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The metric dimension of the incidence graphs of projective and affine planes of small order

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Abstract

The metric dimension μ_q of the incidence graph of a projective plane of order q is known for $q \geq 23$ and $q \leq 5$. The first result uses theoretical (combinatorial and geometric) arguments; the second is provided by computer search. In this paper we determine μ_q for all $q \geq 2$ by combining theoretical methods and computer search based on integer (linear) programming. We also determine the metric dimension of the incidence graphs of finite affine planes.

1 Introduction

A resolving set for a graph G is a set $\mathcal{S} = \{v_1, \ldots, v_k\}$ of vertices such that for any two distinct vertices x and y of G, there exists some $v \in \mathcal{S}$ for which the

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distances $d(x, v)$ and $d(y, v)$ are different; in other words, for every vertex z of G, its distance list $(d(z, v_1), \ldots, d(z, v_k))$ with respect to $\mathcal{S}$ is unique. The size of a smallest resolving set is called the metric dimension of G, and we denote it by $\mu(G)$. A resolving set of size $\mu(G)$ is called a metric basis. Metric dimension was introduced to graph theory in [12, 18], and we also refer the reader to the excellent paper [5] of R. F. Bailey and P. J. Cameron, where the metric dimension of a graph is related to other parameters (the base size of a group, for example); also, its bibliography is a valuable guide to further information on the topic. In this latter paper, distance regular graphs were found particularly interesting to investigate from this viewpoint. Bailey started the systematic study of the metric dimension of imprimitive distance regular graphs [3]. He developed methods to trace back the metric dimension of imprimitive distance regular graphs to that of primitive ones. However, there happen to be a few families for which no meaningful result could be obtained in that way, one of them being the incidence graphs of projective planes. Therefore, Bailey proposed to study this particular case of the general question separately [1]. This was done in [13], where it was shown that any projective plane of order $q \geq 23$ has metric dimension $4q - 4$; moreover, all metric bases are described. Subsequently, studying the metric dimension of graphs related to different finite geometric structures became somewhat prevalent [4, 6, 7, 8]. Bailey also computed the metric dimension of small distance regular graphs [2], and his results (some of which are also covered in [13]) show that the metric dimension of the incidence graphs of projective planes of order 2, 4 and 5 do not follow the general pattern. Motivated by these results, we decided to find the metric dimension of the incidence graphs of projective planes for all the missing cases, and we have obtained the following result, giving a complete answer to the problem.

Theorem 1.1. *The metric dimension μ_q of the incidence graph of a projective plane of order q is*

q	2	3	4	5	7	≥ 8
μ_q	$4q - 3 = 5$	$4q - 4 = 8$	$4q - 6 = 10$	$4q - 5 = 15$	$4q - 5 = 23$	$4q - 4$

Mainly as a corollary of the above theorem, we were able to deduce that the metric dimension of the incidence graph of an affine plane is $3q - 4$ except for $q = 2$, when it is 3 (Theorem 5.2). For $q \geq 13$, this result was already proved in [7].

For the sake of completeness, we include the proof of all cases. The methods we apply here include combinatorial arguments and frequent use of computer search as well. Regarding the latter, we used integer programming models and the MIP solvers GLPK [15] and Gurobi [11]. GLPK was used for modelling in MathProg language, which is very close to writing math formulae, and to solve smaller models (in particular, systems of Diophantine equations and inequalities arising from combinatorial investigations and case analysis), while the much better performance of Gurobi was in store for the harder instances (search for metric bases). We found this approach very comfortable as it requires basically no programming, yet effective enough ([17] and [20], for example, also promote applying LP/IP solvers for certain combinatorial problems).

From now on, we will identify a projective plane with its incidence graph, and hence we may use geometrical and graph theoretical language simultaneously. For basic facts about projective planes we refer the reader to [14]. $\mathrm{PG}(2,q)$ and Π_q denote the Desarguesian and an arbitrary projective plane of order q, respectively. A resolving set $\mathcal{S}$ for Π_q clearly consists of a subset $\mathcal{P}_\mathcal{S}$ of the point set $\mathcal{P}$ of Π_q and a subset $\mathcal{L}_\mathcal{S}$ of the line set $\mathcal{L}$ of Π_q; this will be our standard notation.

The guideline for our study is the following. All projective planes of order at most nine are known and there is no projective plane of order ten; however, for $q \geq 11$ we only know some orders for which no projective plane exists and in no other case we have a complete list of projective planes of a given order (not even for $q = 11$). Hence a complete computer search for the metric bases for projective planes of order 11 or more is not possible, so we need a theoretical approach in this case. Based on the methods of [13], the point sets of metric bases for Π_q tend to have long secants if q is large enough, which is crucial structural information. Thus we will enhance the methods of [13] in order to push down the lower bound to $q \geq 11$ (let us note that in the second author's Bachelor's Thesis [19], the lower bound was pushed down to $q \geq 13$). This is done in Section 3, where many arguments are refinements of those in [13] and [19]. Quite surprisingly, at one point of the proof it was quite helpful to solve an IP model. For $q \leq 9$, trying simply an exhaustive search with Gurobi in itself was disappointingly slow for $q \in \{7, 8, 9\}$. Since for small values of q the presence of long enough secants cannot be proven (in fact, there are some metric bases based on ovals), we had to implement meticulous case studies and many ad hoc arguments to progress, but we also made great use of solving IP models. The details are written in Section 4. The corollary on the metric dimension of affine planes is derived in Section 5. We start with Section 2 containing general combinatorial results and some key lemmas that will be used later on.

2 General arguments

Let us fix some notation and terminology. For a line ℓ and a point P, $[\ell]$ and $[P]$ denote the set of points incident with ℓ and the set of lines incident with P, respectively. For distinct points P and Q, let PQ be the line connecting P and Q. Given a resolving set $\mathcal{S} = \mathcal{P}_\mathcal{S} \cup \mathcal{L}_\mathcal{S}$, a point or a line will be called *inner* if it is an element of $\mathcal{S}$ and *outer* otherwise. Furthermore, n_i will denote the total number of i-secant lines to $\mathcal{P}_\mathcal{S}$ (that is, the number of lines containing exactly i points of $\mathcal{P}_\mathcal{S}$; clearly, $0 \leq i \leq q+1$).

In the incidence graph of a projective plane, the distance of two distinct points is always two (as they have a unique line joining them), as well as the distance of two distinct lines (they admit a unique point of intersection). The distance of a point and a line is one or three, depending on whether they are incident or not, respectively. Hence the distance list of a point P is unique with respect to the resolving set $\mathcal{S}$ if $(\{P\} \cup [P]) \cap \mathcal{S}$ is unique; that is, either $P \in \mathcal{S}$, or $[P] \cap \mathcal{S} \neq [Q] \cap \mathcal{S}$ for any point $Q \notin (\mathcal{S} \cup \{P\})$. The analogous statements holds for lines as well. Hence the next two observations are straightforward; their proofs are written in [13].

Lemma 2.1. *Let $\mathcal{S} = \mathcal{P}_S \cup \mathcal{L}_S$ be a set of vertices in the incidence graph of a finite projective plane. Then any line ℓ intersecting $\mathcal{P}_S$ in at least two points (that is, $|[\ell] \cap \mathcal{P}_S| \geq 2$) has a unique distance list w.r.t. $\mathcal{S}$. Dually, if a point P is covered by at least two lines of $\mathcal{L}_S$ (that is, $|[P] \cap \mathcal{L}_S| \geq 2$), then P has a unique distance list w.r.t. $\mathcal{S}$.*

Proposition 2.2. *$\mathcal{S} = \mathcal{P}_S \cup \mathcal{L}_S$ is a resolving set in a finite projective plane if and only if the following properties hold for $\mathcal{S}$:*

(P1) There is at most one outer line skew to $\mathcal{P}_S$.

(P1') There is at most one outer point not covered by $\mathcal{L}_S$.

(P2) Through every inner point there is at most one outer line tangent to $\mathcal{P}_S$.

(P2') On every inner line there is at most one outer point that is 1-covered by $\mathcal{L}_S$.

From now on, $\mathcal{S} = \mathcal{P}_\mathcal{S} \cup \mathcal{L}_\mathcal{S}$ will always denote an arbitrary resolving set. Let us briefly give the construction of a resolving set of size $4q - 4$. Details can be found in [13].

Proposition 2.3. *Any projective plane of order $q \geq 3$ admits a resolving set of size $4q - 4$.*

Proof. Let P, Q and R be three points not on a line. Let $\mathcal{P}_\mathcal{S} = ([PQ] \cup [PR]) \setminus \{P, Q, R\}$, $\mathcal{L}_\mathcal{S} = ([P] \cup [R]) \setminus \{PQ, PR, RQ\}$. Then $\mathcal{S} = \mathcal{P}_\mathcal{S} \cup \mathcal{L}_\mathcal{S}$ is a resolving set of size $4q - 4$. □

According to Proposition 2.3, in some of the upcoming propositions we assume that $|\mathcal{S}| \leq 4q - 5$ in favor of an indirect proof.

Proposition 2.4. *We have $n_0 \leq |\mathcal{L}_\mathcal{S}| + 1$ and $n_0 + n_1 \leq |\mathcal{S}| + 1$.*

Proof. By Proposition 2.2, we may have one outer skew line and at most $|\mathcal{L}_\mathcal{S}|$ inner skew lines. On each inner point we may have one outer tangent line, so the number of outer skew and outer tangent lines together is at most $1 + |\mathcal{P}_\mathcal{S}|$, while the number of inner skew and inner tangent lines together is at most $|\mathcal{L}_\mathcal{S}|$. □

We give two quick lower bounds on $|\mathcal{S}|$ first. The first one is a direct adaptation of [13, Proposition 10].

Proposition 2.5. *$|\mathcal{P}_S| \geq 2q + 2 - \lfloor 2|\mathcal{S}|/q \rfloor$ and $|\mathcal{L}_\mathcal{S}| \geq 2q + 2 - \lfloor 2|\mathcal{S}|/q \rfloor$, and hence $|\mathcal{S}| \geq 4q + 4 - 2\lfloor 2|\mathcal{S}|/q \rfloor$.*

Proof. The number of lines intersecting $\mathcal{P}_\mathcal{S}$ in at least two points is $q^2+q+1-n_0-n_1$. Using double counting for

$$\Gamma = \{(P,\ell)\colon P \in \mathcal{P}_S, P \in [\ell], |[\ell] \cap \mathcal{P}_S| \geq 2\},$$

we get

$$2(q^2+q+1-n_0-n_1) \leq |\Gamma| \leq |\mathcal{P}_S|(q+1) - n_1,$$

whence

$$\begin{aligned} q|\mathcal{P}_S| &\geq 2q^2+2q+2-2n_0-n_1-|\mathcal{P}_S| \\ &\geq 2q^2+2q+2-(|\mathcal{S}|+|\mathcal{L}_\mathcal{S}|+2)-|\mathcal{P}_S| \\ &= 2q^2+2q-2|\mathcal{S}|, \end{aligned}$$

thus

$$|\mathcal{P}_S| \geq 2q+2-\frac{2|\mathcal{S}|}{q}.$$

As $|\mathcal{P}_\mathcal{S}|$ is an integer, the first assertion is proved. By duality, the second assertion holds as well. □

Corollary 2.6. *If $|\mathcal{S}| \leq 4q-5$, then $|\mathcal{P}_\mathcal{S}| \geq 2q-5$ and $|\mathcal{L}_\mathcal{S}| \geq 2q-5$ (and hence $|\mathcal{S}| \geq 4q-10$), furthermore, $|\mathcal{P}_\mathcal{S}| \leq 2q$, $|\mathcal{L}_\mathcal{S}| \leq 2q$.*

Proposition 2.7. *Suppose that $|\mathcal{S}| \leq 4q-5$ and $q \geq 8$. Then $|\mathcal{P}_S| \geq 2q-4$ and $|\mathcal{L}_\mathcal{S}| \geq 2q-4$, and hence $|\mathcal{S}| \geq 4q-8$, $|\mathcal{P}_\mathcal{S}| \leq 2q-1$ and $|\mathcal{L}_\mathcal{S}| \leq 2q-1$.*

Proof. For an outer point P, let $n_0(P)$ and $n_1(P)$ denote the number of skew and tangent lines to $\mathcal{P}_\mathcal{S}$ through P, and let $\ell(P) = \max\{|\ell \cap \mathcal{P}_\mathcal{S}|\colon P \in \ell \in \mathcal{L}\}$. Clearly,

$$\begin{aligned} \sum_{P \notin \mathcal{P}_\mathcal{S}} n_0(P) &= (q+1)n_0, \\ \sum_{P \notin \mathcal{P}_\mathcal{S}} n_1(P) &= qn_1. \end{aligned}$$

Let $P \notin \mathcal{P}_\mathcal{S}$ be an arbitrary point. Considering all lines through P, we obtain $|\mathcal{P}_\mathcal{S}| \geq \ell(P) + n_1(P) + 2(q - n_0(P) - n_1(P))$, equivalently,

$$\ell(P) \leq |\mathcal{P}_\mathcal{S}| - 2q + n_1(P) + 2n_0(P). \tag{1}$$

Note that if $\ell(P) \leq 1$, then we immediately get $|\mathcal{P}_\mathcal{S}| \leq q+1$, which contradicts Corollary 2.6 under $q \geq 7$, thus for all points P, $\ell(P) \geq 2$. Thus by (1) we get

$$2(q^2+q+1-|\mathcal{P}_\mathcal{S}|) \leq \sum_{P \notin \mathcal{P}_\mathcal{S}} \ell(P) \leq (q^2+q+1-|\mathcal{P}_\mathcal{S}|)(|\mathcal{P}_\mathcal{S}|-2q) + qn_1 + 2(q+1)n_0.$$

Dividing by $q^2+q+1-|\mathcal{P}_\mathcal{S}|$ and rearranging we obtain

$$|\mathcal{P}_\mathcal{S}| \geq 2q+2-\frac{qn_1+2(q+1)n_0}{q^2+q+1-|\mathcal{P}_\mathcal{S}|}.$$

By Proposition 2.4, $n_0 \leq |\mathcal{L}_\mathcal{S}| + 1$ and $n_0 + n_1 \leq |\mathcal{S}| + 1$, and by Corollary 2.6, $|\mathcal{P}_\mathcal{S}| \leq 2q$, $|\mathcal{L}_\mathcal{S}| \leq 2q$. These yield $q^2+q+1-|\mathcal{P}_\mathcal{S}| \geq q^2-q+1$, and $qn_1+2(q+1)n_0 = q(n_0+n_1)+(q+2)n_0 \leq q(4q-4)+(q+2)(2q+1) = 6(q^2-q+1)+7q-4$, thus

$$|\mathcal{P}_\mathcal{S}| \geq 2q+2-6-\frac{7q-4}{q^2-q+1} > 2q-5,$$

provided that $q \geq 8$. By duality, the analogous result for $|\mathcal{L}_\mathcal{S}|$ is also delivered. □

We shall see later that usually most points of $\mathcal{S}$ are contained in two lines, hence the following notation and lemma will be useful.

Notation 2.8. Suppose that e and f are two arbitrarily chosen lines that both intersect $\mathcal{P}_\mathcal{S}$ in at least two points, and let $P = e \cap f$ (see Figure 1 for a guide). Let $E_i = [e] \cap (\mathcal{P}_\mathcal{S} \setminus \{P\})$, $F_i = [f] \cap (\mathcal{P}_\mathcal{S} \setminus \{P\})$, $E_o = [e] \setminus (\mathcal{P}_\mathcal{S} \cup \{P\})$, $F_o = [f] \setminus (\mathcal{P}_\mathcal{S} \cup \{P\})$ (these are the inner and the outer points on e and f, apart from P), $Z = \mathcal{P}_\mathcal{S} \setminus ([e] \cup [f])$. Define k and l by $l = |E_o|$ and $k = |F_o|$. Let $r = |\{\ell \in \mathcal{L} \colon \ell \cap e \in E_o, \ell \cap f \in F_o, \ell \cap Z \neq \emptyset\}|$. Denote by $\tau(Z)$ the total number of lines that intersect Z. Finally, we define the indicator I_P of $P \in \mathcal{P}_\mathcal{S}$, and assign to it the value one if $P \in \mathcal{P}_\mathcal{S}$ and zero otherwise.

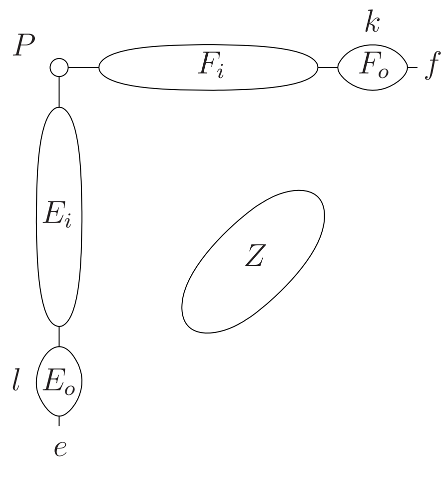

Figure 1: A rough sketch of the structure of $\mathcal{P}_\mathcal{S}$ with respect to two lines.

Lemma 2.9. *Using Notation 2.8, the following estimates hold:*

$$|\mathcal{L}_\mathcal{S}| \geq (k+l-1)q+k+l-kl-I_P-2-(k+l+1)|Z|+r, \text{ and} \quad (2)$$
$$|\mathcal{L}_\mathcal{S}| \geq (k+l-1)q+k+l-kl-I_P-2-\tau(Z), \quad (3)$$

In case of equality in (2) *or* (3), *there is an outer skew line to* $\mathcal{P}_\mathcal{S}$, *and there is an outer tangent line to* $\mathcal{P}_\mathcal{S}$ *on each inner point.*
In case of equality in (2), *every line containing a point of* $E_o \cup F_o \cup \{P\}$ *intersects* Z *in at most one point.*

In case of equality in (3)*, Z does not intersect any line joining a point of E_i with a point of F_i. Furthermore,*

$$\begin{aligned} \tau(Z) &= |Z|q+1 \text{ if and only if } Z \text{ is contained in a line, and} \quad (4)\\ \tau(Z) &\geq (q-1)|Z|+3 \text{ if } Z \text{ is not contained in a line.} \quad (5)\end{aligned}$$

Proof. As 'most' inner points are in $[e] \cup [f]$, we consider Z as if not present for a moment and accumulate all 'possibly problematic' lines with respect to (P1) and (P2) of Proposition 2.2 (tangents and skew lines) into the set $\mathcal{L}_{\text{prob}}$; that is, the lines that contain at least one point of $\{P\} \cup E_o \cup F_o$, different from e and f. Clearly, $|\mathcal{L}_{\text{prob}}| = q - 1 + kq + lq - kl$. Let $\tau'(Z)$ denote the number of lines of $\mathcal{L}_{\text{prob}}$ that intersect Z. Then there are at least $|\mathcal{L}_{\text{prob}}| - \tau'(Z)$ lines in $\mathcal{L}_{\text{prob}}$ that are either tangents or skew lines to $\mathcal{P}_{\mathcal{S}}$. There may be one outer skew line altogether, and one outer tangent line on each inner point in $e \cup f$, thus there may be at most $1 + (q-k) + (q-l) + I_P$ outer lines in $\mathcal{L}_{\text{prob}}$. Hence

$$\begin{aligned} |\mathcal{L}_{\mathcal{S}}| &\geq q - 1 + kq + lq - kl - \tau'(Z) - (1 + (q-k) + (q-l) + I_P)\\ &= (k+l-1)q + k + l - kl - 2 - I_P - \tau'(Z)\end{aligned}$$

and, in case of equality, there must be an outer skew line and an outer tangent line on each point of $(\{P\} \cup E_i \cup F_i) \cap \mathcal{P}_{\mathcal{S}}$. As Z may intersect at most $|Z|$ lines from a pencil $[X]$ for any point $X \in E_o \cup F_o \cup \{P\}$, we see that $\tau'(Z) \leq (k+l+1)|Z| - r$, hence the first estimate holds and, in case of equality, Z must intersect exactly $|Z|$ lines of each of the $k+l+1$ pencils.

However, $\tau'(Z) \leq \tau(Z)$ clearly holds, and in case of equality, every line that intersects Z is in $\mathcal{L}_{\text{prob}}$. If we choose $X \in Z$, we see that X blocks $q+1$ lines (in total, not only of $\mathcal{L}_{\text{prob}}$), and other points $Y \in Z$ block at most q new lines (XY is already blocked); hence $\tau(Z) \leq q+1+(|Z|-1)q = |Z|q+1$ with equality if and only if Z is contained in a line. If Z is not contained in a line, choose $X_1, X_2, X_3 \in Z$ such that they are not collinear. Clearly, these three points block $3q$ lines, and for all other point $Y \in Z$, there are at least two different lines among X_1Y, X_2Y and X_3Y, so Y blocks at most $q-1$ new lines, hence $\tau(Z) \leq 3q+(|Z|-3)(q-1) = |Z|(q-1)+3$. □

Note that

$$|Z| = |\mathcal{P}_{\mathcal{S}}| - (2q-k-l) - I_P. \quad (6)$$

Proposition 2.10. *Suppose that $|\mathcal{S}| \leq 4q-5$ and $|\mathcal{P}_{\mathcal{S}}| \leq 2q-3$. Then $n_{q+1} = n_q = 0$. If $q \geq 3$, then $n_{q-1} = 0$.*

Proof. Let e be a line intersecting $\mathcal{P}_{\mathcal{S}}$ in m points. On each inner point of e there are at least $q - (|\mathcal{P}_{\mathcal{S}}| - m)$ tangents to $\mathcal{P}_{\mathcal{S}}$, one of which may be outer for each point. Similarly, there are at least $q - (|\mathcal{P}_{\mathcal{S}}| - m)$ skew lines on each outer point of e, one of which may be outer (in total). Hence for the number of inner lines we have $|\mathcal{L}_{\mathcal{S}}| \geq (q+1)(q+m-|\mathcal{P}_{\mathcal{S}}|) - m - 1 \geq (q+1)(m-q+3) - m - 1$. If $m \geq q$, we obtain $|\mathcal{L}_{\mathcal{S}}| > 2q$, in contradiction with 2.6. If $m = q-1$ and $|\mathcal{P}_{\mathcal{S}}| \leq 2q-4$, we get a contradiction as well. If $m = q-1$ and $|\mathcal{P}_{\mathcal{S}}| = 2q-3$ then, as $q \geq 3$, $|\mathcal{P}_{\mathcal{S}}| \geq q$

follows, hence we find a line f intersecting ℓ in an inner point and containing another inner point. We use Lemma 2.9 (and Notation 2.8) for these two lines; hence $k = 2$, $2 \le l \le q - 1$, $I_P = 1$, $|Z| = l - 2$. Substituting these and $\tau(Z) \le |Z|q + 1$ into (3) we obtain $2q - 2 \ge |\mathcal{L}_\mathcal{S}| \ge 2q - 1$, a contradiction. □

Let us formulate the so-called and well-known standard equations.

Lemma 2.11.

$$\sum_{i=0}^{q+1} n_i = q^2 + q + 1, \tag{7}$$

$$\sum_{i=0}^{q+1} i n_i = (q+1)|\mathcal{P}_\mathcal{S}|, \tag{8}$$

$$\sum_{i=0}^{q+1} i(i-1) n_i = |\mathcal{P}_\mathcal{S}|(|\mathcal{P}_\mathcal{S}| - 1). \tag{9}$$

Proof. The first one is trivial; for the second and the third count the size of the sets $\{(P, \ell) \colon P \in \mathcal{P}_\mathcal{S}, \ell \in [P]\}$ and $\{(P, Q, \ell) \colon P \in \mathcal{P}_\mathcal{S}, Q \in \mathcal{P}_\mathcal{S}, P \ne Q, \ell = PQ\}$ in two different ways. □

Finally, let us define an IP model to be referred to as the BASIC IP. It relies on the Diophantine system of equations given by the standard equations and some of the theoretical results. We suppose that $q \ge 3$, $|\mathcal{P}_\mathcal{S}|$ and $|\mathcal{L}_\mathcal{S}|$ are given, and consider the n_is as non-negative integer variables ($0 \le i \le q + 1$). We add the following constraints arising from Lemma 2.11, Proposition 2.4 and Proposition 2.10:

$$\begin{aligned} \sum_{i=0}^{q+1} n_i &= q^2 + q + 1, \\ \sum_{i=0}^{q+1} i n_i &= (q+1)|\mathcal{P}_\mathcal{S}|, \\ \sum_{i=0}^{q+1} i(i-1) n_i &= |\mathcal{P}_\mathcal{S}|(|\mathcal{P}_\mathcal{S}| - 1), \\ n_0 &\le |\mathcal{L}_\mathcal{S}| + 1, \\ n_0 + n_1 &\le |\mathcal{S}| + 1, \\ n_{q-1} + n_q + n_{q+1} &= 0. \end{aligned}$$

The objective function can be chosen in different ways to obtain the information we need[1]; for example, if we need a contradiction, then the model should be infeasible with an arbitrary objective function, or if we need, say, a lower bound on n_0, it should be $\min n_0$.

[1]Certainly, many calculations where we chose to solve BASIC IP with a computer may be carried out by hand, but we found that using GLPK for this task speeds up the process remarkably, as typing and modifying the model appropriately is very easy, and the required computer time was always less than 0.1 seconds.

3 Improvements for $q \geq 11$

General assumptions. We want to prove $\mu(\Pi_q) = 4q-4$ for $q \geq 11$ by contradiction. Hence, throughout this section, we assume that $\mathcal{S} = \mathcal{P}_\mathcal{S} \cup \mathcal{L}_\mathcal{S}$ is a resolving set of size $|\mathcal{S}| \leq 4q-5$. By duality and Proposition 2.7, we may assume $2q-4 \leq |\mathcal{P}_\mathcal{S}| \leq |\mathcal{L}_\mathcal{S}| \leq 2q-1$, and hence $|\mathcal{P}_\mathcal{S}| \leq 2q-3$. The next two propositions are the refinements of [13, Propositions 12 and 13].

Proposition 3.1. *Let $q \geq 11$. If $|\mathcal{S}| \leq 4q-5$, then every line intersects $\mathcal{P}_\mathcal{S}$ in either at most 4 or at least $q-5$ points.*

Proof. Suppose that for a line ℓ, $|[\ell] \cap \mathcal{P}_S| = x$, $2 \leq x \leq q$. For a point $P \in [\ell] \setminus \mathcal{P}_S$, let $n_0(P)$ and $n_1(P)$ denote the number of all (outer and inner as well) skew or tangent lines to $\mathcal{P}_S$ through P, respectively; moreover, let t denote the total number of tangents (outer and inner as well) intersecting ℓ outside $\mathcal{P}_S$. Counting the points of $\mathcal{P}_S$ on ℓ and the other lines through P we get

$$|\mathcal{P}_S| \geq x + t(P) + 2(q - n_1(P) - n_0(P)), \tag{10}$$

equivalently, $2q + x \leq |\mathcal{P}_S| + 2n_0(P) + n_1(P)$. Adding up the inequalities for all $P \in [\ell] \setminus \mathcal{P}_S$, we obtain

$$(q+1-x)(2q+x) \leq (q+1-x)|\mathcal{P}_\mathcal{S}| + 2n_0 + t.$$

Recall $n_0 \leq |\mathcal{L}_\mathcal{S} + 1|$ (Proposition 2.4) and, as there can be at most one outer tangent on each point of $\mathcal{P}_\mathcal{S} \setminus [\ell]$ (Proposition 2.2), we have

$$n_0 + t \quad \leq \quad 1 + (|\mathcal{P}_S| - x) + |\mathcal{L}_S| \tag{11}$$

(here we estimate the number of outer skew / tangent lines in question first, then the number of inner ones), whence $2n_0 + t \leq 2|\mathcal{L}_S| + |\mathcal{P}_S| - x + 2 = 2|\mathcal{S}| - |\mathcal{P}_S| - x + 2$. Combined with the previous inequality we obtain

$$(q+1-x)(2q+x) \leq (q-x)|\mathcal{P}_S| + 8q - x - 8 \leq (q-x)(2q-3) + 8q - x - 8, \tag{12}$$

whence

$$x^2 - (q-1)x + 3q - 8 \geq 0$$

follows. The value of the left hand side is $22 - 2q$ for $x = 5$ and $x = q-6$. Therefore, as x is an integer, we conclude that for $q \geq 12$, $x \leq 4$ or $x \geq q-5$ holds. It remains to show that for $q = 11$, we cannot have a five-secant line.

Suppose to the contrary that $q = 11$ and $|\ell \cap \mathcal{P}_\mathcal{S}| = 5$ for a line ℓ. In this case, equality holds in all estimates above, thus by (10), every line intersecting ℓ in an outer point intersects $\mathcal{P}_\mathcal{S}$ in 0, 1 or 2 points; by $n_0 = |\mathcal{L}_\mathcal{S}| + 1$, every line of $\mathcal{L}_\mathcal{S}$ is skew to $\mathcal{P}_\mathcal{S}$, and there exists an outer skew line; by (11), every inner point not on ℓ is incident with exactly one outer tangent line; and by (12), $|\mathcal{P}_\mathcal{S}| = 2q-3 = 19$ and $|\mathcal{L}_\mathcal{S}| = 2q-2 = 20$. Suppose that there is a line $\ell' \neq \ell$ such that $|\ell' \cap \mathcal{P}_\mathcal{S}| = m \geq 3$.

Then $\ell \cap \ell' =: Q \in \mathcal{P}_\mathcal{S}$. Looking around from Q, as every line of $[Q]$ is outer (of which only one may be a tangent to $\mathcal{P}_\mathcal{S}$), we see that $19 = |\mathcal{P}_\mathcal{S}| \geq m + 4 + (11 - 2)$, hence $m \leq 6$. Thus the following constraints are applicable: $n_0 = 21$, $n_5 \geq 1$, $n_i = 0$ $(7 \leq i \leq 12)$. If we add these to BASIC IP, GLPK immediately shows it to be infeasible. □

Proposition 3.2. *Let $q \geq 11$. If $|\mathcal{S}| \leq 4q - 5$, then there exist two lines intersecting $\mathcal{P}_S$ in at least $q - 5$ points.*

Proof. By Proposition 3.1 every line intersects $\mathcal{S}$ in either at most 4 or at least $q - 5$ points. Let ℓ be the longest secant to $\mathcal{S}$ and $x = |[\ell] \cap \mathcal{P}_S|$. Suppose to the contrary that every line other than ℓ intersects $\mathcal{P}_S$ in at most 4 points. Clearly $x \geq 2$; note that $x \leq 4$ is also possible. By Proposition 2.10, $x \leq q - 2$. For this proof, let n_i denote the number of i-secants to $\mathcal{P}_S$ different from ℓ. For notational convenience, let $n_0 = s$ and $n_1 = t$, and let $b = |\mathcal{P}_S|$. Then the standard equations (adapted to this situation) yield

$$\begin{aligned} \sum_{i=2}^{4} n_i &= q^2 + q + 1 - s - t - 1 = q^2 + q - (s + t), \\ \sum_{i=2}^{4} i n_i &= (q + 1)b - t - x, \\ \sum_{i=2}^{4} i(i - 1)n_i &= b(b - 1) - x(x - 1). \end{aligned}$$

Thus

$$\begin{aligned} 0 &\leq \sum_{i=2}^{4} (i - 2)(4 - i)n_i = -\sum_{i=2}^{4} i(i - 1)n_i + 5\sum_{i=2}^{4} i n_i - 8\sum_{i=2}^{4} n_i \\ &= -b^2 + (5q + 6)b + x(x - 6) + 3(s + t) + 5s - 8(q^2 + q). \end{aligned}$$

We recall $x \leq q-2$ and the estimates of Proposition 2.4: $s \leq |\mathcal{L}_\mathcal{S}|+1 \leq 4q-4-|\mathcal{P}_\mathcal{S}|$; $s + t \leq |\mathcal{S}| + 1 \leq 4q - 4$. Then

$$\begin{aligned} 0 &\leq -b^2 + (5q + 6)b + x(x - 6) + 3(s + t) + 5s - 8(q^2 + q) \\ &\leq -b^2 + (5q + 6)b + (q - 2)(q - 8) + 3(4q - 4) + 5(4q - 4 - b) - 8(q^2 + q) \\ &= -b^2 + (5q + 1)b - 7q^2 + 14q - 16. \end{aligned}$$

As the right-hand side expression is increasing in b under $b \leq (5q + 1)/2$, by substituting $b = 2q - 3$ we obtain $0 \leq -q^2 + 13q - 28$, a contradiction for $q \geq 11$. □

Lemma 3.3. *Let e and f be two lines intersecting $\mathcal{P}_\mathcal{S}$ in at least two points. Then, with Notation 2.8,*

$$r \geq kl - (k + l) - (|\mathcal{S}| - 3q) - 2.$$

Proof. There are $kl - r$ lines joining E_o and F_o skew to $\mathcal{P}_\mathcal{S}$. Through P, there are at least $q - 1 - |Z|$ tangent or skew lines to $\mathcal{P}_\mathcal{S}$ (depending on whether or not $P \in \mathcal{P}_\mathcal{S}$). As there can be only one outer skew line and at most one outer tangent on P (if $P \in \mathcal{P}_\mathcal{S}$), we get $|\mathcal{L}_\mathcal{S}| \geq kl - r + q - 1 - |Z| - 1 - I_P$. We obtain the asserted inequality using $|Z| = |\mathcal{P}_\mathcal{S}| - (2q - k - l) - I_P$ (see (6)), $|\mathcal{S}| = |\mathcal{P}_\mathcal{S}| + |\mathcal{L}_\mathcal{S}|$ and rearranging. □

In the upcoming part of this section, as any superset of a resolving set is clearly a resolving set, we assume $|\mathcal{S}| = 4q - 5$, hence either $|\mathcal{P}_\mathcal{S}| = 2q - 4$ and $|\mathcal{L}_\mathcal{S}| = 2q - 1$, or $|\mathcal{P}_\mathcal{S}| = 2q - 3$ and $|\mathcal{L}_\mathcal{S}| = 2q - 2$. As $|\mathcal{S}| = 4q - 5$, we have $|\mathcal{P}_\mathcal{S}| = |\mathcal{S}| - |\mathcal{L}_\mathcal{S}| = 4q - 5 - |\mathcal{L}_\mathcal{S}|$. Recall $|Z| = |\mathcal{P}_\mathcal{S}| - (2q - k - l) - I_P$ (6). Thus we get

$$|Z| = 2q + k + l - |\mathcal{L}_\mathcal{S}| - 5 - I_P. \tag{13}$$

Since $|Z| \geq 0$ and $|\mathcal{L}_\mathcal{S}| \geq 2q - 2$, we obtain

$$k + l \geq 3 + I_P \geq 3. \tag{14}$$

Lemma 3.4. *Assume $|\mathcal{S}| = 4q - 5$, $q \geq 11$, and let e and f be two lines both intersecting $\mathcal{P}_\mathcal{S}$ in at least $q - 5$ points. Then, with Notation 2.8, we have one of the following possibilities:*

- *$P \notin \mathcal{P}_\mathcal{S}$, $k = 4$, $l = 5$, $q = 11$ and $|\mathcal{L}_\mathcal{S}| = 2q - 2$, or*
- *$P \notin \mathcal{P}_\mathcal{S}$, $k = l = 5$, $q \in \{11, 12\}$ and $|\mathcal{L}_\mathcal{S}| = 2q - 2$, or*
- *$P \in \mathcal{P}_\mathcal{S}$, $k = 5$, $l = 6$, $q = 11$ and $|\mathcal{L}_\mathcal{S}| = 2q - 2$, or*
- *$P \in \mathcal{P}_\mathcal{S}$, $k = l = 6$, $q \in \{11, 12\}$ and if $q = 12$, then $|\mathcal{L}_\mathcal{S}| = 2q - 2$.*

Proof. Clearly, $k, l \leq 6 - I_P$ and, by interchanging the roles of e and f, we may suppose $k \leq l$. Let us recall (2) from Lemma 2.9:

$$|\mathcal{L}_\mathcal{S}| \geq (k + l - 1)q + k + l - kl - 2 - I_P - (k + l + 1)|Z| + r.$$

Let $|\mathcal{L}_\mathcal{S}| = 2q - 2 + \varepsilon$ (where $\varepsilon \in \{0, 1\}$). By (13), $|Z| = 2q + (k + l) - |\mathcal{L}_\mathcal{S}| - 5 - I_P = k + l - 3 - I_P - \varepsilon$. Substituting these we easily obtain

$$(k + l - 3)q \leq (k + l - 3)(k + l) + kl - 3 - (\varepsilon + I_P)(k + l) - r - 2\varepsilon. \tag{15}$$

Recall that (14) claims $k + l \geq 3$. Note that $k + l = 3$ is not possible as in this case $kl - 3 < 0$, hence the right-hand side would be negative. Thus we may assume $k + l \geq 4$.

As $kl \leq (k + l)^2/4$ and $r \geq 0$, we get

$$\begin{aligned} & (k + l - 3)q \\ \leq \; & (k + l - 3)(k + l) + \frac{(k + l)^2}{4} - 3 - (\varepsilon + I_P)(k + l) - 2\varepsilon \\ = \; & (k + l - 3)(k + l) + \frac{(k + l + 3)(k + l - 3)}{4} + \frac{9}{4} - 3 - (\varepsilon + I_P)(k + l) - 2\varepsilon, \end{aligned}$$

so

$$q \le \frac{5(k+l)+3}{4} - \frac{3}{4(k+l-3)} - \frac{(\varepsilon + I_P)(k+l)+2\varepsilon}{k+l-3}.$$

If $k+l \le 8$, this contradicts $q \ge 11$. If $k+l = 9$, it is trivial to calculate that $q = 11$ and $\varepsilon = I_P = 0$ must hold. We suppose $k+l \ge 10$ in what follows. Substitute $r \ge kl+3-k-l-q$ (cf. Lemma 3.3) into (15) to obtain

$$(k+l-4)q \le (k+l)(k+l-2) - 6 - (\varepsilon + I_P)(k+l) - 2\varepsilon. \tag{16}$$

If $k+l = 10$ we get $6q \le 74 - 10(\varepsilon + I_P) - 2\varepsilon$, which yields $q \in \{11, 12\}$ and $\varepsilon = I_P = 0$. If $k+l \ge 11$, then $l = 6$, whence $P \in \mathcal{P}_\mathcal{S}$ follows; that is, $I_P = 1$. Substituting $k+l = 11$ and $I_P = 1$ into (16) we get $7q \le 82 - 13\varepsilon$, hence $q = 11$ and $\varepsilon = 0$. Repeating this with $k+l = 12$ we get $8q \le 102 - 14\varepsilon$, hence either $q = 12$ and $\varepsilon = 0$, or $q = 11$. □

Theorem 3.5. *The metric dimension of any projective plane of order $q \ge 11$ is $4q-4$.*

Proof. Suppose to the contrary that $\mathcal{S}$ is a resolving set of size $4q-5$. Then Proposition 3.2 assures the existence of two 'long' secants (with at least $q-5$ inner points on each). Lemma 3.4 immediately gives a contradiction for $q \ge 13$, so we assume $q \in \{11, 12\}$.

We examine the case $q = 11$ first. Considering the possible cases according to Lemma 3.4, we see that there can be at most one 7-secant and all other 'long' secants (if there are any) are 6-secants. Consider BASIC IP extended with $n_6 + n_7 \ge 2$ (Proposition 3.2), $n_5 = 0$ (Proposition 3.1), $n_7 \le 1$ (Lemma 3.4), $n_8 = n_9 = 0$ (Lemma 3.4) with the objective function $\min n_6 + n_7$, where either $|\mathcal{L}_\mathcal{S}| = 2q-2$ or $|\mathcal{L}_\mathcal{S}| = 2q-1$. For $|\mathcal{L}_\mathcal{S}| = 2q-1$ the model is infeasible, while for $|\mathcal{L}_\mathcal{S}| = 2q-2$ the minimum of $n_6 + n_7$ is 5. Counting the points of $\mathcal{P}_\mathcal{S}$ on these lines we get $2q-3 = 19 = |\mathcal{P}_\mathcal{S}| \ge 6 + (6-1) + (6-2) + (6-3) + (6-4) = 20$, a contradiction.

Let us now examine $q = 12$. Then $|\mathcal{L}_\mathcal{S}| = 2q-2 = 22$, $|\mathcal{P}_\mathcal{S}| = 2q-3 = 21$, and Proposition 3.4 yields that there can be at most one 8-secant and all other 'long' lines are 7-secants. Similarly as in the previous case, we solve BASIC IP extended with the constraints $n_7 + n_8 \ge 2$, $n_5 = n_6 = 0$, $n_8 \le 1$, $n_9 = n_{10} = 0$ for the objective function $\min n_7 + n_8$. This gives $n_7 + n_8 \ge 4$, whence $21 = |\mathcal{P}_\mathcal{S}| \ge 7 + 7 - 1 + 7 - 2 + 7 - 3 = 22$, a contradiction. □

4 Planes of very small order: $2 \le q \le 9$

Let us recall that the general construction of size $4q-4$ works for $q \ge 3$, and that Proposition 2.5 yields $|\mathcal{S}| \ge 4q + 4 - 2\lfloor 2|\mathcal{S}|/q \rfloor$. If Π_q is a given projective plane of order q, one may try to compute the metric dimension of the incidence graph G of Π_q using a simple IP model as follows. Let $\mathcal{P}$ and $\mathcal{L}$ be the set of points and lines on Π_q, respectively. Let $\mathcal{S}$ be a resolving set for G. As seen at the beginning of Section 2, the distance lists of two distinct points P and P' with respect to $\mathcal{S}$ are different if and only if $P \in \mathcal{S}$ or $P' \in \mathcal{S}$ or $\mathcal{S}$ contains a line from $([P] \cup [P']) \setminus \{PP'\}$; dually, the analogous statement holds for lines.

Define the binary variables x_v for all $v \in V = \mathcal{P} \cup \mathcal{L}$, which shall indicate whether or not v is in the resolving set. For all $P, P' \in \mathcal{P}$, $P \neq P'$ and $\ell, \ell' \in \mathcal{L}$, $\ell \neq \ell'$ add the constraints

$$\begin{aligned} x_P + x_{P'} + \sum_{\ell \in ([P] \cup [P']) \setminus PP'} x_\ell &\geq 1, \\ x_\ell + x_{\ell'} + \sum_{P \in ([\ell] \cup [\ell']) \setminus \ell \cap \ell'} x_P &\geq 1. \end{aligned}$$

Then the metric dimension of Π_q is $\min \sum_{v \in V} x_v$ (this is the objective function). We will refer to this model as RESSET IP in the following. Let us remark that an IP formulation for finding the metric dimension of a general graph is used in [10]. If we apply it to the incidence graph of projective planes and remove the constraints that are trivially satisfied due to our graph being bipartite, we obtain RESSET IP.

Observe that RESSET IP contains $2(q^2+q+1)$ variables and $2\binom{q^2+q+1}{2}$ constraints (plus $4(q^2+q+1)$ more due the the variables being binary), which both grow very quickly in q. Also, RESSET IP makes no use of the symmetries of the plane. $\mathrm{PG}(2,q)$ as a cyclic projective plane may be constructed via perfect difference sets, which can be found in [9]. Non-Desarguesian planes of small order can be found on the web page of Eric Moorhouse [16], where the planes are given by listing the lines as subsets of the point set identified with $\{0, 1, \ldots, q^2 + q\}$.

Next we give the results in detail. In some cases, we added some notes just to complement the computer results. Let us note that we have used a standard PC for the computations.

4.1 $\mu(\mathrm{PG}(2, 2)) = 5 = 4q - 3$

We leave the proof of $\mu(\mathrm{PG}(2, 2)) = 5$ to the interested reader. It is fast both by hand and by computer.

4.2 $\mu(\mathrm{PG}(2, 3)) = 8 = 4q - 4$

Gurobi solves RESSET IP in much less than a second; however, it can be proved quickly by hand as well, as for $|\mathcal{S}| = 4q - 5 = 7$, Proposition 2.5 gives $7 \geq 16 - 2\lfloor 14/3 \rfloor = 8$, a contradiction.

4.3 $\mu(\mathrm{PG}(2, 4)) = 10 = 4q - 6$

It took only 0.17 seconds for Gurobi to solve RESSET IP; however, the lower bound is quick again. For $|\mathcal{S}| = 4q - 7 = 9$, Proposition 2.5 gives $9 \geq 20 - 2\lfloor 18/4 \rfloor = 12$, a contradiction. As for an upper bound, we give an attractive construction using hyperovals.

An oval (or hyperoval) in a projective plane of order q is a set of $q+1$ (or $q+2$) points in general position; that is, no three of them are collinear. In $\mathrm{PG}(2,q)$, ovals exist for every value of q, while hyperovals exist if and only if q is even. Clearly, all the $q+1$ lines through a point of a hyperoval $\mathcal{O}$ must contain exactly one other point of $\mathcal{O}$, so hyperovals do not admit tangents. Hence on a point $P \notin \mathcal{O}$, there are $(q+2)/2$ two-secants and hence $q/2$ skew lines to $\mathcal{O}$, and there are exactly $\binom{q+2}{2}$ two-secants to $\mathcal{O}$.

Thus a hyperoval $\mathcal{O}$ in $\mathrm{PG}(2,4)$ has six points, $\binom{6}{2} = 15$ secants and $4^2+4+1-15 = 6$ skew lines, through any point $P \notin \mathcal{O}$ there pass exactly two skew lines; thus the set $\mathcal{O}^d$ of skew lines form a dual hyperoval. Now let $P \in \mathcal{O}$ and $\ell \in \mathcal{O}^d$ be arbitrary, and let $\mathcal{P}_S = \mathcal{O} \setminus \{P\}$, $\mathcal{L}_S = \mathcal{O}^d \setminus \{\ell\}$, $\mathcal{S} = \mathcal{P}_S \cup \mathcal{L}_S$. Clearly, ℓ is the only outer skew line to $\mathcal{P}_S$, and there is precisely one tangent line on every point $R \in \mathcal{P}_S$ (namely PR). Thus P1 and P2 hold. Dually, P1' and P2' also hold, thus $\mathcal{S}$ is a resolving set of size ten.

4.4 $\mu(\mathrm{PG}(2,5)) = 15 = 4q-5$.

Gurobi solved RESSET IP in 15 seconds. In this case, Proposition 2.5 gives only $|\mathcal{S}| \geq 4q-6$, so to prove the lower bound, the computer search is much more effective than trying to improve it theoretically. As for the upper bound, we give a construction based on ovals in detail, since we also find it reasonably simple and attractive.

Take an oval $\mathcal{O}$. Then $\mathcal{O}$ contains six points, and there are six tangents, $\binom{6}{2} = 15$ two-secants and $31-6-15 = 10$ skew lines to $\mathcal{O}$. Through a point of $\mathcal{O}$, there is one tangent and five 2-secants, through an internal point of $\mathcal{O}$, there are three skew lines and three 2-secants, and through an external point of $\mathcal{O}$, there are two tangents, two skew lines and two 2-secants (with respect to $\mathcal{O}$). Let ℓ be a tangent to $\mathcal{O}$, let P be one of the five external points to $\mathcal{O}$ on e, and let Q be an internal point of $\mathcal{O}$ on one of the skew lines through P. We define $\mathcal{P}_\mathcal{S}$ and $\mathcal{L}_\mathcal{S}$ to be $\mathcal{O} \cup \{Q\}$ and the set of the eight skew lines on the external points of e different from P, respectively.

Let us now check the requirements of Proposition 2.2 to see that $\mathcal{P}_\mathcal{S} \cup \mathcal{L}_\mathcal{S}$ is a resolving set for $\mathrm{PG}(2,5)$ of size 15. (P1): Clearly, there is one outer skew line to $\mathcal{P}_\mathcal{S}$; namely, PQ. (P1'): On each point not in $\mathcal{O} \cup [\ell]$ there are at least two skew lines to $\mathcal{O}$, and each intersects ℓ in an external point of $\mathcal{O}$. Hence at most one of these skew lines is not in $\mathcal{L}_\mathcal{S}$, thus the only outer point not covered by $\mathcal{L}_\mathcal{S}$ is P. (P2): On a point of $\mathcal{O}$, there is precisely one tangent to $\mathcal{P}_\mathcal{S}$ (the unique tangent to $\mathcal{O}$); on Q, there are three tangents to $\mathcal{O}$, but two of them are in $\mathcal{L}_\mathcal{S}$. (P2'): As all inner lines are skew to $\mathcal{O}$, they contain three internal and three external points of $\mathcal{O}$. Let $e \in \mathcal{L}_\mathcal{S}$. Then the internal points of e are covered by at least two lines of $\mathcal{L}_\mathcal{S}$, as well as the external point $\ell \cap e$. Hence the only problem could be if the remaining two external points on e both laid on the skew lines through P (and so the only line of $\mathcal{L}_\mathcal{S}$ covering them would be e). In this case, each of the three lines joining P to an internal point on e would be a 2-secant to $\mathcal{O}$ (recall that there are two skew lines on P, and a tangent line cannot contain an internal point), hence the number of 2-secants on P would be three; but it is two, a contradiction.

4.5 $\mu(\mathrm{PG}(2,7)) = 23 = 4q - 5$.

When trying to solve RESSET IP, Gurobi found a construction of size 23 in about five minutes, but the best lower bound it could obtain after running for 332 hours was 20. Proposition 2.5 gives only $|\mathcal{S}| > 4q - 8 = 20$, so we need another approach to suffice. One possible way is to use combinatorial arguments and symmetries of the plane to reduce the search space.

Suppose now that $\mathcal{S}$ is a resolving set of size $4q - 6 = 22$. Proposition 2.5 gives $|\mathcal{P}_\mathcal{S}| \geq 2q + 2 - \lfloor 2|S|/q \rfloor = 10$. BASIC IP (with arbitrary objective function) is infeasible with $|\mathcal{P}_\mathcal{S}| = 10$, $|\mathcal{L}_\mathcal{S}| = 12$, hence we may assume $|\mathcal{P}_\mathcal{S}| = |\mathcal{L}_\mathcal{S}| = 11 = 2q - 3$. Let ℓ be a longest secant to $\mathcal{P}_\mathcal{S}$ with $t = |\ell \cap \mathcal{P}_\mathcal{S}|$. Proposition 2.10 assures $n_6 = n_7 = n_8 = 0$, so $t \leq 5$. As BASIC IP is infeasible if we add the constraint $n_4 + n_5 = 0$, we obtain $t \geq 4$.

Suppose that every line joining a point of $\mathcal{P}_\mathcal{S} \setminus [\ell]$ to a point of $[\ell] \setminus \mathcal{P}_\mathcal{S}$ is a tangent to $\mathcal{P}_\mathcal{S}$. Then we find $(11-t)(8-t)$ tangents, from which at most $11-t$ may be outer (by (P2) of Proposition 2.2), thus we find $11 = |\mathcal{L}_\mathcal{S}| \geq (11-t)(8-t) - (11-t) = 77 - 18t + t^2 \geq 12$ (recall $t \in \{4,5\}$), a contradiction. Hence we find a line ℓ' which contains at least two inner points X and Y, and meets ℓ in an outer point O. Let A and B be two outer points of $[\ell] \setminus \{O\}$. As the group of collineations of $\mathrm{PG}(2,q)$ is transitive on quadruples in general position, we may fix any such quadruple to take the role of (X,Y,A,B), which also fixes $O = XY \cap AB$; so we may add the constraints $x_X + x_Y = 2$, $x_A + x_B + x_O = 0$ to RESSET IP. Let C, D, E, F, G denote the five further points of ℓ. There are four collineations which setwise stabilize both $\{X,Y\}$ and $\{A,B\}$, hence ℓ as well, and their only non-trivial action on $\{C,D,E,F,G\}$ is, after suitable re-labelling, $\varphi = (C)(DE)(FG)$ in cycle notation. As $t \geq 4$, we may add the constraint $x_C + x_D + x_E + x_F + x_G \geq 4$. Applying φ, we may assume $D \in \mathcal{P}_\mathcal{S}$ and $F \in \mathcal{P}_\mathcal{S}$, thus we may add the constraints $x_D = 1$ and $x_F = 1$. Moreover, as ℓ is a longest secant to $\mathcal{P}_\mathcal{S}$, for each line r we may add the constraint $\sum_{P \in r} x_P \leq \sum_{P \in \ell} x_P$. By the assumption $|\mathcal{P}_\mathcal{S}| = |\mathcal{L}_\mathcal{S}| = 11$, we may also add $\sum_{P \in \mathcal{P}} x_P = 11$ and $\sum_{\ell \in \mathcal{L}} x_\ell = 11$. Then RESSET IP extended with these additional constraints is proved infeasible by Gurobi in about 110 minutes.

As for the upper bound, we give the construction of size 23 found by Gurobi, where (x,y), (m) and (∞) denote the affine point with coordinates x and y, the common point of the lines of slope m and that of the vertical lines, respectively, and $[m,b]$, $[c]$ and ℓ_∞ denote the lines of equation $y = mx + b$, $x = c$, and the line at infinity, respectively:

$$\begin{aligned}
\mathcal{P}_\mathcal{S} &= \{(0,0),(0,1),(0,3),(0,6),(2,0),(2,4),(6,4),(2,5),(3,5),(0),(1)\},\\
\mathcal{L}_\mathcal{S} &= \{[2,2],[2,4],[2,5],[3,2],[3,4],[4,2],[4,5],[5,5],[6,4],[6,5],[4],[5]\}.
\end{aligned}$$

Let us note that there is one line (namely $[0]$) which intersects $\mathcal{P}_\mathcal{S}$ in four points, all other lines contain at most three inner points; there are six but no seven points in

$\mathcal{P}_{\mathcal{S}}$ in general position; $n_0 = 13 = |\mathcal{L}_{\mathcal{S}}| + 1$, so all lines of $\mathcal{L}_{\mathcal{S}}$ are skew to $\mathcal{P}_{\mathcal{S}}$; $n_1 = 10$ and $n_3 = 8$. We do not know of a resolving set of size 23 with a more straightforward structure.

4.6 $\mu(\mathrm{PG}(2, 8)) = 28 = 4q - 4$.

In this case, we found it easier to enhance the theoretical methods and perform detailed case analysis than to reduce the search space in due extent. Let us formulate a detailed and actualized version of Lemma 2.9 with no restrictions on the order of the plane (for the sake of future use).

Lemma 4.1. *Suppose that $|\mathcal{P}_{\mathcal{S}}| = 2q - 3$. Let e and f be two lines, both intersecting $|\mathcal{P}_{\mathcal{S}}|$ in at least two points. Then, with Notation 2.8,*

$$|Z| = k + l - 3 - I_P, \quad (17)$$
$$0 \le |\mathcal{L}_{\mathcal{S}}| + (k+l)^2 + kl + q - (q + 3 + I_P)(k + l) - 1 - r, \quad (18)$$
$$0 \le |\mathcal{L}_{\mathcal{S}}| + (k+l)^2 + |\mathcal{S}| - 2q + 1 - (q + 2 + I_P)(k + l), \quad (19)$$
$$0 \le |\mathcal{L}_{\mathcal{S}}| + kl - (k + l) - (2 + I_P)q + I_P + 3, \text{ and} \quad (20)$$
$$0 \le |\mathcal{L}_{\mathcal{S}}| + kl - 2(k + l) - (2 + I_P)q + 2I_P + 8 \text{ if } Z \text{ is not contained in a line.} \quad (21)$$

Proof. We simply substitute $|\mathcal{P}_{\mathcal{S}}| = 2q - 3$ into $|Z| = |\mathcal{P}_{\mathcal{S}}| - (2q - k - l) - I_P$ (6) to obtain (17). $|\mathcal{P}_{\mathcal{S}}| = 2q - 3$ and (17) applied for (2) of Lemma 2.9 yields (18). Then (19) follows from (18) and $r \ge kl - (k + l) - (|\mathcal{S}| - 3q) - 2$ (Lemma 3.3). Finally, (20) and (21) follow from $|\mathcal{P}_{\mathcal{S}}| = 2q - 3$ and (17) applied for (3), taking into account (4) and (5) (which also yield $\tau(Z) \le |Z|q + 1$). □

Note that e and f are $q + I_P - k$ and $q + I_P - l$ secants.

Throughout the proofs we will obtain or assume more and more restrictions on the n_is which then can be added to BASIC IP. We will refer to BASIC IP extended with all appropriate new constraints by the term BASIC IP+.

Let us now fix $q = 8$. Suppose to the contrary that $\mathcal{S}$ is a resolving set with $|\mathcal{S}| = 4q - 5 = 27$. Proposition 2.5 gives $|\mathcal{P}_{\mathcal{S}}| \ge 12$. BASIC IP with $|\mathcal{P}_{\mathcal{S}}| = 12$ has a unique integer solution, namely $n_0 = 16$, $n_1 = 11$, $n_2 = 45$, $n_3 = n_4 = n_5 = n_6 = 0$, $n_7 = 1$. Let ℓ be the 7-secant line, let $W = \mathcal{P}_{\mathcal{S}} \setminus [\ell]$, $|W| = 5$. Then W cannot have three or more collinear points as there are no other three or longer secants. Looking around from a point of W, among the four lines containing another point of W there are some intersecting ℓ in an inner point, hence resulting in a 3-secant, a contradiction. Thus $|\mathcal{P}_{\mathcal{S}}| = 13 = 2q - 3$, $|\mathcal{L}_{\mathcal{S}}| = 14 = 2q - 2$.

Let us substitute $q = 8$ and $|\mathcal{L}_{\mathcal{S}}| = 14$ into (18) and (20) of Lemma 4.1 to obtain

$$0 \le 21 + (k+l)^2 + kl - (11 + I_P)(k + l) - r, \text{ and} \quad (22)$$
$$0 \le 1 + kl - (k + l) - 7I_P. \quad (23)$$
$$(24)$$

Recall that Proposition 2.10 assures $n_7 = n_8 = n_9 = 0$.

Case 1: $n_6 \geq 1$**.**

Let us assume $n_6 \geq 1$, and let e be a 6-secant. Choose an arbitrary line f that is a t-secant to $\mathcal{P}_\mathcal{S}$, $2 \leq t \leq 6$. Suppose first $e \cap f \notin \mathcal{P}_\mathcal{S}$; thus $I_P = 0$, $k = 2$, $l = 8 - t$. Then (22) with $r \geq 0$ gives $(l-4)(l-1) - 1 \geq 0$, whence $l \geq 5$ and so $t \leq 3$ follows. Suppose now $e \cap f \in \mathcal{P}_\mathcal{S}$; thus $I_P = 1$, $k = 3$, $l = 9 - t$. Then (23) gives $2l - 9 \geq 0$, whence $l \geq 5$ and so $t \leq 4$ follows. Furthermore, (21) yields $l \geq 6$, that is, $t \leq 3$. Summing up, we conclude that $t \leq 4$, and if $t = 4$, then f intersects e in an inner point and Z is collinear. This also yields that we have $n_6 = 1$ and $n_5 = 0$.

Suppose now that $n_4 \geq 1$, and let f be a 4-secant. Then $I_P = 1$, $k = 3$, $l = 5$ by the previous observations yield, and the $|Z| = 4$ points outside $e \cup f$ are contained in a line, say, h. As h cannot intersect e or f in an inner point (it would be a 5-secant then), it must intersect both in an outer point, and hence it is a 4-secant intersecting e in an outer point, a contradiction. Thus $n_4 = 0$.

The above arguments show that, under the assumption $n_6 \geq 1$, $n_4 \leq 73(1 - n_6)$, $n_5 \leq 73(1 - n_6)$ and $n_6 = 1$ can be added to BASIC IP+ Minimizing the objective function n_0 gives $\min n_0 = 15 = |\mathcal{L}_\mathcal{S}| + 1$, so all lines of $\mathcal{L}_\mathcal{S}$ must be skew to $\mathcal{P}_\mathcal{S}$. Let W be an outer point on e. As $n_4 = 0$ and $|\mathcal{P}_\mathcal{S} \setminus [e]| = 7$ is not divisible by three, there must be a line g through W intersecting $|\mathcal{P}_\mathcal{S}| \setminus [e]$ in t points, $t \in \{1, 2\}$. Let $U \in \mathcal{P}_\mathcal{S} \cap [h]$. Then the three lines joining U with an outer point of e may cover at most six points of $\mathcal{P}_\mathcal{S} \setminus [e]$, hence there is a line f through U that intersects $\mathcal{P}_\mathcal{S}$ in at least, and thus exactly three points. In this case $|Z| = 5$, hence there are at least two lines on the inner point P that are tangents to $\mathcal{P}_\mathcal{S}$; at most one of them may be outer, so $\mathcal{L}_\mathcal{S}$ should contain a tangent line, a contradiction, and thus $n_6 = 0$ must hold.

Case 2: $n_6 = 0$**.**

Adding this single constraint to BASIC IP+, we get $n_5 \leq 3$ and $n_0 \geq 13$.

`Case 2.1:` $n_5 = 3$. Suppose $n_5 = 3$, and let e_1, e_2 and e_3 be the 5-secants to $\mathcal{P}_\mathcal{S}$. Suppose that $e_1 \cap e_2 \notin \mathcal{P}_\mathcal{S}$. Then, as $|\mathcal{P}_\mathcal{S}| = 13 = 2 \cdot 5 + 3$, e_3 must intersect both e_1 and e_2 in an inner point; say, R_1 and R_2, respectively. Then there are three tangents to $\mathcal{P}_\mathcal{S}$ through R_1 (the three lines joining R_1 with the three outer points on e_2 different from $e_1 \cap e_2$), at most one of which may be outer; hence there are two inner tangents on R_1. The same is true for R_2, whence $14 = |\mathcal{L}_\mathcal{S}| \geq n_0 - 1 + 4 \geq 16$, a contradiction.

Thus any two of e_1, e_2 and e_3 must intersect in an inner point. Let R_1, R_2, and R_3 be the three (inner) points of intersections of the three 5-secants. Clearly, on each of them there are exactly 4 tangents, so we see that $\mathcal{L}_\mathcal{S}$ contains at least 9 tangents. But then $14 = |\mathcal{L}_\mathcal{S}| \geq n_0 - 1 + 9 \geq 21$, a contradiction. Hence $n_5 \leq 2$.

`Case 2.2:` $n_5 = 2$. Let e and f be the two 5-secants. Assume $P \in \mathcal{P}_\mathcal{S}$ first. Then $|Z| = 4$, hence there are at least 3 tangents on P. As $n_0 \geq 13$, we get $14 = |\mathcal{L}_\mathcal{S}| \geq 13 - 1 + 3 - 1 = 14$, so $n_0 = 13$, and $\mathcal{L}_\mathcal{S}$ contains exactly 12 skew lines and two tangents on P. Adding $n_5 = 2$ and $n_0 = 13$ to BASIC IP+, it has only one solution, yielding $n_3 = 0$ and $n_4 = 3$. If Z is collinear, then $\mathcal{P}_\mathcal{S}$ is contained in the three sides of a triangle, yielding only one 4-secant (the side containing Z), a contradiction. If no three points of Z are collinear, then $\tau(Z) = 4 \cdot 9 - \binom{4}{2}$ gives a contradiction in (3) of Lemma 2.9. Thus Z contains three points on a line h and one more point not on h. In this case $\tau(Z) = 31$ gives equality in (3), hence Z cannot intersect any line joining a point of E_i with one of F_i. As $n_3 = 0$, h must intersect e in an outer point and f in an inner point (we may interchange e and f if necessary). Let $X \in F_i \cap [h]$. Then there are four lines joining X with a point of E_o. One of them is h, another one may be blocked by the point in $Z \setminus [h]$, so the remaining two lines are tangents to $\mathcal{P}_\mathcal{S}$. As there can be only one outer tangent line in an inner point, one of these lines must be in $|\mathcal{L}_\mathcal{S}|$ but, as seen before, there are exactly two tangents in $\mathcal{L}_\mathcal{S}$ which meet $\mathcal{P}_\mathcal{S}$ in P, a contradiction. Hence $P \notin \mathcal{P}_\mathcal{S}$.

We may assume now $P \notin \mathcal{P}_\mathcal{S}$. Then $k = l = |Z| = 3$, and (22), coming from (2) of Lemma 2.9, gives equality with $r = 0$, hence $r = 0$ and all lines containing a point of $E_o \cup F_o \cup \{P\}$ intersect Z in at most one point. This yields that $n_0 = 13$ (there are four skew lines in $[P]$, and $|E_o| \cdot |F_o| - r = 9$ between E_o and F_o). Hence, again, BASIC IP+ gives the same one and only solution as above.

A 4-secant line to $\mathcal{P}_\mathcal{S}$ must contain at least two points of Z. As $n_4 = 3$, the points of Z cannot be collinear (in this case, only one line could contain at least two points of Z), hence each 4-secants contains two points of Z, one of E_i and one of F_i. Let X_1, X_2 and X_3 be the three points of E_i covered by a 4-secant line; as $|Z| = 3$, these must be pairwise distinct. Among the three lines joining X_i to F_o, at most one may be blocked by Z, hence at least two of them are tangents; thus there is at least one inner tangent line on X_i, $i \in \{1, 2, 3\}$. This gives $|\mathcal{L}_\mathcal{S}| \geq n_0 - 1 + 3 = 15$, a contradiction, hence $n_5 \leq 1$.

`Case 2.3:` $n_5 = 1$. Now BASIC IP+ contains $n_7 = n_6 = 0$ and $n_5 = 1$ as extra constraints and it gives $n_0 \geq 14$, $n_4 \geq 2$. Let e be a 5-secant and let f be a 4-secant. Suppose first $P \in \mathcal{P}_\mathcal{S}$. Then $k = 4$, $l = 5$, $|Z| = 5$. Thus on $[P]$, there are at least two tangents to $\mathcal{P}_\mathcal{S}$, so one of them is in $\mathcal{L}_\mathcal{S}$; as $14 = |\mathcal{L}_\mathcal{S}| \geq n_0 - 1 + 1 \geq 14$, we obtain that $n_0 = 14$, and $\mathcal{L}_\mathcal{S}$ consists of 13 skew lines and one tangent line through P. With $n_0 = 14$, BASIC IP+ gives the unique solution $n_4 = 4$, $n_3 = 2$, $n_2 = 38$, $n_1 = 14$. Let $h \neq f$ be a 4-secant. Suppose that $h \cap e \in E_i =: X$. If $h \cap f \in F_i$, then $|[h] \cap Z| = 2$, so at most three among the five lines joining X with F_o may be blocked by Z, hence there must be an inner tangent on X, a contradiction. Also, if $h \cap f \in F_o$, then $|[h] \cap Z| = 3$, so at most two among the four lines joining X with F_o, different from h, may be blocked by Z, and we get a similar contradiction. Clearly, $P \in h$ is not possible (as there would be too many inner tangents on P). Thus each of the three 4-secants different from f must intersect e in an outer point. Counting the inner points on e, f and the remaining three 4-secants we thus obtain $13 = |\mathcal{P}_\mathcal{S}| \geq 5 + 3 + 3 + 2 + 1 = 14$, a contradiction. Hence no 4-secant may intersect e in an inner point; thus we may assume $P \notin \mathcal{P}_\mathcal{S}$.

Suppose now $n_4 \geq 3$. Counting $|\mathcal{P}_\mathcal{S}|$ on e and three 4-secants (none of which may cover an inner point of e) we obtain $13 = |\mathcal{P}_\mathcal{S}| \geq 5 + 4 + 3 + 2 = 14$. Thus $n_4 = 2$.

With this, the BASIC IP+ yields $n_0 = 15$, $n_1 = 13$, and $n_3 = 7$, hence $\mathcal{L}_\mathcal{S}$ contains only skew lines and, as no tangent may be inner and $n_1 = 13 = |\mathcal{P}_\mathcal{S}|$, each point of $\mathcal{P}_\mathcal{S}$ is on exactly one tangent line.

Let f and h be 4-secants, and let $R = h \cap f$, $X = h \cap e$. As seen above, $X \in E_o$. If $R \in F_o$, then $Z = [h] \cap \mathcal{P}_\mathcal{S}$, and $15 = n_0 = (|[P]| - 2 - |Z|) + (|E_o| \cdot |F_o| - 1) = 3 + 11$, a contradiction. If $R = P$, then there are $15 = n_0 = (|[P]| - 3) + (|E_o| \cdot |F_o| - r)$ skew lines to $\mathcal{P}_\mathcal{S}$, whence $r = 3$. As $|Z| = 4 > r$, there is a point in Z that does not meet any line joining a point of E_o to one of F_o, hence there is no tangent line on this point of $\mathcal{P}_\mathcal{S}$, a contradiction. Thus $R \in F_i$. Then $|[h] \cap Z| = 3$; let W be the unique point of $Z \setminus [h]$. Let $Y \in e$, and let us estimate the number of 3-secants on Y. If $Y = P$, then there are no 3-secants through Y. If $Y \in E_i$, then there is exactly one tangent on Y, so exactly three of the four lines joining Y to a point of F_o are blocked by Z. Either each of them is blocked once, and hence the line joining Y to the fourth point of Z is the only 3-secant on Y, or one of them is blocked twice, and thus it is the only 3-secant on Y. Hence each point of E_i is on exactly one 3-secant line. Let $Y \in E_o$. If $Y = X$ or $Y = e \cap RW$, then each line though Y (different from f) contains either at most one point of Z or it is h, hence none of them is a 3-secant. If Y is the third point of E_o, then the only line through it that may be a 3-secant is YW. Thus the number of 3-secant lines is at most six, in contradiction with $n_3 = 7$, so $n_5 = 1$ cannot hold.

`Case 2.4:` $n_5 = 0$. In this case, BASIC IP+ gives $n_0 = 15$ and $n_4 \geq 5$. As $n_0 = |\mathcal{L}_\mathcal{S}| + 1$, all lines of $\mathcal{L}_\mathcal{S}$ are skew to $\mathcal{P}_\mathcal{S}$. Take four 4-secants. If any two of them intersects in an outer point, then $|\mathcal{P}_\mathcal{S}| \geq 16$, a contradiction. Hence we may assume that e and f are two 4-secants that intersect in an inner point; let $[e] \cap \mathcal{P}_\mathcal{S} = \{P_1, P_2, P_3, P_4\}$, $[f] \cap \mathcal{P}_\mathcal{S} = \{P_4, P_5, P_6, P_7\}$. As the collineation group of $\mathrm{PG}(2, 8)$ is transitive on the quadruple of points in general position, we may move P_1, P_2, P_5 and P_6 to any quadruple in general position, hence we may fix e and f arbitrarily and add the following constraints to the RESSET IP:

$$\begin{aligned} \forall \ell \in \mathrm{PG}(2,8)\colon \sum_{P \in \ell} x_P \ &\leq \ 4(1 - x_\ell) \\ \sum_{1 \leq i \leq 7} x_{P_i} \ &= \ 5 \\ \sum_{P \in e} x_P \ &= \ 4 \\ \sum_{P \in f} x_P \ &= \ 4 \end{aligned}$$

Gurobi solves this extended model in about 11 seconds and shows $\sum_{v \in \mathcal{P} \cup \mathcal{L}} x_v \geq 28$; thus a resolving set of size 27 cannot exist.

4.7 $\mu(\Pi_9) = 32 = 4q - 4$.

Suppose to the contrary that $\mathcal{S}$ is a resolving set of size $4q-5 = 31$. Proposition 2.5 gives $|\mathcal{P}_\mathcal{S}| \geq 14$. Suppose $|\mathcal{P}_\mathcal{S}| = 14$, $|\mathcal{L}_\mathcal{S}| = 17$. Then BASIC IP gives $n_8 = 1$ and $n_0 = 18$, hence all lines of $\mathcal{L}_\mathcal{S}$ are skew. However, through an inner point of the 8-secant line, there are at least $9-(|\mathcal{P}_\mathcal{S}|-8) = 3$ tangents to $\mathcal{P}_\mathcal{S}$, none of which may be inner, a contradiction. Hence we may assume $|\mathcal{P}_\mathcal{S}| = 15 = 2q-3$, $|\mathcal{L}_\mathcal{S}| = 16 = 2q-2$. Lemma 4.1 then gives

$$0 \leq (k+l)^2 + kl - (12+I_P)(k+l) - r + 24, \quad (25)$$
$$0 \leq (k+l)^2 - (11+I_P)(k+l) + 30, \quad (26)$$
$$0 \leq kl - 2(k+l) + 6 - 9I_P, \text{ if } Z \text{ is not collinear.} \quad (27)$$

Recall that Proposition 2.10 assures $n_8 = n_9 = n_{10} = 0$.

Case 1: $n_7 \geq 1$.

Suppose that e is a 7-secant. Let f be a line intersecting e in an outer point, and suppose it contains at least two points of $\mathcal{P}_\mathcal{S}$. Then $I_P = 0$, $k = 2$ and $r \geq 0$ with (25) yield $(l-5)(l-1)-1 \geq 0$. As $l \geq 2$, this yields $l \geq 6$; hence a line intersecting e in an outer point may contain at most three points of $\mathcal{P}_\mathcal{S}$.

Let f be a line intersecting e in an inner point and containing another point of $\mathcal{L}_\mathcal{S}$. Then $k = 3$, $I_P = 1$; these with (25) give $(l-5)(l-1)-1 \geq 0$, whence $l \geq 6$ follows; thus a line intersecting e in an inner point contains at most four points of $\mathcal{P}_\mathcal{S}$. The observations so far yield in case of $n_7 \geq 1$ we have $n_7 = 1$, $n_6 = n_5 = 0$. Suppose now that f is a 4-secant with $e \cap f \in \mathcal{P}_\mathcal{S}$. Then (27) with $I_P = 1$, $k = 3$, $l = 6$ gives a contradiction; hence, in such case, Z must be collinear. As $|Z| = 5$, this gives a line containing at least 5 points of $\mathcal{P}_\mathcal{S}$ different from e, a contradiction. Hence we also have $n_4 = 0$. However, adding the constraints $n_7 = 1$, $n_6 = n_5 = n_4 = 0$ to BASIC IP+, it becomes infeasible. We conclude that $n_7 = 0$.

Case 2: $n_7 = 0$, $n_6 \geq 1$.

Suppose now that e is a 6-secant. Let f be another 6-secant. If $e \cap f \in \mathcal{P}_\mathcal{S}$, then $I_P = 1$, $k = l = 4$ gives a contradiction with (26). If $e \cap f \notin \mathcal{P}_\mathcal{S}$, then $I_P = 0$, $k = l = 3$ and $r \geq 0$ gives a contradiction with (25). Hence $n_6 = 1$. BASIC IP+ yields $n_0 \geq 15$ and $n_5 \leq 3$. Assume now that f is a 5-secant to $\mathcal{P}_\mathcal{S}$ and $e \cap f \in \mathcal{P}_\mathcal{S}$. Then there are at least $10 - 2 - |Z| = 3$ tangents on P to $\mathcal{P}_\mathcal{S}$, hence at least two of these must be in $\mathcal{L}_\mathcal{S}$.

`Case 2.1:` $n_6 = 1$, $n_5 = 3$. Suppose now $n_5 = 3$, and let h_1, h_2 and h_3 be the three 5-secants to $\mathcal{P}_\mathcal{S}$. Counting the points of $\mathcal{P}_\mathcal{S}$ on e and the 5-secants we get $15 = |\mathcal{P}_\mathcal{S}| \geq 6+4+3+2 = 15$, hence any two of these four lines must intersect in an inner point. Let $P_i = e \cap h_i$. Then, as $\mathcal{L}_\mathcal{S}$ contains at least two tangent lines on P_i, $1 \leq i \leq 3$, so at least six tangents altogether, we get $16 = |\mathcal{L}_\mathcal{S}| \geq n_0 - 1 + 6 \geq 20$, a contradiction. Thus $n_5 \leq 2$.

`Case 2.2:` $n_6 = 1$, $n_5 = 2$. Suppose now $n_5 = 2$. Then BASIC IP+ gives $n_0 \geq 16$. If a 5-secant meets e in an inner point, then $16 = |\mathcal{L}_\mathcal{S}| \geq n_0 - 1 + 2 = 17$, a contradiction; hence both 5-secants intersect e in an outer point. As $|\mathcal{P}_\mathcal{S}| = 15$, it follows that the two 5-secants meet in an inner point R, and $\mathcal{P}_\mathcal{S}$ is contained in the triangle formed by e and the two 5-secants, which yields $n_4 = 0$. Also, the number of tangents to $\mathcal{P}_\mathcal{S}$ through R is at least $10 - 2 - 6 = 2$, so $\mathcal{L}_\mathcal{S}$ must contain at least one tangent; hence $16 = |\mathcal{L}_\mathcal{S}| \geq n_0 - 1 + 1 \geq 16$ gives $n_0 = 16$. Adding this to BASIC IP+ we obtain $n_4 \geq 2$, a contradiction. Hence $n_5 \leq 1$.

`Case 2.3:` $n_6 = 1$, $n_5 = 1$. Suppose now $n_5 = 1$, and let f be the 5-secant. Then BASIC IP+ gives $n_0 = 17$, hence $\mathcal{L}_\mathcal{S}$ consists of 16 skew lines, so $P = e \cap f \notin \mathcal{P}_\mathcal{S}$; also we obtain from BASIC IP+ $n_1 \geq 14$. These mean that all but one point of $\mathcal{P}_\mathcal{S}$ is on exactly one (outer) tangent to $\mathcal{P}_\mathcal{S}$. Note that the tangents through the points of Z meet both e and f in an outer point; that is, either meet both in P, or join a point of E_o with one of F_o. Let t be the number of lines on P blocked by Z. Then $17 = n_0 = |[P]| - 2 - t + |E_o| \cdot |F_o| - r = 20 - t - r$, where $t \geq 1$ and $r \geq 0$. If $t = 1$, then the points of Z are on a line h containing P; clearly, h is not tangent. The $r = 2$ lines joining a point of E_o with one of F_o blocked by Z cover at most two points of Z, hence there are two points of Z that are not incident with any tangent line, a contradiction. If $t = 2$, then $r = 1$ and Z is covered by two lines containing P, call them h_1 and h_2. If, say, h_1 is a 3-secant, then only one of the inner points on h_1 can be covered by the $r = 1$ line joining E_o to F_o and blocked by Z, hence there are at least two inner points without a tangent line, a contradiction. If $t = 3$ and $r = 0$, then there are three lines through P covering Z, one of which is a 2-secant; through the two inner points on this line there are no tangents. Thus $n_5 = 0$.

`Case 2.4:` $n_6 = 1$, $n_5 = 0$. BASIC IP+ yields $n_0 = 17$, $n_1 = 15$ and $n_4 = 6$. Thus all lines of $\mathcal{L}_\mathcal{S}$ are skew, and each point of $\mathcal{P}_\mathcal{S}$ is on exactly one tangent line. Suppose that a 4-secant intersects e in an inner point P. Then there are at least $|[P]| - 2 - (|\mathcal{P}_\mathcal{S}| - 9) = 2$ tangents on P, a contradiction. Then counting the points of $\mathcal{P}_\mathcal{S}$ on e and four 4-secants we get $15 = |\mathcal{P}_\mathcal{S}| \geq 6+4+3+2+1 = 16$, a contradiction. Thus we conclude that $n_6 = 0$.

Case 3: $n_7 = n_6 = 0$, $n_5 \geq 1$.

BASIC IP+ gives $n_0 \geq 16$, hence $\mathcal{L}_\mathcal{S}$ contains at least $n_0 - 1 = 15$ skew lines and at most one tangent. Suppose that $n_5 \geq 4$, and let h_i, $1 \leq i \leq 4$ be distinct 5-secants. Suppose that two 5-secants, say, h_3 and h_4 meet in an inner point P. Then there are at least $|[P]| - 2 - (|\mathcal{P}_\mathcal{S}| - 9) = 2$ tangents on P; it follows that $\mathcal{L}_\mathcal{S}$ contains 15 skew lines and one tangent through P, and there are two 5-secants, two tangents and six 2-secants on P. From these we can see that any pair of 5-secants, except maybe for $\{h_3, h_4\}$, must intersect in an outer point. Then counting the points of $\mathcal{P}_\mathcal{S}$ on h_1, h_2, h_3 and h_4 we get $15 = |\mathcal{P}_\mathcal{S}| \geq 5 + 5 + 5 + 4 = 19$, a contradiction. Hence $n_5 \leq 3$. BASIC IP+ then yields $n_0 = 17$, $n_4 = 3$, and $n_5 = 3$. Now all lines in $\mathcal{L}_\mathcal{S}$ must be skew; hence, as seen before, any two 5-secants meet in an outer point, whence $15 = |\mathcal{P}_\mathcal{S}| \geq 5 + 5 + 5$, so $\mathcal{P}_\mathcal{S}$ is contained in the triangle formed by the three

5-secants. However, this contradicts $n_4 \geq 1$. It follows that $n_5 = 0$.

Case 4: $n_7 = n_6 = n_5 = 0$.

In this case, BASIC IP+ is infeasible, which gives a final contradiction with $|\mathcal{S}| = 4q - 5$, yielding $\mu(\Pi_9) = 4q - 4$.

5 Corollary on the metric dimension of affine planes

Let A_q denote (the incidence graph of) an arbitrary affine plane of order q. In [7], it was shown that $\mu(A_q) = 3q - 4$ if $q \geq 13$. The upper bound is due to a construction proving $\mu(A_q) \leq 3q - 4$ for all $q \geq 3$, whereas the lower bound was proved by showing an intrinsic link between resolving sets for affine and projective planes. For the sake of a compact presentation, we deliver this connection as the combination of three results of [7].

Result 5.1 ([7], Propositions 2.4, 2.5, and proof of Theorem 2.6)**.** *Assume that A_q, $q \geq 5$, contains a resolving set $\mathcal{S}_A$ of size $|\mathcal{S}_A| \leq 3q - 4$. Then the projective closure of A_q admits a resolving set $\mathcal{S} = \mathcal{P}_\mathcal{S} \cup \mathcal{L}_\mathcal{S}$ of size $|\mathcal{S}_A| + q$. Furthermore, $|\mathcal{L}_\mathcal{S}| \geq 2q - 3$ and there is a q-secant line to $\mathcal{P}_\mathcal{S}$.*

Using Result 5.1, Theorem 1.1 and computer search, we are able to prove the following.

Theorem 5.2. *The metric dimension of an affine plane of order q is $3q - 4$ except for $q = 2$, in which case it is 3.*

Proof. Recall $\mu(A_q) \leq 3q - 4$ for all $q \geq 3$. Suppose now that an affine plane A_q of order $q \geq 5$ admits a resolving set of size $|\mathcal{S}_A| \leq 3q - 5$. Apply Result 5.1 to obtain a resolving set $\mathcal{S} = \mathcal{P}_\mathcal{S} \cup \mathcal{L}_\mathcal{S}$ of size $|\mathcal{S}| \leq 4q - 5$ for a projective plane of order q.

If $q \geq 8$, this contradicts Theorem 1.1, thus $\mu(A_q) = 3q - 4$ must hold.

Suppose now $q \in \{5, 7\}$. Result 5.1 also yields $|\mathcal{L}_\mathcal{S}| \geq 2q - 3$ and that $\mathcal{P}_\mathcal{S}$ admits a q-secant line ℓ. By Proposition 2.10, this is impossible unless $|\mathcal{P}_\mathcal{S}| \geq 2q - 2$, in which case $|\mathcal{P}_\mathcal{S}| = 2q - 2$ and $|\mathcal{L}_\mathcal{S}| = 2q - 3$ follow. On the unique outer point P of ℓ, there can be at most $|\mathcal{P}_\mathcal{S}| - q = q - 2$ lines besides ℓ that meet $\mathcal{P}_\mathcal{S}$, so we find two skew lines r_1 and r_2 on P. Using the symmetries of the plane, we may fix P, ℓ, r_1 and r_2. Add $\sum_{Q \in \mathcal{P}} x_Q = 2q - 2$, $\sum_{r \in \mathcal{L}} x_r = 2q - 3$, $x_P = 0$, $\sum_{Q \in \ell} x_Q = q$, $\sum_{Q \in r_1} x_Q = 0$, $\sum_{Q \in r_2} x_Q = 0$ to RESSET IP. Gurobi proves this model infeasible for $q = 5$ and $q = 7$ in less than a second and in about two minutes, respectively. Hence $\mu(A_q) = 3q - 4$ holds for $q \in \{5, 7\}$ as well.

For $q \in \{2, 3, 4\}$, either an appropriate modification of RESSET IP or the general IP model of [10] can be solved very quickly to see that $\mu(A_2) = 3$, and $\mu(A_q) = 3q - 4$ for $q \in \{3, 4\}$. This completes the proof. □

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