chi'_s(H).$$

Moreover, this bound is tight.

Proof. Let $f_i : E(G) \to \{0, 1, \ldots, k_G - 1\}$, $0 \leq i \leq t_G - 1$, be star compatible colorings of G and $f_H : E(H) \to \{k_G, k_G+1, \ldots, k_G+\chi'_s(H)-1\}$ be a star edge coloring of H. Also, let c_H be a proper vertex coloring of H using colors $\{0, 1, \ldots, \chi(H) - 1\}$. We define edge coloring f of $G \square H$ as follows. For every edge ab of G and every vertex x of H, let

$$f((a, x)(b, x)) = f_{c_H(x)}(ab),$$

that is well-defined, since $t_G \geq \chi(H)$.

For every edge xy of H and every vertex a of G, let

$$f((a, x)(a, y)) = f_H(xy).$$

Note that edge coloring f uses $k_G+\chi'_s(H)$ different colors. Since the colors of edges in G and H are different and colorings f_H and f_i, $0 \leq i \leq t_G-1$, are star edge colorings, the edge coloring f is a proper edge coloring and every 4-path (or 4-cycle) with two incident edges in G or H uses at least three different colors. Thus, we only need to consider the case where we have a path with edges $\alpha = (a, x)(b, x)$, $\beta = (b, x)(b, y)$ and $\gamma = (b, y)(c, y)$, respectively. In such a case, we have $f(\alpha) = f_{c_H(x)}(ab)$ and $f(\gamma) = f_{c_H(y)}(bc)$. On the other hand, $c_H(x) \neq c_H(y)$ because $xy \in E(H)$, and consequently star edge colorings $f_{c_H(x)}$ and $f_{c_H(y)}$ are star compatible. Therefore, $f(\alpha) \neq f(\gamma)$. This shows that f is a star edge coloring of $G \square H$.

Now, we prove the tightness of the bound. Let $G \cong P_2$ and $H \cong C_n$, where $n > 4$ is even. Note that for paths and cycles, there exists a star edge coloring with at most 3 colors except for C_5 which requires 4 colors (see proof of Theorem 5.1 in [2]). Therefore, since $\chi'_s(C_n) = 3$ and P_2 is $(2, 2)$-star colorable, we conclude that $\chi'_s(P_2 \square C_n) \leq 5$. Up to symmetry and permutation, there are only four different 4-star edge colorings of $P_2 \square P_3$ that are shown in Figure 1. It is easy to check that none of the colorings can be extended to a 4-star edge coloring of $P_2 \square C_n$. Hence, for every $n \geq 5$, $\chi'_s(P_2 \square C_n) \geq 5$, which implies the bound is tight. □

Note that if G is a (k_G, t_G)-star colorable graph, then it is also (ak_G, at_G)-star colorable for natural number a. In particular, every graph G is $(a\chi'_s(G), a)$-star colorable and therefore, G is $(\chi'_s(G)\chi(H), \chi(H))$-star colorable, for every graph H. Thus, using Theorem 2.1, we have the following corollary.

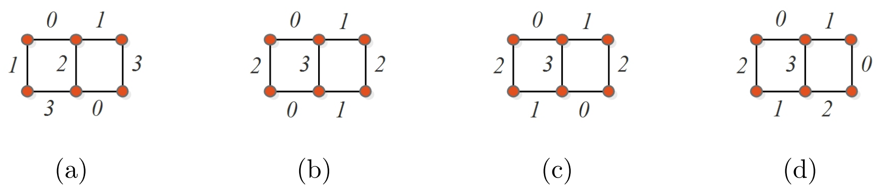


Figure 1: All 4-star edge colorings of $P_2 \square P_3$.

Corollary 2.1. *For every two graphs G and H, we have*

$$\chi'_s(G \square H) = \chi'_s(H \square G) \leq \min\{\chi'_s(G)\chi(H) + \chi'_s(H), \chi'_s(H)\chi(G) + \chi'_s(G)\}.$$

In the following theorem we obtain star compatible colorings for the Cartesian product of two graphs.

Theorem 2.2. *Let G be (k_G, t_G)-star colorable graph and let H be (k_H, t_H)-star colorable graph. If $t_G \geq \chi(H)$ and $t_H \geq \chi(G)$, then $G \square H$ is $(k_G + k_H, \min\{t_G, t_H\})$-star colorable.*

Proof. Suppose that $g_i : E(G) \to \{0, 1, \ldots, k_G - 1\}$, $0 \leq i \leq t_G - 1$, are star compatible colorings of G and $h_i : E(H) \to \{k_G, k_G + 1, \ldots, k_G + k_H - 1\}$, $0 \leq i \leq t_H - 1$, are star compatible colorings of H. Also, let $c_G : V(G) \to \{0, 1, \ldots, \chi(G) - 1\}$ be a proper vertex coloring of G and $c_H : V(H) \to \{0, 1, \ldots, \chi(H) - 1\}$ be a proper vertex coloring of H. If $t = \min\{t_G, t_H\}$, then for each i, $0 \leq i \leq t - 1$, we define edge coloring f_i of $G \square H$ as follows. For every edge ab of G and for every vertex x of H, let $m_i(x) = (c_H(x) + i) \pmod{t_G}$ and

$$f_i((a, x)(b, x)) = g_{m_i(x)}(ab).$$

Also, for every edge xy of H and for every vertex a of G, let $n_i(a) = (c_G(a) + i) \pmod{t_H}$ and

$$f_i((a, x)(a, y)) = h_{n_i(a)}(xy).$$

By the same argument as in proof of Theorem 2.1, every f_i, $0 \leq i \leq t - 1$, is a star edge colorings of $G \square H$. It suffices to prove that edge colorings $f_1, \ldots, f_{t-1}$ are pairwise compatible as follows. For each vertex (a, x), consider colorings f_i and f_j, where $0 \leq i < j \leq t - 1$. We can easily see that

$$\mathcal{C}_{f_i}((a, x)) = \mathcal{C}_{g_{m_i(x)}}(a) \cup \mathcal{C}_{h_{n_i(a)}}(x),$$

and

$$\mathcal{C}_{f_j}((a, x)) = \mathcal{C}_{g_{m_j(x)}}(a) \cup \mathcal{C}_{h_{n_j(a)}}(x).$$

Also, for $i \neq j$, we have $m_i(x) \neq m_j(x)$ and $n_i(a) \neq n_j(a)$. Therefore, $\mathcal{C}_{g_{m_i(x)}}(a) \cap \mathcal{C}_{g_{m_j(x)}}(a) = \emptyset$ and $\mathcal{C}_{h_{n_i(a)}}(x) \cap \mathcal{C}_{h_{n_j(a)}}(x) = \emptyset$. Moreover, since the star edge colorings of G and H use different sets of colors, $\mathcal{C}_{g_{m_i(x)}}(a) \cap \mathcal{C}_{h_{n_j(a)}}(x) = \emptyset$ and $\mathcal{C}_{g_{m_j(x)}}(a) \cap \mathcal{C}_{h_{n_i(a)}}(x) = \emptyset$. Thus, we conclude that $\mathcal{C}_{f_i}((a, x)) \cap \mathcal{C}_{f_j}((a, x)) = \emptyset$, as desired. $\square$

Corollary 2.2. *If G is a (k_G, t_G)-star colorable graph and $t_G \geq \chi(G)$, then G^d is $(d \cdot k_G, t_G)$-star colorable.*

Proof. If $d = 2$, then by Theorem 2.2, graph G^2 is $(2k_G, t_G)$-star colorable. Now, suppose that $d > 2$ and G^{d-1} is $((d-1) \cdot k_G, t_G)$-star colorable. Then using Theorem 2.2 and by induction on d, we conclude that $G^d = G^{d-1} \square G$ is $(d \cdot k_G, t_G)$-star colorable. □

In order to study the star chromatic index of the Cartesian product of paths and cycles with an arbitrary graph, in the following theorems we present some star compatible colorings of paths and cycles.

Theorem 2.3. *For all natural numbers $n, r \geq 2$, P_n is $(2r, r)$-star colorable.*

Proof. Let $V(P_n) = \{0, 1, \ldots, n-1\}$ and $E(P_n) = \{xy : 0 \leq x \leq n-2, y = x+1\}$. We define edge colorings $f_i : E(P_n) \to \{0, 1, \ldots, 2r-1\}$, $0 \leq i \leq r-1$, as follows. For each $xy \in E(P_n)$ with $x < y$, let

$$f_i(xy) = x + 2i \pmod{2r}.$$

Clearly, each f_i, $0 \leq i \leq r-1$, is a star edge colorings of P_n. For every i and j, $0 \leq i < j \leq r-1$, and every $x \in V(P_n)$, we have

$$\mathcal{C}_{f_i}(x) \cap \mathcal{C}_{f_j}(x) = \begin{cases} \{2i\} \cap \{2j\} & \text{if } x = 0, \\ \{x-1+2i, x+2i\} \cap \{x-1+2j, x+2j\} & \text{if } 0 < x < n-1, \\ \{n-2+2i\} \cap \{n-2+2j\} & \text{if } x = n-1. \end{cases}$$

Since $0 \leq i < j \leq r-1$, $2i \not\equiv 2j \pmod{2r}$. Therefore, $\mathcal{C}_{f_i}(x) \cap \mathcal{C}_{f_j}(x) = \emptyset$, and consequently edge colorings f_i and f_j are pairwise star compatible. □

Theorem 2.4. *For all natural numbers $n, r \geq 2$, we have the following statements.*

(i) If $n \geq 4$ is even, then C_n is $(2r, r)$-star colorable.

(ii) If $n \geq 2r+1$ is odd, then C_n is $(2r+1, r)$-star colorable.

(iii) If $n \geq 3$ is odd, then C_n is $(2r + \lceil \frac{2r}{n-1} \rceil, r)$-star colorable.

Proof. Let $V(C_n) = \{0, 1, \ldots, n-1\}$ and $E(C_n) = \{xy : 0 \leq x \leq n-1, y = x+1 \pmod{n}\}$.

(i) Let $n \geq 4$ be even. Two cases may occur:

Case 1. $r = 2$.
Since n is even, two cases may be considered:

- $n = 0 \pmod 4$.
  Define edge colorings f_0 and f_1 of C_n as follows:

$$f_0 : \underbrace{\underline{0,1,2,3}, \underline{0,1,2,3}, \ldots, \underline{0,1,2,3}}_{n}, \qquad f_1 : \underbrace{\underline{2,3,0,1}, \underline{2,3,0,1}, \ldots, \underline{2,3,0,1}}_{n}.$$

- $n = 2 \pmod 4$.
  Define edge colorings f_0 and f_1 of C_n as follows:

$$f_0 : \underbrace{\underline{0,1,2,3}, \underline{0,1,2,3}, \ldots, \underline{0,1,2,3}}_{n-2}, \underline{2,1}, \; f_1 : \underbrace{\underline{2,3,0,1}, \underline{2,3,0,1}, \ldots, \underline{2,3,0,1}}_{n-2}, \underline{0,3}.$$

It is easy to see that edge colorings f_0 and f_1 are star compatible in both cases.

Case 2. $r > 2$.
Define edge colorings $f_i : E(C_n) \to \{0, 1, \ldots, 2r-1\}$, $0 \le i \le r-1$, as follows. For every $xy \in E(C_n)$ ($y = x+1 \pmod n$), let

$$f_i(xy) = \begin{cases} x + 2i \pmod{2r} & \text{if } 0 \le x \le n-2, \\ n+1+2i \pmod{2r} & \text{if } x = n-1. \end{cases}$$

For every i, $0 \le i \le r-1$, f_i is a star edge coloring of C_n. Otherwise, suppose that there exists a bi-colored 4-path (or 4-cycle), say $P : v_1v_2v_3v_4v_5$. Since every three consecutive edges in $C_n \setminus \{0n-1\}$, that $0n-1$ denotes the edge between vertex 0 and vertex $n-1$, have three different colors, it is enough to consider two cases: $v_1 = n-3$, $v_2 = n-2$, $v_3 = n-1$, $v_4 = 0$, $v_5 = 1$, and $v_1 = n-2$, $v_2 = n-1$, $v_3 = 0$, $v_4 = 1$, $v_5 = 2$. In the first case, $f_i(v_1v_2) = n - 3 + 2i \pmod{2r}$, and $f_i(v_3v_4) = n+1+2i \pmod{2r}$. Since $2r \ge 6$, we have $n-3 \ne n+1 \pmod{2r}$, which is a contradiction. In the second case, $f_i(v_1v_2) = f_i(v_3v_4)$, and $f_i(v_2v_3) = f_i(v_4v_5)$. It implies that $n-2+2i = 2i \pmod{2r}$, and $n+1+2i = 1+2i \pmod{2r}$, which is a contradiction. Thus, every f_i, $0 \le i \le r-1$, is a star edge coloring of C_n.

We now show that these star edge colorings are pairwise star compatible. For every i and j, $0 \le i < j \le r-1$, and every $x \in V(C_n)$, we have

$$\mathcal{C}_{f_i}(x) \cap \mathcal{C}_{f_j}(x) = \begin{cases} \{n{+}1{+}2i, 2i\} \cap \{n{+}1{+}2j, 2j\} & \text{if } x = 0, \\ \{x{-}1{+}2i, x{+}2i\} \cap \{x{-}1{+}2j, x{+}2j\} & \text{if } 1 {\le} x {\le} n-2, \\ \{n{-}2{+}2i, n{+}1{+}2i\} \cap \{n{-}2{+}2j, n{+}1{+}2j\} & \text{if } x = n-1. \end{cases}$$

Since $0 \le i < j \le r-1$ and n is even, $2i \ne 2j \pmod{2r}$ and $n+1 \pmod{2r}$ is odd. Therefore, we conclude that $\mathcal{C}_{f_i}(x) \cap \mathcal{C}_{f_j}(x) = \emptyset$, as desired.

(ii) Let $n \ge 2r+1$ be odd. First, we consider the case $r = 2$. Since n is odd, we have two possibilities: either $n = 1 \pmod 4$, or $n = 3 \pmod 4$. If $n = 1 \pmod 4$, then we provide edge colorings f_0 and f_1 of C_n with the following patterns.

$$f_0 : \underbrace{\underline{0,1,2,3}, \underline{0,1,2,3}, \ldots, \underline{0,1,2,3}}_{n-5}, \underline{0,1,2,4,3}, \quad f_1 : \underline{4}, \underbrace{\underline{3,0,1,2}, \underline{3,0,1,2}, \ldots, \underline{3,0,1,2}}_{n-1}.$$

If $n = 3 \pmod 4$, then we provide edge colorings f_0 and f_1 of C_n with the following patterns.

$$f_0 : \underbrace{\underline{0,1,2,3},\underline{0,1,2,3},\ldots,\underline{0,1,2,3}}_{n-3},\underline{0,4,2}, \quad f_1 : \underline{4,3},\underbrace{\underline{0,1,2,3},\underline{0,1,2,3},\ldots,\underline{0,1,2,3}}_{n-3},\underline{1}.$$

It is then easy to see that in each case edge colorings f_0 and f_1 are star compatible.

Now, we consider the case $r > 2$. Assume that $n-1 = 2rp+u$, where $u \in \{0, 2, 4, \ldots, 2(r-1)\}$ and $p \geq 1$. Let

$$b = \frac{n-1-2r}{2} = \frac{2r(p-1)+u}{2},$$

and for every $0 \leq i \leq r-1$, $u_i = b - 2i - 2 \pmod{2r}$, and

$$x_i = \begin{cases} 2r & \text{if } u_i = 0, \\ u_i & \text{if } u_i \neq 0. \end{cases}$$

For every i, $0 \leq i \leq r-1$, we define ordered $(b+1)$-tuples T_i (where each entry in the tuples represents a color), as follows. For $0 \leq i \leq r-2$, let

$$T_i = (\underbrace{\underline{2r-2i-2, 2r-2i-1, \ldots, 2r}}_{2i+3}, \underbrace{\underline{1,2,\ldots,2r},\underline{1,2,\ldots,2r},\ldots,\underline{1,2,\ldots,x_i}}_{b-2i-2}).$$

Also, we define

$$T_{r-1} = (\underline{2r}, \underbrace{\underline{1,2,\ldots,2r},\underline{1,2,\ldots,2r},\ldots,\underline{1,2,\ldots,x_{r-1}}}_{b}).$$

We denote the ℓ-th entry of T_i by T_i^ℓ. For every i, $0 \leq i \leq r-1$, we provide edge coloring f_i of C_n with the following pattern. Let

$$f_0 : \underbrace{\underline{T_0^b, T_0^{b-1}, \ldots, T_0^1}}_{b}, \underbrace{\underline{a_0, a_1, \ldots, a_{2r-1}}}_{2r}, \underbrace{\underline{T_{r-1}^1, T_{r-1}^2, \ldots T_{r-1}^b}}_{b}, \underline{q_0}.$$

For $1 \leq i \leq r-1$, let

$$f_i : \underbrace{\underline{T_i^b, T_i^{b-1}, \ldots, T_i^1}}_{b}, \underbrace{\underline{a_i, a_{i+1}, \ldots, a_{i+2r-1}}}_{2r}, \underbrace{\underline{T_{i-1}^1, T_{i-1}^2, \ldots T_{i-1}^b}}_{b}, \underline{q_i}.$$

In this pattern, $a_i = (2r-1)i, a_{i+1} = (2r-1)i+1, \ldots, a_{i+2r-1} = (2r-1)(i+1)$ (arithmetic is done modulo $2r+1$) and q_i is determined as follows.

If $b = 0$, then $q_i = a_{i+2r} = (2r-1)(i+1)+1 \pmod{2r+1}$. If $b > 1$, then $q_i = T_i^{b+1}$. If $b = 1$, then q_i is adjacent to T_{i-1}^1 and T_i^1. Since T_{i-1}^1 and T_i^1 are even, it is reasonable to choose q_i from $S_i = \{1, 3, \ldots, 2r-1\} \setminus \{a_i, a_{i+2r-1}\}$. Now, to determine the value of q_i, we describe a bipartite graph $G(X, Y)$, as follows. Let $X = \{S_0, S_1, \ldots, S_{r-1}\}$ and $Y = \{1, 3, \ldots, 2r-1\}$. Note that $|X| = |Y|$. Also, vertex $S_\alpha \in X$ is adjacent to vertex $s_\beta \in Y$ if and only if s_β belongs to S_α. For each i, $0 \leq i \leq r-1$, $a_i \pmod{2r+1}$ and $a_{i+2r-1} \pmod{2r+1}$ are odd. Moreover, $a_0 \pmod{2r+1}$ is even and $a_{2r-1} \pmod{2r+1}$ is odd. By definition of S_i, we have the following facts.

- For every i, $0 \leq i \leq r-1$, $|S_i| \geq r-2$.
- For every i, $0 \leq i \leq r-2$, $|S_i \cup S_{i+1}| \geq r-1$.
- For every i, $0 \leq i \leq r-3$, $|S_i \cup S_{i+2}| = r$.

Thus, X satisfies the marriage condition and by the Hall's marriage theorem [4], graph $G(X,Y)$ has a perfect matching. In a perfect matching of $G(X,Y)$, we take the label of the vertex that is matched to S_i as q_i.

It is easy to see that every f_i, $0 \leq i \leq r-1$, is a proper edge colorings of C_n. To prove that every f_i, $0 \leq i \leq r-1$, is a star edge coloring, it suffices to show that every 4-path P in C_n is not bi-colored. For every j and k, where $0 \leq j \leq 2r-3$ and $1 \leq k \leq b-2$, we have $a_j \neq a_{j+2}$ and $T_i^k \neq T_i^{k+2}$; thus if P has at least three consecutive edges with colors from $\{a_i, \ldots, a_{i+2r-1}\}$ or $\{T_i^1, \ldots, T_i^b\}$, then P is not bi-colored. Hence, we consider the following cases.

If P is bi-colored with colors $T_i^2, T_i^1, a_i, a_{i+1}$, or q_i, T_i^1, a_i, a_{i+1}, then $a_{i+1} = T_i^1$. Therefore, $(2r-1)i+1 = 2r-2i-2 \pmod{2r+1}$. Hence, $(2r+1)i = 2r-3 \pmod{2r+1}$, which is a contradiction.

If P is bi-colored with colors $a_{i+2r-2}, a_{i+2r-1}, T_{i-1}^1, T_{i-1}^2$, or $a_{i+2r-2}, a_{i+2r-1}, T_{i-1}^1, q_i$, then $a_{i+2r-2} = T_{i-1}^1$. Therefore, $(2r-1)(i+1)-1 = 2r-2(i-1)-2 \pmod{2r+1}$. Hence, $(2r+1)(i+1) = 2 \pmod{2r+1}$, which is a contradiction.

If P is bi-colored with colors $a_{i+2r-1}, T_{i-1}^1, q_i, T_i^1$, or $T_{i-1}^1, q_i, T_i^1, a_i$, then $T_{i-1}^1 = T_i^1$. Therefore, $2r-2(i-1)-2 = 2r-2i-2 \pmod{2r+1}$, which is a contradiction.

If P is bi-colored with colors $a_{i+2r-1}, q_i, a_i, a_{i+1}$, or $a_{i+2r-2}, a_{i+2r-1}, q_i, a_i$, then $a_{i+2r-1} = a_i$. Therefore, $(2r-1)(i+1) = (2r-1)i \pmod{2r+1}$. Hence, $2r-1 = 0 \pmod{2r+1}$, which is a contradiction.

If $b > 1$ and P is bi-colored with colors $T_{i-1}^b, q_i, T_i^b, T_i^{b-1}$, or $T_{i-1}^{b-1}, T_{i-1}^b, q_i, T_i^b$, then $T_{i-1}^b = T_i^b$, which is a contradiction.

Thus, every f_i, $0 \leq i \leq r-1$, is a star edge coloring. Now, we show that the $f_1, \ldots, f_{r-1}$ are pairwise star compatible. Let $c = n-b$, and for every vertex $x \in V(C_n)$, let $d_x = x-b$. Thus, for all natural numbers i and j, $0 \leq i < j \leq r-1$, we have

$$\mathcal{C}_{f_i}(x) \cap \mathcal{C}_{f_j}(x) = \begin{cases} \{a_i, a_{i+2r}\} \cap \{a_j, a_{j+2r}\} & \text{if } x=0, b=0, \\ \{T_i^b, q_i\} \cap \{T_j^b, q_j\} & \text{if } x=0, b>0, \\ \{T_i^{b-x}, T_i^{b-(x-1)}\} \cap \{T_j^{b-x}, T_j^{b-(x-1)}\} & \text{if } 0<x<b, \\ \{T_i^1, a_i\} \cap \{T_j^1, a_j\} & \text{if } x=b, b>0, \\ \{a_{i+d_x-1}, a_{i+d_x}\} \cap \{a_{j+d_x-1}, a_{j+d_x}\} & \text{if } b<x<c-1, \\ \{a_{i+2r-1}, q_i\} \cap \{a_{j+2r-1}, q_j\} & \text{if } x=c-1, b=0, \\ \{a_{i+2r-1}, T_{i-1}^1\} \cap \{a_{j+2r-1}, T_{j-1}^1\} & \text{if } x=c-1, b>0, \\ \{T_{i-1}^{x-c+1}, T_{i-1}^{x-c+2}\} \cap \{T_{j-1}^{x-c+1}, T_{j-1}^{x-c+2}\} & \text{if } c \leq x < n-1, b>1, \\ \{T_{i-1}^b, q_i\} \cap \{T_{j-1}^b, q_j\} & \text{if } x=n-1, b>0. \end{cases}$$

For every s, $0 \leq s \leq r-1$, it is obvious that $|T_i^s - T_j^s| \geq 2$ is even and $|a_i - a_j| \geq 2$. Thus, we have $\mathcal{C}_{f_i}(x) \cap \mathcal{C}_{f_j}(x) = \emptyset$. As an example, in Figure 2, three pairwise star compatible colorings of C_{15} by seven colors are shown. Here, $T_0 = (4,5,6,1,2)$, $T_1 = (2,3,4,5,6)$, $T_2 = (6,1,2,3,4)$, $r=3$, $b=4$, $p=2$, and $u=2$.

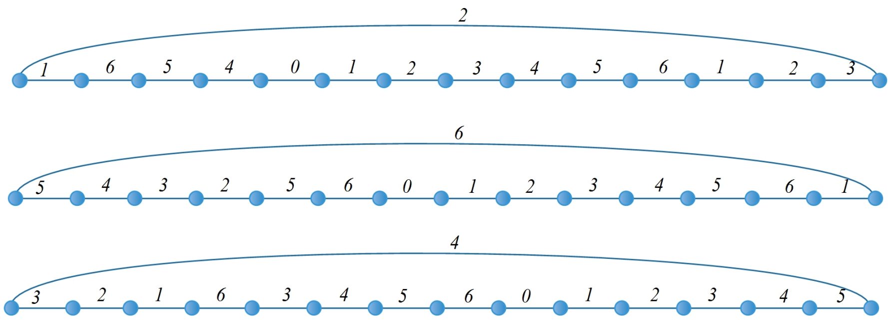


Figure 2: Three compatible star edge colorings of C_{15}.

(iii) If $n \geq 2r+1$, then by assertion (ii) we are done. Thus, let $3 \leq n = 2p+1 < 2r+1$ and $a = \left\lceil \frac{2r}{n-1} \right\rceil - 1$. We show that C_n is $(2r+1+a, r)$-star colorable. By applying assertion (ii) a times, we can provide ap pairwise star compatible colorings of C_n with $(2p+1)a$ colors. Note that each set of p pairwise star compatible colorings uses $2p+1$ new colors. Since

$$2r+1+a-a(2p+1) = 2(r-ap)+1 \leq n,$$

by assertion (ii), we can present $(r-ap)$ pairwise star compatible colorings with $2(r-ap)+1$ colors. Therefore, we provide $ap+(r-ap)=r$ pairwise star compatible colorings of C_n with $(2p+1)a+2(r-ap)+1 = 2r+a+1$ colors, as desired. □

By Theorems 2.1 and 2.4, we have the following corollary.

Corollary 2.3. *For every graph G and a natural number n, we have the following statements.*

(i) *If $n \geq 2$, then $\chi'_s(G \square P_n) \leq \chi'_s(G \square C_{2n}) \leq \chi'_s(G) + 2\chi(G)$.*

(ii) *If $n \geq 2\chi(G)+1$ is odd, then $\chi'_s(G \square C_n) \leq \chi'_s(G) + 2\chi(G) + 1$.*

(iii) *If $n \geq 3$ is odd, then $\chi'_s(G \square C_n) \leq \chi'_s(G) + 2\chi(G) + \left\lceil \frac{2\chi(G)}{n-1} \right\rceil \leq \chi'_s(G) + 2\chi(G) + 3$.*

3 Cartesian product of paths and cycles

In this section, we study the star chromatic index of grids, hypercubes, and toroidal grids. We first obtain the star chromatic index of 2-dimensional grids, and then we extend this result in order to get an upper bound on the star chromatic index of

d-dimensional grids and d-dimensional hypercubes, $d \geq 3$. Also, we obtain some upper bounds for the star chromatic index of $P_m \Box C_n$, $C_m \Box C_n$, and d-dimensional toroidal grids.

Theorem 3.1. *For all natural numbers $2 \leq m \leq n$, we have*

$$\chi'_s(P_m \Box P_n) = \begin{cases} 3 & \textit{if } m = n = 2, \\ 4 & \textit{if } m = 2, n \geq 3, \\ 5 & \textit{if } m = 3, n \in \{3, 4\}, \\ 6 & \textit{otherwise.} \end{cases}$$

Proof. Let $V(P_m \Box P_n) = \{(i,j) : 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$. If $m = n = 2$, then $\chi'_s(P_m \Box P_n) = \chi'_s(C_4) = 3$. By symmetry, we consider the following cases.

Case 1. $m = 2$ and $n \geq 3$.
It is not difficult to see that there is no 3-star edge coloring of $P_2 \Box P_3$. Hence, for every $n \geq 3$, $\chi'_s(P_2 \Box P_n) \geq \chi'_s(P_2 \Box P_3) > 3$. Now, consider the edge coloring $f_{2,n}$ of $P_2 \Box P_n$ as follows. For every j, $0 \leq j \leq n-2$, let

$$f_{2,n}((i,j)(i,j+1)) = \begin{cases} j \pmod 4 & \text{if } i = 0, \\ j+3 \pmod 4 & \text{if } i = 1. \end{cases}$$

For every j, $0 \leq j \leq n-1$, let

$$f_{2,n}((0,j)(1,j)) = j + 1 \pmod 4.$$

Since $f_{2,n}$ has a repeating pattern, it suffices to check that $f_{2,7}$ is a star edge coloring to see that there is no bi-colored 4-path (4-cycle) in $P_2 \Box P_n$. The edge coloring $f_{2,7}$, shown in Figure 3, is clearly a 4-star edge coloring. Therefore, for every $n \geq 3$, $\chi'_s(P_2 \Box P_n) = 4$.

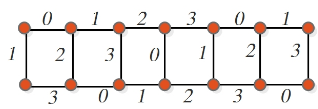


Figure 3: A 4-star edge coloring of $P_2 \Box P_7$.

Case 2. $m = 3$ and $n \in \{3, 4\}$.
By checking all possibilities, it can be seen that there is no 4-star edge coloring of $P_3 \Box P_3$. Therefore, $\chi'_s(P_3 \Box P_4) \geq \chi'_s(P_3 \Box P_3) > 4$. In Figure 4, 5-star edge colorings of $P_3 \Box P_3$ and $P_3 \Box P_4$ are presented. Thus, $\chi'_s(P_3 \Box P_3) = \chi'_s(P_3 \Box P_4) = 5$.

Case 3. $m = n = 4$ or $m \geq 3$ and $n \geq 5$.
In this case, we first show that $\chi'_s(P_m \Box P_n) \geq 6$. For this purpose, we construct all possible 5-star edge colorings of $P_3 \Box P_4$ and then we show it is impossible to extend

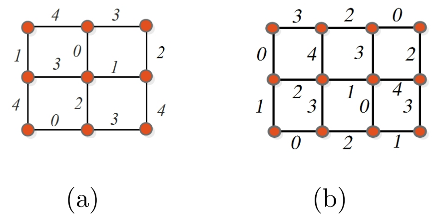


Figure 4: A 5-star edge coloring of $P_3 \square P_3$ and a 5-star edge coloring of $P_3 \square P_4$.

these edge colorings to a 5-star edge coloring of $P_m \square P_n$, when $m = n = 4$ or $m \geq 3$, $n \geq 5$. Consider path $P : (0,0)(1,0)(2,0)(2,1)$ in $P_3 \square P_4$. In a 5-star edge coloring of this graph, edges $(0,0)(1,0)$ and $(2,0)(2,1)$ either have the same color or not. It can be checked that in each case, there is only one 5-star edge coloring of $P_3 \square P_4$. Up to symmetry and permutation, the 5-star edge coloring of $P_3 \square P_4$ is unique and is shown in Figure 4(b).

Now, we try to extend the coloring of $P_3 \square P_4$ to a 5-star edge coloring of $P_3 \square P_5$ or $P_4 \square P_4$. In Figure 5, all possibilities to obtain the desired colorings are depicted. It turns out that it is impossible and therefore $\chi'_s(P_m \square P_n) > 5$, when $m = n = 4$, or $m \geq 3$ and $n \geq 5$.

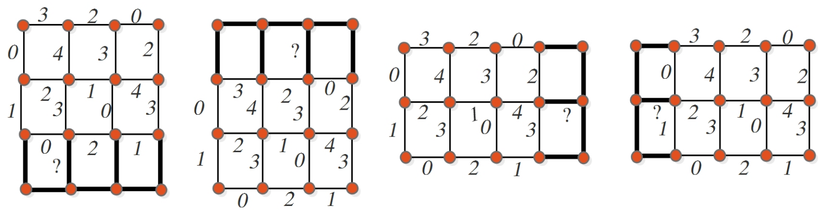


Figure 5: There is no 5-star edge coloring of $P_4 \square P_4$ and $P_3 \square P_5$.

Define the edge coloring $f_{m,n} : E(P_m \square P_n) \to \{0, 1, \ldots, 5\}$ as follows. For every i and j, where $0 \leq i \leq m-2$ and $0 \leq j \leq n-1$, let

$$f_{m,n}((i,j)(i+1,j)) = \begin{cases} i \pmod 4 & \text{if } j = 0 \pmod 2, \\ i+3 \pmod 4 & \text{if } j = 1 \pmod 2. \end{cases}$$

For every i and j, where $0 \leq i \leq m-1$ and $0 \leq j \leq n-2$, let

$$f_{m,n}((i,j)(i,j+1)) = \begin{cases} 4 + (i \pmod 2) & \text{if } j = 1 \pmod 4, \\ 5 - (i \pmod 2) & \text{if } j = 3 \pmod 4, \\ i+1 \pmod 4 & \text{otherwise.} \end{cases}$$

Since $f_{m,n}$ has a repeating pattern, it suffices to check that $f_{7,6}$ is a 6-star edge coloring. The edge coloring $f_{7,6}$ is shown in Figure 6; we can see that there is no bi-colored 4-path (4-cycle) in $P_7 \square P_6$. $\square$

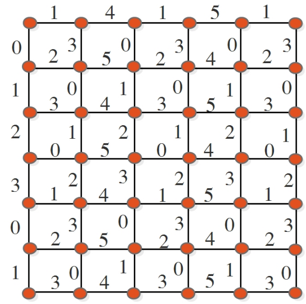


Figure 6: A 6-star edge coloring of $P_7 \square P_6$.

By Corollary 2.2, Theorems 2.1, 2.3, and 3.1, we can obtain an upper bound on the star chromatic index of d-dimensional grids as follows.

Corollary 3.1. *If $G_{\ell_1,\ell_2,\ldots,\ell_d}$ is a d-dimensional grid, $d \geq 2$, then*

$$\chi'_s(G_{\ell_1,\ell_2,\ldots,\ell_d}) \leq 4d - 2.$$

Moreover, for $d = 2$ and $\ell_1, \ell_2 \geq 4$, this bound is tight.

Proof. By Theorem 3.1, if $d = 2$, then $\chi'_s(G_{\ell_1,\ell_2}) = \chi'_s(P_{\ell_1} \square P_{\ell_2}) \leq 6$, and the equality holds for $\ell_1, \ell_2 \geq 4$. By Theorem 2.3 and a similar argument as in the proof of Corollary 2.2, we conclude that $G_{\ell_1,\ell_2,\ldots,\ell_{d-2}}$ is $(4(d-2), 2)$-star colorable. Thus, by Theorems 2.1 and 3.1, we have

$$\chi'_s(G_{\ell_1,\ell_2,\ldots,\ell_d}) \leq 4(d-2) + \chi'_s(P_{\ell_1} \square P_{\ell_2}) \leq 4d - 2.$$

□

Corollary 3.2. *If Q_d is the d-dimensional hypercube with $d \geq 3$, then*

$$\chi'_s(Q_d) \leq 2d - 2.$$

Moreover, this bound is tight for $d = 3$ and $d = 4$.

Proof. It is known that for every natural number $d \geq 2$, $Q_d = Q_{d-1} \square P_2$, $\chi(Q_d) = 2$, and $\chi'_s(Q_3) = 4$ (see proof of Theorem 5.1 in [2]). Therefore, by Corollary 2.1, we have

$$\chi'_s(Q_d) \leq \chi'_s(P_2)\chi(Q_{d-1}) + \chi'_s(Q_{d-1}) \leq 2 + \chi'_s(Q_{d-1}).$$

Thus, by induction on d, it follows that $\chi'_s(Q_d) \leq 2(d-3) + \chi'_s(Q_3) = 2d - 2$.

Since $Q_4 = C_4 \square C_4$ and $\chi'_s(P_4 \square P_4) = 6$, we conclude that $\chi'_s(Q_4) \geq 6$, which implies the equality. Therefore, for $d = 3$ and $d = 4$ the upper bound is tight. □

Note that, for all natural numbers m and n, $P_m \square P_n$ is a subgraph of $P_m \square C_n$ and $C_m \square C_n$. Hence, $\chi'_s(P_m \square P_n)$ that is determined in Theorem 3.1, is a lower bound for $\chi'_s(P_m \square C_n)$ and $\chi'_s(C_m \square C_n)$. In the following theorem, we give some upper bounds for the star chromatic index of the Cartesian product of paths and cycles.

Theorem 3.2. *For every natural number m,*

(i) *if n is an even natural number, then $\chi'_s(P_m \square C_n) \leq 7$;*

(ii) *if n is an odd natural number, then $\chi'_s(P_m \square C_n) \leq 8$.*

Proof. (i) If $n \geq 4$ is even, then C_n is $(4, 2)$-star colorable. Therefore, by Theorem 2.1, we have

$$\chi'_s(P_m \square C_n) \leq 4 + \chi'_s(P_m) = 7.$$

(ii) If $n \geq 5$ is odd, then C_n is $(5, 2)$-star colorable and by Theorem 2.1, $\chi'_s(P_m \square C_n) \leq 8$. If $n = 3$, then we define edge coloring $g_{m,3}$ as follows. For every i and j, where $0 \leq i \leq m-1$ and $0 \leq j \leq 2$, let

$$g_{m,3}((i,j)(i,j+1 \pmod n)) = \begin{cases} j & \text{if } i = 0 \pmod 2, \\ j+3 & \text{if } i = 1 \pmod 2. \end{cases}$$

For every i and j, where $0 \leq i \leq m-2$ and $0 \leq j \leq 2$, let

$$g_{m,3}((i,j)(i+1,j)) = \begin{cases} j+1 & \text{if } i = j \pmod 3, \\ 6 & \text{if } i = j+1 \pmod 3, \\ 7 & \text{if } i = j+2 \pmod 3. \end{cases}$$

Since $g_{m,3}$ has a repeating pattern, it suffices to check that $g_{4,3}$ is a star edge coloring. Edge coloring $g_{4,3}$ is shown in Figure 7, and it is easy to see that there is no bi-colored 4-path (4-cycle) in $P_4 \square C_3$. Thus, we conclude that $\chi'_s(P_m \square C_3) \leq 8$, for every $m \geq 2$. $\square$

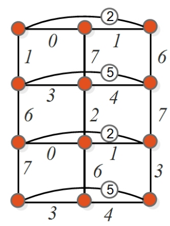


Figure 7: An 8-star edge coloring of $P_4 \square C_3$.

By Theorems 2.1 and 2.4, we give some upper bounds on the star chromatic index of the Cartesian product of two cycles.

Theorem 3.3. *For all natural numbers $m, n \geq 3$, we have the following statements.*

(i) *If m and n are even, then $\chi'_s(C_m \square C_n) \leq 7$.*

(ii) *If $m \neq 3$ is odd and n is even, then $\chi'_s(C_m \square C_n) \leq 8$.*

(iii) *If $m = 3$ and n even, then $\chi'_s(C_m \square C_n) \leq 9$.*

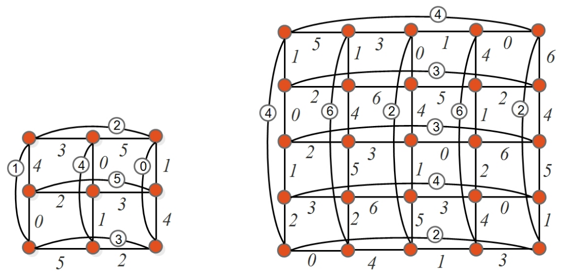
(iv) *If m and n are odd, then $\chi'_s(C_m \square C_n) \leq 10$.*

Proof. (i) Let m and n be even. Thus, $\chi'_s(C_m) = 3$ and by Theorem 2.4(i), C_m is $(4, 2)$-star colorable. Applying Theorem 2.1, we conclude that $\chi'_s(C_m \square C_n) \leq 4 + \chi'_s(C_n) = 7$.

(ii) Let $m > 3$ be odd and n be even. By Theorem 2.4(ii), C_m is $(5, 2)$-star colorable. Therefore, by Theorem 2.1, $\chi'_s(C_m \square C_n) \leq 5 + \chi'_s(C_n) = 8$, as desired.

(iii) Let $m = 3$ and n be even. By Theorem 2.4(iii), C_3 is $(6, 2)$-star colorable. Then, by Theorem 2.1, $\chi'_s(C_3 \square C_n) \leq 6 + \chi'_s(C_n) = 9$.

(iv) Let m and n be odd. A 6-star edge coloring of $C_3 \square C_3$ and a 7-star edge coloring of $C_5 \square C_5$ are shown in Figure 8.



(a) (b)

Figure 8: A 6-star edge coloring of $C_3 \square C_3$ and a 7-star edge coloring of $C_5 \square C_5$.

Now, assume that m and n are not both 3 or 5. Without loss of generality, assume that $m > 3$. If $m > 5$ and $n = 5$, then $\chi'_s(C_m) = 3$ and, by Theorem 2.4(ii), C_n is $(7, 3)$-star colorable. Hence, $\chi'_s(C_m \square C_5) \leq 7 + \chi'_s(C_m) = 10$. In other cases, $\chi'_s(C_n) = 3$ and C_m is $(7, 3)$-star colorable, or vice versa. Thus, $\chi'_s(C_m \square C_n) \leq 10$. □

Remark. By giving some more complex pattern for the star edge coloring of $P_m \square C_n$ and $C_m \square C_n$, we have found that $\chi'_s(P_m \square C_n) \leq 7$ and $\chi'_s(C_m \square C_n) \leq 8$, for all natural numbers m and n [9].

We propose the following conjecture.

Conjecture 3.1. *For all natural numbers m, n,*

$$\chi'_s(C_m \Box C_n) \leq 7.$$

By Corollary 2.2, Theorems 2.1, 2.4, and 3.3, we obtain the following upper bounds on the star chromatic index of d-dimensional toroidal grids.

Corollary 3.3. *For all natural numbers $d \geq 2$ and $\ell_1, \ell_2, \ldots, \ell_d \geq 3$, we have the following statements.*

(i) *The toroidal grid $T_{2\ell_1, 2\ell_2, \ldots, 2\ell_d}$ is $(4d, 2)$-star colorable and $\chi'_s(T_{2\ell_1, 2\ell_2, \ldots, 2\ell_d}) \leq 4d - 1$.*

(ii) *If every $\ell_i > 3$, $1 \leq i \leq d$, then the toroidal grid $T_{\ell_1, \ell_2, \ldots, \ell_d}$ is $(7d, 3)$-star colorable and $\chi'_s(T_{\ell_1, \ell_2, \ldots, \ell_d}) \leq 7d - 4$.*

Proof. By Theorem 2.4, every even cycle is $(4, 2)$-star colorable and every odd cycle, except C_3, is $(7, 3)$-star colorable. Then, by Corollary 2.2, $T_{2\ell_1, 2\ell_2, \ldots, 2\ell_d}$ is $(4d, 2)$-star colorable and $T_{\ell_1, \ell_2, \ldots, \ell_d}$ is $(7d, 3)$-star colorable. Thus, by Theorems 2.1 and 3.3, we have

$$\chi'_s(T_{2\ell_1, 2\ell_2, \ldots, 2\ell_d}) \leq 4(d-2) + \chi'_s(C_{2\ell_{d-1}} \Box C_{2\ell_d}) \leq 4d - 1,$$

and

$$\chi'_s(T_{\ell_1, \ell_2, \ldots, \ell_d}) \leq 7(d-2) + \chi'_s(C_{\ell_{d-1}} \Box C_{\ell_d}) \leq 7d - 4.$$

□

4 Conclusion

In this paper, we have found a tight upper bound for the star chromatic index of the Cartesian product of two arbitrary graphs G and H. Then, we determined the exact value of $\chi'_s(P_m \Box P_n)$ for all natural numbers $m, n \geq 2$. Moreover, we presented upper bounds for the star chromatic index of the Cartesian product of a path and a cycle and the Cartesian product of two cycles. We conjectured that, for all natural numbers m, n, $\chi'_s(C_m \Box C_n) \leq 7$. Finally, we have obtained upper bounds for the star chromatic index of grids, hypercubes, and toroidal grids.

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