AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **78(2)** (2020), Pages 297–313

Decompositions of complete equipartite graphs into cycles of lengths 3 and 6

P. Paulraja R. Srimathi

Department of Mathematics
Kalasalingam Academy of Research and Education
Krishnankoil-626126
India
ppraja56@gmail.com gsrimathi66@gmail.com

Abstract

Let G be an even graph. If α and β are non-negative integers such that $3\alpha + 6\beta = |E(G)|$, then we say that (α, β) is an *admissible pair* for G. If G admits a decomposition into α cycles of length 3 and β cycles of length 6 for every admissible pair (α, β), then we say that G admits a $\{C_3^\alpha, C_6^\beta\}$*-decomposition*. In this paper, it is proved that for $\lambda \geq 1$, with m and n at least three, the λ-fold complete m-partite graph in which each partite set has n vertices admits a $\{C_3^\alpha, C_6^\beta\}$-decomposition if and only if $\lambda n(m-1) \equiv 0 \pmod 2$ and $3\alpha+6\beta = \lambda\binom{m}{2}n^2$. This is a companion result to Huang and Fu [$(4,5)$-Cycle systems of complete multipartite graphs, *Taiwanese J. Math.* 16 (2012), 999–1006]. A similar result has also been obtained by the authors for the tensor product of complete graphs in [*Discuss. Math. Graph Theory*, doi:10.7151/dmgt.2178 (in press)].

1 Introduction

In this paper, graphs are assumed to be loopless, connected and finite. Let C_k denote the *cycle of length* k. The cycle C_3 is called a *triangle*. The complete graph on n vertices is denoted by K_n and $\overline{K}_n$ denotes the complement of K_n. A *bowtie* is a pair of triangles with a common vertex and we denote it by $(a, b, c) \cup (a, d, e)$. If $H_1, H_2, \ldots, H_k$ are edge-disjoint subgraphs of the graph G such that $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_k)$, then we say that $H_1, H_2, \ldots, H_k$ *decompose* G and we write this as $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$. If each $H_i \cong H$, $1 \leq i \leq k$, then we say that H decomposes G and we denote it by $H|G$. If each $H_i \cong C_m$, the cycle of length m, then we write $C_m|G$ and in this case we say that G has a C_m*-decomposition* or an m*-cycle decompositon*. A graph G is $\{H_1, H_2\}$*-decomposable* if the edge set $E(G)$ of G can be partitioned into $E_1, E_2, \ldots, E_k$ such that for every $i \in \{1, 2, \ldots, k\}$, $\langle E_i \rangle \simeq H_1$ or $\langle E_i \rangle \simeq H_2$. Let G be an even graph. For integers α and β the pair

ISSN: 2202-3518

(α, β) is an *admissible pair* for the graph G if $3\alpha + 6\beta = |E(G)|$. If G admits a decomposition into α cycles of length 3 and β cycles of length 6 for every admissible pair (α, β), then G has a $\{C_3^\alpha, C_6^\beta\}$*-decomposition*. The graph obtained by replacing each edge of G by λ parallel edges is denoted by $G(\lambda)$.

The problem of decomposing $K_{2n+1}(\lambda)$ and $K_{2n}(\lambda) - I$, where I is a perfect matching, into cycles of varying lengths is completely settled by Bryant el al. [13, 14]. Recently, Bahmanian and Šajna [3] posed the following problem:

Determine the necessary and sufficient conditions on the parameters λ, m, n, and $a_1, a_2, \ldots, a_t$ for the complete equipartite multigraph $(K_m \circ \overline{K}_n)(\lambda)$ to admit a decomposition into cycles of lengths $a_1, a_2, \ldots, a_t$. The necessary conditions (see [3]) are the following:

(1) $2 \le a_i \le mn$ for all $i = 1, 2, \ldots, t$;

(2) if $m = 2$, then $a_1, a_2, \ldots, a_t$ are all even;

(3) $\sum_{i=1}^{t} a_i = mn\left\lfloor \frac{\lambda n(m-1)}{2} \right\rfloor$;

(4) if λ is odd, then $\sum_{a_i \ge 3} a_i \ge n^2\binom{m}{2}$; and

(5) if λ is even, then $\max\{a_i : i = 1, 2, \ldots, t\} \le \frac{1}{2}\lambda n^2\binom{m}{2} - t + 2$.

Decomposing the graph $(K_m \circ \overline{K}_n)(\lambda)$ into triangles is considered in [18] and C_k-decompositions of $K_m \circ \overline{K}_n$, $k \in \{4, 6, 8, p, 2p, 3p, p^2, mn\}$, where p is a prime, are considered in [6, 22, 24, 25, 28, 29, 30]. Existence of C_k-decompositions of the graphs $K_3 \circ \overline{K}_n$, $K_4 \circ \overline{K}_n$ and $K_5 \circ \overline{K}_n$ are proved in [16], [8] and [9], respectively. Decomposition of $K_{n,n} \cong K_2 \circ \overline{K}_n$ into cycles of even length is studied in [34]. Decomposition of $K_m \circ \overline{K}_n$ into k-cycles, where k is small, is dealt with in [32, 33]. Recently, irrespective of the parity of k, the authors of [15] actually solve the existence problem for a C_k-decomposition of $(K_m \circ \overline{K}_n)(\lambda)$ whose cycle-set can be partitioned into 2-regular graphs containing all the vertices except those belonging to one part. Existence of a C_5-decomposition of $(K_m \circ \overline{K}_n)(\lambda)$ and a C_p-decomposition of $(K_m \circ \overline{K}_n)(\lambda)$, where p is a prime, are obtained in [10] and [31], respectively.

Huang and Fu [21] obtained a necessary and sufficient condition for the existence of a $\{C_4^\alpha, C_5^\beta\}$-decomposition of the complete equipartite graph, $K_m \circ \overline{K}_n$. Similarly, the present authors obtained a necessary and sufficient condition for the existence of a $\{C_3^\alpha, C_6^\beta\}$-decomposition of the λ-fold tensor product of the complete graphs, $(K_m \times K_n)(\lambda)$; see [27]. Bahmanian and Šajna [3] showed that if $K_m(\lambda n)$ has a decomposition into cycles of lengths $k_1, k_2, \ldots, k_t$ (plus a perfect matching if $\lambda n(m-1)$ is odd), then $(K_m \circ \overline{K}_n)(\lambda)$ has a decomposition into cycles of lengths $k_1 n, k_2 n, \ldots, k_t n$ (plus a perfect matching if $\lambda n(m-1)$ is odd). Billington [5] obtained necessary and sufficient conditions for the existence of a $\{C_3^\alpha, C_4^\beta\}$-decomposition of $K_{a,b,c}$, $a \le b \le c$, where $K_{a,b,c}$ is the complete tripartite graph with parts of size a, b and c, respectively. Recently, Ganesamurthy and Paulraja, in [17], have discussed the existence of a $\{C_3^\alpha, C_6^\beta\}$-decomposition of the graph $K_{a,b,c}$, $a \le b \le c$.

In this paper, we obtain a necessary and sufficient condition for the existence of a $\{C_3^\alpha, C_6^\beta\}$-decomposition of $(K_m \circ \overline{K}_n)(\lambda)$.

We prove the following main theorem.

Theorem 1.1. *Let $m, n \geq 3$, $\alpha, \beta \geq 0$ and let $\lambda \geq 1$. The graph $(K_m \circ \overline{K}_n)(\lambda)$ admits a $\{C_3^\alpha, C_6^\beta\}$-decomposition if and only if $\lambda n(m-1) \equiv 0 \pmod 2$ and $3\alpha + 6\beta = \lambda\binom{m}{2}n^2$.*

2 Preliminaries and some known results

The *wreath product* (also called the *lexicographic product*) of the graphs G_1 and G_2, denoted by $G_1 \circ G_2$, has vertex set $V(G_1) \times V(G_2)$ in which $(x_1, y_1)(x_2, y_2)$ is an edge whenever x_1x_2 is an edge in G_1 or, $x_1 = x_2$ and y_1y_2 is an edge in G_2, see Figure 1. Let $V(G) = \{x^1, x^2, \ldots, x^m\}$ and $V(H) = \{1, 2, \ldots, n\}$. For $x^i \in V(G)$, $x^i \times V(H) = \{(x^i, j) \mid j \in \{1, 2, \ldots, n\}\}$; we denote (x^i, j) by x^i_j. For $1 \leq i \leq m$, the set $X^i = \{x^i_1, x^i_2, \ldots, x^i_n\}$ is the i^{th} *layer* (of vertices) of $G \circ H$, corresponding to the vertex x^i of $V(G)$, see Figure 1. The graph $K_m \circ \overline{K}_n$ is the complete m-partite graph in which each partite set has exactly n vertices.

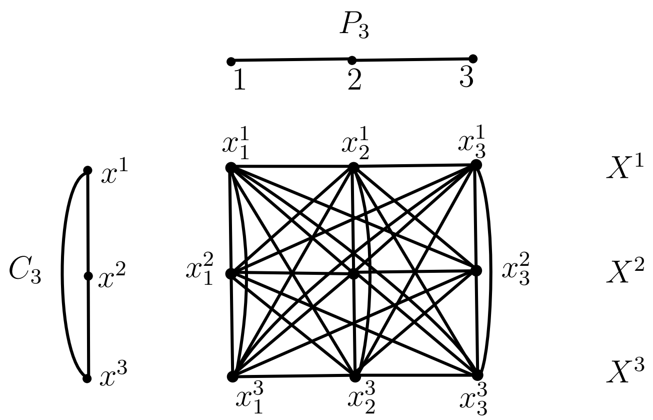

Figure 1: The graph $C_3 \circ P_3$.

A *latin square* of order n, denoted by L_n, is an $n \times n$ array, each cell of which contains exactly one of the symbols in $\{1, 2, \ldots, n\}$ such that each row and each column of the array contains each of the symbols in $\{1, 2, \ldots n\}$ exactly once. A latin square is said to be *idempotent* if the cell (i, i) contains the symbol i, $1 \leq i \leq n$. As in [7], a cell (i, j) of a partial latin square is termed "empty" if it contains no entry and "filled" otherwise. We represent a *partial latin square* L by a set of ordered triples (i, j, k), where the entry k occurs in row i and column j. A *quasigroup* of order n is a pair $(Q, *)$, where Q is a set of size n and "$*$" is a binary operation on Q such that for every pair of elements $a, b \in Q$, the equations $a * x = b$ and $y * a = b$ have unique solutions. Let $Q = \{1, 2, \ldots, 2m\}$ and let $H = \{\{1, 2\}, \{3, 4\}, \ldots, \{2m-1, 2m\}\}$; the two element sets $\{2i-1, 2i\} \in H$ are *holes* of the quasigroup. A *quasigroup* with *holes* H is a quasigroup $(Q, *)$ of order $2m$ in which for each $h = \{2i-1, 2i\} \in H$, $(h, *)$ is a subquasigroup of $(Q, *)$, see [23].

Let $\mathcal{C} = \{C^1, C^2, \ldots, C^r\}$ be any family of nonempty sets. The *intersection graph* of $\mathcal{C}$, denoted by $\Omega(\mathcal{C})$, is the graph having $\mathcal{C}$ as vertex set with C^i adjacent to C^j if and only if $i \neq j$ and $C^i \cap C^j \neq \emptyset$; see [4].

We use the following theorems to prove our results.

Theorem 2.1. [1] *For every n and $n \neq 6, 8$, there exists a PBD $(n, \{3, 4, 5\})$, where PBD denotes pairwise balanced design.*

Theorem 2.2. [17] *Let $K_{a,b,c}$ be the complete tripartite graph with $a \leq b \leq c$ and let $K_{a,b,c} \neq K_{1,1,c}$, when $c \equiv 1 \,(mod\ 6)$ and $c > 1$. If $a \equiv b \equiv c \,(mod\ 6)$, then $K_{a,b,c}$ admits a $\{C_3^{\alpha}, C_6^{\beta}\}$-decomposition for any $\alpha \equiv a \,(mod\ 2)$, with $0 \leq \alpha \leq ab$.*

Theorem 2.3. [19] *Let $k \geq 3$ be an odd integer.*
(i) If $m \geq 3$, then $K_m \circ \overline{K}_{2k}$ can be decomposed into cycles of length k.
(ii) If $m \geq 3$ is odd, then $K_m \circ \overline{K}_k$ can be decomposed into cycles of length k.

Theorem 2.4. [26] *Let $m \geq 3$, $n \geq 3$ and $\lambda \geq 1$. If $\mathcal{C}$ is a C_p-decomposition of $(K_m \circ \overline{K}_n)(\lambda)$, where $3 \leq p \leq mn$, is a prime, then $\Omega(\mathcal{C})$, the intersection graph of C_p-decomposition of $(K_m \circ \overline{K}_n)(\lambda)$, is Hamiltonian.*

From the proof of Theorem 6.1 of [11], we have the following theorem.

Theorem 2.5. [11]
(i) If $n \equiv 11 \,(mod\ 12)$, then $K_n \setminus E(K_{11})$ has a bowtie decomposition.
(ii) If $n \equiv 5 \,(mod\ 12)$, then $K_n \setminus E(K_5)$ has a bowtie decomposition.

Theorem 2.6. [11]
(i) If $n \equiv 1$ or $9 \,(mod\ 12)$, then K_n can be decomposed into bowties.
(ii) If $n \equiv 3$ or $7 \,(mod\ 12)$, then K_n can be decomposed into bowties and one C_3.
(iii) If $n \equiv 0, 2, 6,$ or $8 \,(mod\ 12)$, then $K_n - I$, where I is a perfect matching, has a bowtie decomposition.

Theorem 2.7. [31] *Let G be a connected even multigraph on $k \geq 3$ edges with maximum degree $\Delta(G) = \Delta$ and vertex chromatic number $\chi(G) = \chi$. Then for all $n \geq \Delta/2$, the graph $G \circ \overline{K}_n$ admits a decomposition into cycles of length k whenever there exist at least $\chi - 2$ mutually orthogonal latin squares of order n.*

3 A $\{C_3^{\alpha}, C_6^{\beta}\}$-decomposition of $K_m \circ \overline{K}_n$

In this section, we prove Theorem 1.1 when $\lambda = 1$. Throughout the paper, B denotes the bowtie.

Lemma 3.1. *For the bowtie B, the graph $B \circ \overline{K}_n$, $n \geq 2$, admits a $\{C_3^\alpha, C_6^\beta\}$-decomposition.*

Proof. The graph $B \circ \overline{K}_n = (K_3 \oplus K_3) \circ \overline{K}_n = K_{n,n,n} \oplus K_{n,n,n}$. Hence by Theorem 2.2, $B \circ \overline{K}_n$ admits the required decomposition except when n is odd and $(\alpha, \beta) = (0, n^2)$. For this pair $(0, n^2)$, a $\{C_3^0, C_6^{n^2}\}$-decomposition of $B \circ \overline{K}_n$ follows from Theorem 2.7. □

Lemma 3.2. *If $K_m \circ \overline{K}_n$, $n \geq 2$ has a C_3-decomposition, then*
(i) it has a bowtie decomposition whenever $\binom{m}{2}n^2 \equiv 0$ (mod 2);
(ii) it has a decomposition into bowties and one C_3 whenever $\binom{m}{2}n^2 \equiv 1$ (mod 2).

Proof. Let $\mathcal{C}$ denote the set of all cycles of length three in a C_3-decomposition of $K_m \circ \overline{K}_n$. Let $\Omega(\mathcal{C})$ be the intersection graph of $\mathcal{C}$. Then $\Omega(\mathcal{C})$ has Hamiltonian cycle C, by Theorem 2.4. If the Hamilton cycle C has even length, then $\Omega(\mathcal{C})$ has a 1-factor; each edge of the 1-factor corresponds to a bowtie of $K_m \circ \overline{K}_n$; the bowties corresponding to a 1-factor of $\Omega(\mathcal{C})$ yield a bowtie decomposition of $K_m \circ \overline{K}_n$. If the Hamilton cycle C is of odd length, then $\mathcal{C}$ admits a almost perfect matching M. The ends of each of the edges of M yield a bowtie of $K_m \circ \overline{K}_n$ and the M-unsaturated vertex of $\Omega(\mathcal{C})$ corresponds to a triangle of $K_m \circ \overline{K}_n$. □

Remark 3.3. An ordered triple (i, j, k) stands for the $(i, j)^{\text{th}}$ entry of a latin square, k. We write the entries of a partial latin square by ordered triples in the following lemma; for example, the three triples (x_i, y_l, z), (x_k, y_j, z) and (x_k, y_l, w) represent the following partial latin square, where r_{x_i} represents the row x_i and c_{y_j} represents the column y_j of the latin square.

	c_{y_j}	c_{y_l}
r_{x_i}		z
r_{x_k}	z	w

It is well-known that a latin square of order n gives rise to a decomposition of $K_{n,n,n}$ into triangles. The edge induced subgraph of $K_{n,n,n}$ corresponding to the above partial latin square is isomorphic to $K_{2,2,2} - E(K_3)$; since, for example, the entry w gives the triangle (x_k, y_l, w) and the two entries z give the two triangles namely, (x_k, y_j, z) and (x_i, y_l, z). It is easy to observe that $K_{2,2,2} - E(K_3)$ can be decomposed into a C_3 and a C_6.

Remark 3.4. By the definition of the wreath product of graphs, each K_2 in K_r yields a complete bipartite graph $K_{st,st}$ in $K_r \circ \overline{K}_{st}$. Similarly, each K_2 in K_r yields a $K_{s,s}$ in $K_r \circ \overline{K}_s$; this $K_{s,s}$ in $K_r \circ \overline{K}_s$, in turn, gives a copy of the graph $K_{st,st}$ in $(K_r \circ \overline{K}_s) \circ \overline{K}_t$. As K_r is the complete graph, $K_r \circ \overline{K}_{st} \cong (K_r \circ \overline{K}_s) \circ \overline{K}_t$.

Lemma 3.5. *The graphs $K_4 \circ \overline{K}_6$ and $K_8 \circ \overline{K}_6$ admit $\{C_3^\alpha, C_6^\beta\}$-decompositions.*

Proof. (i) Clearly,

$$\begin{aligned}
K_4 \circ \overline{K}_6 &= (K_4 \circ \overline{K}_2) \circ \overline{K}_3, \text{ by Remark 3.4,}\\
&= (K_8 - I) \circ \overline{K}_3, \text{ where } I \text{ is a perfect matching,}\\
&= (B \oplus B \oplus B \oplus B) \circ \overline{K}_3, \text{ by Theorem 2.6,}\\
&\qquad\qquad \text{where } B \text{ denotes the bowtie,}\\
&= B \circ \overline{K}_3 \oplus B \circ \overline{K}_3 \oplus B \circ \overline{K}_3 \oplus B \circ \overline{K}_3,
\end{aligned}$$

and now apply Lemma 3.1.

(ii) Next consider the graph $K_8 \circ \overline{K}_6$ and the commutative quasigroup $(Q, *)$ on the set $Q = \{1, 2, \ldots, 16\}$, with holes $H_i = \{2i-1, 2i\}$, given below. For our convenience, we assume that the vertex set of $K_8 \circ \overline{K}_6$ is $Q \times Z_3$. For each pair $x, y \in Q$, with $x < y$ and with $\{x, y\}$ not a hole, we give three triangles in $K_8 \circ \overline{K}_6$, namely, $((x, i), (y, i), (x \circ y, i-1))$, $i \in Z_3$.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}	c_{16}
r_1			11	7	3	9	4	15	6	16	14	13	12	5	8	10
r_2			12	13	15	7	6	10	11	8	9	3	4	16	5	14
r_3					14	13	9	16	7	15	5	2	6	1	10	8
r_4					9	16	1	12	5	14	15	8	2	10	11	6
r_5							10	13	4	7	1	16	8	11	2	12
r_6							2	11	1	12	8	10	3	15	14	4
r_7									3	5	16	14	15	12	13	11
r_8									14	2	6	4	5	9	1	3
r_9											2	15	16	8	12	13
r_{10}											13	6	11	4	3	1
r_{11}													10	3	4	7
r_{12}													1	7	9	5
r_{13}															7	9
r_{14}															6	2
r_{15}																
r_{16}																

Figure 2: A quasigroup of order 16, with holes.

Now we will obtain a $\{C_3^\alpha, C_6^\beta\}$-decomposition of $K_8 \circ \overline{K}_6$. For $\beta \leq 24$, we proceed as follows: consider the 8 partial latin squares, each having 3 cells (given below), of $(Q, *)$ in Figure 2.

$\{(r_1, c_3, 11), (r_2, c_3, 12), (r_2, c_9, 11)\}$, $\{(r_1, c_4, 7), (r_1, c_6, 9), (r_2, c_6, 7)\}$,
$\{(r_1, c_{12}, 13), (r_2, c_4, 13), (r_2, c_{12}, 3)\}$, $\{(r_1, c_5, 3), (r_1, c_8, 15), (r_2, c_5, 15)\}$,
$\{(r_1, c_7, 4), (r_1, c_9, 6), (r_2, c_7, 6)\}$, $\{(r_1, c_{16}, 10), (r_2, c_8, 10), (r_2, c_{16}, 14)\}$,
$\{(r_1, c_{10}, 16), (r_2, c_{10}, 8), (r_2, c_{14}, 16)\}$, $\{(r_1, c_{14}, 5), (r_1, c_{15}, 8), (r_2, c_{15}, 5)\}$.

Observe that in each of the three cells of the above partial latin squares, two of the cells have the same entry and hence the edge induced subgraph corresponding to the partial latin square is isomorphic to $K_{2,2,2} - E(K_3)$.

As pointed out earlier, for every pair of elements $x, y \in Q$ with $x < y$ and with $\{x, y\}$ not a hole, we have three triangles, namely, $((x, i), (y, i), (x \circ y, i-1))$, $i \in Z_3$. For example, consider the partial latin square $\{(r_1, c_3, 11), (r_2, c_3, 12), (r_2, c_9, 11)\}$. The pair of elements 1 and 3 of Q and $i = 0, 1, 2$ yield the three triangles

$((1,0),(3,0),(11,2))$, $((1,1),(3,1),(11,0))$ and $((1,2),(3,2),(11,1))$. Similarly, for the pair of elements 2 and 3 we have three triangles, $((2,0),(3,0),(12,2))$, $((2,1),(3,1),(12,0))$ and $((2,2),(3,2),(12,1))$. Finally, for the pair of elements 2 and 9, the three triangles are $((2,0),(9,0),(11,2))$, $((2,1),(9,1),(11,0))$ and $((2,2),(9,2),(11,1))$. The union of these nine triangles gives three copies of $K_{2,2,2}-E(K_3)$; for example, for the pairs of elements of Q, namely, $\{1,3\}$, $\{2,3\}$ and $\{2,9\}$ when $i=0$, yield the three triangles $((1,0),(3,0),(11,2))$, $((2,0),(3,0),(12,2))$ and $((2,0),(9,0),(11,2))$ and their union is isomorphic to $K_{2,2,2}-E(K_3)$. For $i=1$, the union of the triangles $((1,1),(3,1),(11,0))$, $((2,1),(3,1),(12,0))$ and $((2,1),(9,1),(11,0))$ yields a copy of $K_{2,2,2}-E(K_3)$. Finally, for $i=2$, the union of the triangles $((1,2),(3,2),(11,1))$, $((2,2),(3,2),(12,1))$ and $((2,2),(9,2),(11,1))$ gives a copy of $K_{2,2,2}-E(K_3)$. Thus each of the eight partial latin squares listed above yields three subgraphs isomorphic to $K_{2,2,2}-E(K_3)$ in $K_8\circ\overline{K}_6$. Hence the eight partial latin squares yield 24 copies of $K_{2,2,2}-E(K_3)$ in $K_8\circ\overline{K}_6$, and each of the cells not in these eight partial latin squares gives three triangles. This completes the proof for $\beta\le 24$, because each copy of $K_{2,2,2}-E(K_3)$ is decomposable into three copies of C_3 or, one C_3 and one C_6.

Next we suppose that $\beta\ge 25$.

$$\begin{aligned}\text{The graph } K_8\circ\overline{K}_6 &= (K_3\oplus K_3\oplus K_4\oplus K_{2,2,3})\circ\overline{K}_6\\ &= K_3\circ\overline{K}_6\oplus K_3\circ\overline{K}_6\oplus K_4\circ\overline{K}_6\oplus K_{12,12,18}.\end{aligned}$$

Now apply Theorem 2.2 to each of the graphs $K_3\circ\overline{K}_6\cong K_{6,6,6}$, $K_{2,2,3}\circ\overline{K}_6\cong K_{12,12,18}$ and (i) of this lemma to the graph $K_4\circ\overline{K}_6$ to obtain a $\{C_3^\alpha, C_6^\beta\}$-decomposition of $K_8\circ\overline{K}_6$. □

As mentioned in the introduction, in the following lemma $X^i=\{x_1^i,x_2^i,\dots,x_n^i\}$ denotes the i^{th} layer of $K_5\circ\overline{K}_3$.

Lemma 3.6. *The graph $K_5\circ\overline{K}_3$ admits a $\{C_3^\alpha, C_6^\beta\}$-decomposition.*

Proof. First we consider the proof for $\beta\le 9$. Let G_1, G_2, G_3 and H be the subgraphs of $K_5\circ\overline{K}_3$ induced by the edges in the union of cycles in $(x_1^1,x_3^2,x_1^3)\cup(x_1^1,x_3^3,x_1^5)\cup(x_3^2,x_1^4,x_1^5)$, $(x_1^1,x_1^4,x_3^5)\cup(x_1^2,x_1^3,x_3^5)\cup(x_1^2,x_3^3,x_1^4)$, $(x_3^1,x_1^2,x_1^5)\cup(x_3^1,x_1^3,x_1^4)\cup(x_1^3,x_3^4,x_1^5)$ and (x_1^1,x_1^2,x_3^4), respectively, where (x_a^i,x_b^j,x_c^k) denotes a cycle of length 3 in $K_5\circ\overline{K}_3$. Each of the subgraphs G_i, $1\le i\le 3$, of $K_5\circ\overline{K}_3$, is isomorphic to $K_{2,2,2}-E(K_3)$. Let $\rho=(1\,2\,3)$ be a permutation on $\{1,2,3\}$. If ρ acts on the subscripts of the vertices of $V(K_5\circ\overline{K}_3)$, then $G_i,\rho(G_i),\rho^2(G_i)$, $1\le i\le 3$, and $H,\rho(H),\rho^2(H)$, decompose the graph $K_5\circ\overline{K}_3$ into nine isomorphic copies of $K_{2,2,2}-E(K_3)$ and three copies of C_3. As the graph $K_{2,2,2}-E(K_3)$ is decomposable into three copies of C_3, or one C_3 and one C_6, the result follows for $\beta\le 9$.

Next we suppose that $\beta\ge 10$.

$$\begin{aligned}\text{The graph } K_5\circ\overline{K}_3 &= (B\oplus C_4)\circ\overline{K}_3\\ &= B\circ\overline{K}_3\oplus C_4\circ\overline{K}_3\\ &= B\circ\overline{K}_3\oplus K_{6,6}.\end{aligned}$$

As $C_6 \mid K_{6,6}$, now apply Lemma 3.1 to complete the proof. □

Lemma 3.7. *The graph $K_{11} \setminus E(K_5)$ has a decomposition into bowties and one C_3.*

Proof. Let $V(K_{11}) = \{x^1, x^2, \ldots, x^{11}\}$, $V(K_5) = \{x^1, x^2, \ldots, x^5\}$ and $E(K_5) = E(\langle x^1, x^2, \ldots, x^5\rangle)$. A bowtie decomposition of $K_{11} \setminus E(K_5)$ is

$(x^1, x^{11}, x^{10}) \cup (x^{10}, x^9, x^2)$, $(x^7, x^6, x^1) \cup (x^1, x^8, x^9)$,
$(x^8, x^7, x^2) \cup (x^2, x^6, x^{11})$, $(x^{10}, x^6, x^3) \cup (x^3, x^9, x^7)$,
$(x^{11}, x^7, x^4) \cup (x^4, x^6, x^9)$, $(x^8, x^6, x^5) \cup (x^5, x^9, x^{11})$,
$(x^4, x^8, x^{10}) \cup (x^{10}, x^7, x^5)$

and the leave C_3 is (x^3, x^8, x^{11}). □

Next we supply the proof of Theorem 1.1 when $\lambda = 1$.

Proof of Theorem 1.1.

Let $\lambda = 1$. The proof of the necessity is obvious and so we prove the sufficiency. By hypothesis, $3 \mid \binom{m}{2}$ or $3 \mid n^2$.

Case 1: $\binom{m}{2} \equiv 0 \pmod 3$.

Clearly, $m \equiv 0$, 1,3 or 4 (mod 6).

Subcase 1.1: m is odd.

Then $m \equiv 1$ or 3 (mod 6). If $m \equiv 1$ or 9 (mod 12), then the graph

$$\begin{aligned} K_m \circ \overline{K}_n &= (B \oplus B \oplus \cdots \oplus B) \circ \overline{K}_n, \text{ by Theorem 2.6, where } B \text{ is the bowtie,} \\ &= B \circ \overline{K}_n \oplus B \circ \overline{K}_n \oplus \cdots \oplus B \circ \overline{K}_n \end{aligned}$$

and apply Lemma 3.1 to complete the proof.

If $m \equiv 3$ or 7 (mod 12), then $K_m \circ \overline{K}_n = (K_3 \oplus B \oplus B \oplus \cdots \oplus B) \circ \overline{K}_n$, by Theorem 2.6. Now apply Theorem 2.2 and Lemma 3.1 to the graphs $K_3 \circ \overline{K}_n \cong K_{n,n,n}$ and $B \circ \overline{K}_n$, respectively.

Subcase 1.2: m is even.

Then $m \equiv 0$ or 4 (mod 6). Then by hypothesis, n is even, say $2n'$, where $n' \geq 2$ as $n \geq 4$. As $m \equiv 0$ or 4 (mod 6), $2m \equiv 0,\ 8 \pmod{12}$. The graph

$$\begin{aligned} K_m \circ \overline{K}_n &= K_m \circ \overline{K}_{2n'} \\ &= (K_m \circ \overline{K}_2) \circ \overline{K}_{n'} \\ &= (K_{2m} - I) \circ \overline{K}_{n'} \\ &= (B \oplus B \oplus \cdots \oplus B) \circ \overline{K}_{n'}, \text{ by Theorem 2.6,} \\ &= B \circ \overline{K}_{n'} \oplus B \circ \overline{K}_{n'} \oplus B \circ \overline{K}_{n'}. \end{aligned}$$

Now apply Lemma 3.1 to the graph $B \circ \overline{K}_{n'}$.

Case 2: $n \equiv 0 \pmod 3$.

By Case 1, we assume that $3 \nmid \binom{m}{2}$. Then $m \equiv 2$ or 5 (mod 6).

Subcase 2.1: m is odd.

Then $m \equiv 5$ or 11 (mod 12). First we assume that $n \equiv 0 \pmod 6$. Let $n = 6k$, $k \geq 1$. If $m \equiv 5 \pmod{12}$, then the graph

$$\begin{aligned} K_m \circ \overline{K}_n &= K_m \circ \overline{K}_{6k} \\ &= (K_m \circ \overline{K}_3) \circ \overline{K}_{2k} \\ &= (B \oplus B \oplus \cdots \oplus B) \circ \overline{K}_{2k}, \text{ by Theorem 2.3 and Lemma 3.2,} \\ &= B \circ \overline{K}_{2k} \oplus B \circ \overline{K}_{2k} \oplus \cdots \oplus B \circ \overline{K}_{2k}. \end{aligned}$$

Now apply Lemma 3.1 to the graph $B \circ \overline{K}_{2k}$ to get a desired decomposition.

If $m \equiv 11 \pmod{12}$, then the graph

$$\begin{aligned} K_m \circ \overline{K}_n &= (K_m \circ \overline{K}_3) \circ \overline{K}_{2k} \\ &= (K_3 \oplus B \oplus B \oplus \cdots \oplus B) \circ \overline{K}_{2k}, \text{ by Theorem 2.3 and Lemma 3.2,} \\ &= K_3 \circ \overline{K}_{2k} \oplus B \circ \overline{K}_{2k} \oplus B \circ \overline{K}_{2k} \oplus \cdots \oplus B \circ \overline{K}_{2k}. \end{aligned}$$

The required decomposition follows by Lemma 3.1.

Next we assume that $n \equiv 3 \pmod 6$. Let $n = 6k + 3$, $k \geq 0$. First we prove the result for $k = 0$; when $k = 0$, $n = 3$. If $m \equiv 5 \pmod{12}$, then the graph

$$\begin{aligned} K_m \circ \overline{K}_3 &= (K_5 \oplus B \oplus B \oplus \cdots \oplus B) \circ \overline{K}_3, \text{ by Theorem 2.5,} \\ &= K_5 \circ \overline{K}_3 \oplus B \circ \overline{K}_3 \oplus B \circ \overline{K}_3 \oplus \cdots \oplus B \circ \overline{K}_3. \end{aligned}$$

Now the desired decomposition follows by Lemmas 3.1 and 3.6. If $m \equiv 11 \pmod{12}$, then $m = 12k' + 11$. For $k' \geq 0$, the graph

$$\begin{aligned} K_m \circ \overline{K}_3 &= (K_{11} \oplus B \oplus B \oplus \cdots \oplus B) \circ \overline{K}_3, \text{ by Theorem 2.5,} \\ &= (K_5 \oplus K_3 \oplus B \oplus B \oplus \cdots \oplus B) \circ \overline{K}_3, \text{ by Lemma 3.7,} \\ &= K_5 \circ \overline{K}_3 \oplus K_3 \circ \overline{K}_3 \oplus B \circ \overline{K}_3 \oplus \cdots \oplus B \circ \overline{K}_3. \end{aligned}$$

Now the result follows by Theorem 2.2, and Lemmas 3.1 and 3.6.

Next we suppose that $k \geq 1$. If $m \equiv 5 \pmod{12}$, then the graph

$$\begin{aligned} K_m \circ \overline{K}_n &= K_m \circ \overline{K}_{6k+3} \\ &= (K_m \circ \overline{K}_3) \circ \overline{K}_{2k+1} \\ &= (B \oplus B \oplus \cdots \oplus B) \circ \overline{K}_{2k+1}, \text{ by Theorem 2.3 and Lemma 3.2,} \\ &= B \circ \overline{K}_{2k+1} \oplus B \circ \overline{K}_{2k+1} \oplus \cdots \oplus B \circ \overline{K}_{2k+1}. \end{aligned}$$

The desired decomposition follows by Lemma 3.1. If $m \equiv 11 \pmod{12}$, then the graph

$$\begin{aligned} K_m \circ \overline{K}_n &= (K_m \circ \overline{K}_3) \circ \overline{K}_{2k+1} \\ &= (K_3 \oplus B \oplus \cdots \oplus B) \circ \overline{K}_{2k+1}, \text{ by Theorem 2.3 and Lemma 3.2,} \\ &= K_3 \circ \overline{K}_{2k+1} \oplus B \circ \overline{K}_{2k+1} \oplus B \circ \overline{K}_{2k+1} \oplus \cdots \oplus B \circ \overline{K}_{2k+1}. \end{aligned}$$

Now the result follows by Theorem 2.2 and Lemma 3.1.

Subcase 2.2: m is even.

Then by the assumption, 3 does not divide $\binom{m}{2}$, $m \equiv 2$ or 8 (mod 12) and hence $m \neq 6$. Then the hypothesis implies n is even; let $n = 6k$, $k \geq 1$, as by assumption $n \equiv 0$ (mod 3). As K_m can be decomposed into copies of K_3, K_4 and K_5, for $m \neq 6, 8$, by Theorem 2.1, $K_m \circ \overline{K}_{6k}$ is decomposed into copies of $K_3 \circ \overline{K}_{6k}$, $K_4 \circ \overline{K}_{6k}$ and $K_5 \circ \overline{K}_{6k}$. We show that, for $m \neq 8$, each of the graphs $K_3 \circ \overline{K}_{6k}$, $K_4 \circ \overline{K}_{6k}$ and $K_5 \circ \overline{K}_{6k}$ has a $\{C_3^\alpha, C_6^\beta\}$-decomposition. By Theorem 2.2, the graph $K_3 \circ \overline{K}_{6k} = K_{6k,6k,6k}$ has a $\{C_3^\alpha, C_6^\beta\}$-decomposition. Moreover, the graph

$$\begin{aligned} K_4 \circ \overline{K}_{6k} &= (K_4 \circ \overline{K}_2) \circ \overline{K}_{3k} \\ &= (K_8 - I) \circ \overline{K}_{3k} \\ &= (B \oplus B \oplus B \oplus B) \circ \overline{K}_{3k}, \text{ by Theorem 2.6,} \\ &= B \circ \overline{K}_{3k} \oplus B \circ \overline{K}_{3k} \oplus B \circ \overline{K}_{3k} \oplus B \circ \overline{K}_{3k} \end{aligned}$$

and hence by Lemma 3.1, $K_4 \circ \overline{K}_{6k}$ has a $\{C_3^\alpha, C_6^\beta\}$-decomposition. Further, the graph

$$\begin{aligned} K_5 \circ \overline{K}_{6k} &= (K_5 \circ \overline{K}_3) \circ \overline{K}_{2k} \\ &= (B \oplus B \oplus B \oplus B \oplus B) \circ \overline{K}_{2k}, \text{ by Theorem 2.3 and Lemma 3.2,} \\ &= B \circ \overline{K}_{2k} \oplus B \circ \overline{K}_{2k} \oplus B \circ \overline{K}_{2k} \oplus B \circ \overline{K}_{2k} \oplus B \circ \overline{K}_{2k}. \end{aligned}$$

Now apply Lemma 3.1 to the graph $B \circ \overline{K}_{2k}$ to have a $\{C_3^\alpha, C_6^\beta\}$-decomposition of $K_5 \circ \overline{K}_{6k}$.

Finally, we prove the result for $m = 8$. If $k = 1$, then apply Lemma 3.5. If $k \geq 2$, then the graph

$$\begin{aligned} K_8 \circ \overline{K}_{6k} &= (K_8 \circ \overline{K}_6) \circ \overline{K}_k \\ &= (B \oplus B \oplus \cdots \oplus B) \circ \overline{K}_k, \text{ by Theorem 2.3 and Lemma 3.2,} \\ &= B \circ \overline{K}_k \oplus B \circ \overline{K}_k \oplus \cdots \oplus B \circ \overline{K}_k. \end{aligned}$$

The result now follows by Lemma 3.1. □

4 A $\{C_3^\alpha, C_6^\beta\}$-decomposition of $(K_m \circ \overline{K}_n)(\lambda)$

In Section 3, existence of a $\{C_3^\alpha, C_6^\beta\}$-decomposition of $(K_m \circ \overline{K}_n)(\lambda)$, when $\lambda = 1$, is obtained. In this section, we prove the same for $\lambda > 1$.

Lemma 4.1. *The graph $K_8(2)$ admits a decomposition into bowties and two copies of K_5.*

Proof. Let $V(K_8(2)) = \{x^1, x^2, \ldots, x^8\}$. The induced subgraphs $\langle x^1, x^2, x^3, x^4, x^5\rangle$ and $\langle x^1, x^2, x^6, x^7, x^8\rangle$ of $K_8(2)$ are isomorphic to $K_5(2)$ and they have a pair of edges

x^1x^2 in common. From these two $K_5(2)$, consider two edge-disjoint K_5's, one from each copy. The remaining edges are partitioned into bowties as follows:

$(x^3, x^6, x^1) \cup (x^1, x^4, x^7)$, $(x^1, x^8, x^5) \cup (x^5, x^3, x^7)$,
$(x^3, x^6, x^2) \cup (x^2, x^4, x^7)$, $(x^2, x^8, x^5) \cup (x^5, x^6, x^7)$,
$(x^4, x^6, x^8) \cup (x^8, x^3, x^7)$, $(x^5, x^6, x^4) \cup (x^4, x^3, x^8)$.

This gives a required decomposition. □

Lemma 4.2. *If $n \geq 2$, then the graph $(K_3 \circ \overline{K}_n)(2) \simeq K_3(2) \circ \overline{K}_n$ has a C_6-decomposition.*

Proof. The proof follows from Theorem 2.7. □

Let M denote the multigraph having four vertices, six edges, a pair of edge-disjoint triangles and a pair of vertices each having degree two; it is denoted by $M = (a, b, c) \cup (b, d, c)$. The graph $K_4(2)$ can be decomposed into two copies of M. This graph M is used in the next Lemma 4.3.

Lemma 4.3. *If $n \geq 2$, then the graph $M \circ \overline{K}_n$ admits a $\{C_3^\alpha, C_6^\beta\}$-decomposition.*

Proof. Let $M = (x^1, x^2, x^3) \cup (x^2, x^4, x^3)$. The graph $M \circ \overline{K}_n \simeq (K_3 \oplus K_3) \circ \overline{K}_n = (K_3 \circ \overline{K}_n) \oplus (K_3 \circ \overline{K}_n)$. Then by Theorem 2.2 we obtain the required $\{C_3^\alpha, C_6^\beta\}$-decomposition except when n is odd and $\alpha = 0$. In this case the required decomposition follows from Theorem 2.7. □

Lemma 4.4. *If $n \geq 3$, then the graph $(K_4 \circ \overline{K}_n)(2)$ has a $\{C_3^\alpha, C_6^\beta\}$-decomposition.*

Proof. Let n be even. The graph $(K_4 \circ \overline{K}_n)(2) \simeq K_4(2) \circ \overline{K}_n = (K_3 \oplus K_3 \oplus K_3 \oplus K_3) \circ \overline{K}_n$ as $K_4(2)$ has a K_3-decomposition. As n is even, the result follows by Theorem 2.2. If n is odd, then the graph $(K_4 \circ \overline{K}_n)(2) \simeq K_4(2) \circ \overline{K}_n = (M \oplus M) \circ \overline{K}_n$, where M is the graph defined just above Lemma 4.3. The result follows by Lemma 4.3. □

A bowtie system of order n is a pair (S, B), where B is a collection of edge disjoint bowties which partition the edge set of K_n, and S is the vertex set of K_n. As in [12], we define a 2-*perfect bowtie* system as follows: "If $t = \{(a, b, c) \cup (a, d, e)\}$ is a bowtie we will denote by $2t$ the set of two bowties $\{(a, b, e) \cup (a, c, d), (a, b, d) \cup (a, c, e)\}$. A bowtie system (S, B) is said to be 2-*perfect* provided it is possible, for each $t \in B$, to select a bowtie from $2t$ so that the resulting collection B^* of bowties gives a bowtie system (S, B^*)." It is clear that, if a graph G admits a 2-perfect bowtie decomposition, then obviously G has a bowtie decomposition.

Combining the results of [2], [12] and [20], we have the following Theorem 4.5.

Theorem 4.5. [2, 12, 20] *Existence of a λ-fold bowtie system of $K_n(\lambda)$ for various values of n and λ is given in the table below:*

$\lambda\,(\text{mod } 6)$	$n \geq 5$	*Leave*
$1, 5\,(mod\ 6)$	$1\ or\ 9\,(mod\ 12)$	$\emptyset$
	$3\ or\ 7\,(mod\ 12)$	K_3
$2, 4\,(mod\ 6)$	$0\ or\ 1\,(mod\ 3)$	$\emptyset$
$3\,(mod\ 6)$	$1\,(mod\ 4)$	$\emptyset$
$0\,(mod\ 6)$	*all* n	$\emptyset$

Lemma 4.6. *Let $m, n \geq 3$ and let $\alpha, \beta \geq 0$. The graph $(K_m \circ \overline{K}_n)(2)$ admits a $\{C_3^\alpha, C_6^\beta\}$-decomposition if and only if $3(\alpha + 2\beta) = m(m-1)n^2$.*

Proof. The necessity follows by the edge divisibility condition and we prove the sufficiency. By hypothesis, $m \equiv 0$ or 1 (mod 3) or $n \equiv 0$ (mod 3).

Case 1. $m \equiv 0\ or\ 1\,(mod\ 3)$

First, if $m = 3$ and n is even, then $(K_3 \circ \overline{K}_n)(2) = (K_3 \circ \overline{K}_n) \oplus (K_3 \circ \overline{K}_n)$; now apply Theorem 2.2. If $m = 3$ and n is odd, then apply Theorem 2.2 if $\alpha \neq 0$. If $\alpha = 0$, apply Lemma 4.2. Next, if $m = 4$, then the lemma follows by Lemma 4.4. If $m \geq 5$, then $(K_m \circ \overline{K}_n)(2) \simeq (K_m(2) \circ \overline{K}_n) = (B \oplus B \oplus \cdots \oplus B) \circ \overline{K}_n$, by Theorem 4.5. Proof for this case now follows by Lemma 3.1.

Case 2. $n \equiv 0\,(mod\ 3)$

Let $n = 3k$. Because of Case 1, we assume $m \equiv 2$ (mod 3). As K_m can be decomposed into copies of K_3, K_4 and K_5, for $m \neq 8$, by Theorem 2.1, the graph $(K_m \circ \overline{K}_n)(2)$ can be decomposed into copies of $(K_3 \circ \overline{K}_n)(2)$, $(K_4 \circ \overline{K}_n)(2)$ and $(K_5 \circ \overline{K}_n)(2)$. Now we prove that each of these graphs admits a $\{C_3^\alpha, C_6^\beta\}$-decomposition. Clearly, $(K_3 \circ \overline{K}_n)(2)$ has a required decomposition, by Case 1 above. Next, $(K_4 \circ \overline{K}_n)(2) = (K_4 \circ \overline{K}_{3k})(2)$ also admits a desired decomposition, by Lemma 4.4. The graph $(K_5 \circ \overline{K}_n)(2) = (K_5 \circ \overline{K}_{3k})(2) = (K_5 \circ \overline{K}_{3k}) \oplus (K_5 \circ \overline{K}_{3k})$. If $k = 1$, then apply Lemma 3.6. For $k \geq 2$, the graph $(K_5 \circ \overline{K}_{3k})(2) \simeq (B \oplus B \oplus \cdots \oplus B) \circ \overline{K}_k$, by Theorem 2.3 and Lemma 3.2; apply Lemma 3.1 to $B \circ \overline{K}_k$. The graph

$$
\begin{aligned}
(K_8 \circ \overline{K}_n)(2) &= (K_8(2) \circ \overline{K}_n) \\
&= (K_5 \oplus K_5 \oplus B \oplus \cdots \oplus B) \circ \overline{K}_{3k} \text{ by Lemma 4.1,} \\
&= (K_5 \circ \overline{K}_3) \circ \overline{K}_k \oplus (K_5 \circ \overline{K}_3) \circ \overline{K}_k \oplus B \circ \overline{K}_{3k} \oplus \ldots B \circ \overline{K}_{3k} \\
&= B \circ \overline{K}_k \oplus B \circ \overline{K}_k \oplus \cdots \oplus B \circ \overline{K}_k, \text{ by Theorem 2.3} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{and Lemma 3.2.}
\end{aligned}
$$

Now apply Lemma 3.1, to get a required decomposition. ☐

Let L'_m denote the cells above the main diagonal of an idempotent latin square L_m.

Theorem 4.7. [23] *For all odd $m \geq 3$, the graph $K_m(3)$ has a C_3-decomposition.*

It is well-known that each cell of L'_m corresponds to a triangle of $K_m(3)$ and all the cells of L'_m give a C_3-decomposition of $K_m(3)$. By suitably pairing the cells of L'_m, we obtain the following lemma.

Lemma 4.8. *If $m \geq 5$ and $m \equiv 3 \pmod 4$, then the graph $K_m(3)$ has a decomposition into the graphs B, M, $C_3(2)$ and one C_3.*

Proof. Let $m = 4k + 3$ and let $V(K_m(3)) = \{x^1, x^2, \dots, x^m\}$. Also, let L_m be an idempotent commutative latin square of order m. In L'_m, each odd row has even number of cells. Now we pair the consecutive cells, from the left, of the odd rows of L'_m; the entries of each such pair yields one of the graphs B, M or $C_3(2)$. Again in L'_m, each even row has odd number of cells. We pair consecutive cells, from the left, of each of the even rows of L'_m so that the last cell is left out in the pairing. In this pairing also the entries corresponding to the pair of cells yield any one of the graphs B, M or $C_3(2)$.

After the above pairing, there are $2k + 1$ cells left out in the last column of L'_m. Pair these cells, from top to bottom, except the cell in the $(n-1)^{\text{th}}$ row. The entries in such a pairing again yield any one of the graphs B, M or $C_3(2)$. Finally the entry corresponding to the cell in the last column and $n-1^{\text{th}}$ row of L'_m yield a triangle. □

In the proof of the following lemma, let $(L_m, *)$ denote a commutative quasigroup of order m with holes and let L''_m denote the cells of $(L_m, *)$ which lie above the holes.

Lemma 4.9. *If $m \geq 6$ is even, then the graph $(K_m - I)(3)$ has a bowtie decomposition, where I is a perfect matching of K_m.*

Proof. Let $(L_m, *)$ be a commutative quasigroup of order m with holes $H = \{\{1, 2\}, \{3, 4\}, \dots, \{m-1, m\}\}$. Corresponding to H, consider the matching $x^1x^2, x^3x^4, \dots, x^{m-1}x^m$ of K_m. In L''_m, each row has an even number of cells and we pair the consecutive cells of each of the rows from left to right as in the above lemma. One can easily check that the entries in the consecutive cells of L''_m yield only a bowtie, using the fact that $(L_m, *)$ is a commutative quasigroup with holes. □

Lemma 4.10. *Let $m, n \geq 3$ and let $\alpha, \beta \geq 0$. The graph $(K_m \circ \overline{K}_n)(3)$ admits a $\{C_3^\alpha, C_6^\beta\}$-decomposition if and only if $n(m-1) \equiv 0 \pmod 2$ and $\alpha + 2\beta = \binom{m}{2}n^2$.*

Proof. The necessity follows by the degree condition and the edge divisibility condition. Now we prove the sufficiency in two cases.

Case 1: m is odd.

Then $m \equiv 1$ or $3 \pmod 4$. If $m \equiv 1 \pmod 4$, then the graph

$$\begin{aligned} (K_m \circ \overline{K}_n)(3) &\simeq K_m(3) \circ \overline{K}_n \\ &= (B \oplus B \oplus \cdots \oplus B) \circ \overline{K}_n, \text{ by Theorem 4.5,} \\ &= B \circ \overline{K}_n \oplus B \circ \overline{K}_n \oplus \cdots \oplus B \circ \overline{K}_n. \end{aligned}$$

Now apply Lemma 3.1 to $B \circ \overline{K}_n$ to get a required decomposition.

Next, let $m \equiv 3 \pmod 4$. If $m = 3$ and n is even, then $(K_3 \circ \overline{K}_n)(3) = (K_3 \circ \overline{K}_n) \oplus (K_3 \circ \overline{K}_n) \oplus (K_3 \circ \overline{K}_n)$; apply Theorem 2.2 to the graph $K_3 \circ \overline{K}_n$. If $m = 3$ and n is odd, then $(K_3 \circ \overline{K}_n)(3) = (K_3 \circ \overline{K}_n)(2) \oplus (K_3 \circ \overline{K}_n)$; and now apply Lemma 4.6

and Theorem 2.2 to the graphs $(K_3 \circ \overline{K}_n)(2)$ and $K_3 \circ \overline{K}_n$, respectively. If $m \geq 5$, then the graph $K_m(3) \circ \overline{K}_n$ can be decomposed into copies of $B \circ \overline{K}_n$, $M \circ \overline{K}_n$, $C_3(2) \circ \overline{K}_n$ and $C_3 \circ \overline{K}_n$ as $K_m(3)$ can be decomposed into copies of B, M, $C_3(2)$ and one copy of C_3 by Lemma 4.8. Now apply Lemmas 3.1, 4.3, 4.6 and Theorem 2.2 to the graphs $B \circ \overline{K}_n$, $M \circ \overline{K}_n$, $C_3(2) \circ \overline{K}_n$ and $K_3 \circ \overline{K}_n$, respectively, to get a required decomposition.

Case 2: m is even.

In this case n is even; let $n = 2k$, $k \geq 2$. The graph

$$\begin{aligned}(K_m \circ \overline{K}_n)(3) &\simeq (K_m \circ \overline{K}_{2k})(3)\\ &= (K_m \circ \overline{K}_2)(3) \circ \overline{K}_k\\ &= (K_{2m} - I)(3) \circ \overline{K}_k\\ &= (B \oplus B \oplus \cdots \oplus B) \circ \overline{K}_k, \text{ by Lemma 4.9,}\\ &= B \circ \overline{K}_k \oplus B \circ \overline{K}_k \oplus \cdots \oplus B \circ \overline{K}_k.\end{aligned}$$

Now apply Lemma 3.1 to $B \circ \overline{K}_k$ to get a desired decomposition. □

Lemma 4.11. *Let $m, n \geq 3$ and let $\alpha, \beta \geq 0$. The graph $(K_m \circ \overline{K}_n)(6)$ admits a $\{C_3^{\alpha}, C_6^{\beta}\}$-decomposition if and only if $\alpha + 2\beta = m(m-1)n^2$.*

Proof. The necessity follows by the edge divisibility condition and we prove the sufficiency. If $m = 3$, then $(K_3 \circ \overline{K}_n)(6) = (K_3 \circ \overline{K}_n)(2) \oplus (K_3 \circ \overline{K}_n)(2) \oplus (K_3 \circ \overline{K}_n)(2)$. Now apply Lemma 4.6 to the graph $(K_3 \circ \overline{K}_n)(2)$ to get a required decomposition of $(K_3 \circ \overline{K}_n)(6)$. If $m = 4$, then $(K_4 \circ \overline{K}_n)(6) = (K_4 \circ \overline{K}_n)(2) \oplus (K_4 \circ \overline{K}_n)(2) \oplus (K_4 \circ \overline{K}_n)(2)$, and apply Lemma 4.4 to get a required decomposition of $(K_4 \circ \overline{K}_n)(6)$.

Now we assume that $m \geq 5$. Then the graph

$$\begin{aligned}(K_m \circ \overline{K}_n)(6) &\simeq K_m(6) \circ \overline{K}_n\\ &= (B \oplus B \oplus \cdots \oplus B) \circ \overline{K}_n, \text{ by Theorem 4.5,}\\ &= B \circ \overline{K}_n \oplus B \circ \overline{K}_n \oplus \cdots \oplus B \circ \overline{K}_n.\end{aligned}$$

Now apply Lemma 3.1 to get a required decomposition. □

Now we complete the proof of Theorem 1.1 when $\lambda > 1$.

Proof of Theorem 1.1.

Let $\lambda \equiv t \pmod 6$ and let $\lambda = 6k + t$, $t \in \{0, 1, 2, 3, 4, 5\}$.

If $\lambda \equiv 0 \pmod 6$, then the graph $(K_m \circ \overline{K}_n)(6k) = (K_m \circ \overline{K}_n)(6) \oplus (K_m \circ \overline{K}_n)(6) \oplus \cdots \oplus (K_m \circ \overline{K}_n)(6)$ and by Lemma 4.11 the required decomposition follows.

If $\lambda \equiv 3 \pmod 6$, then the graph $(K_m \circ \overline{K}_n)(6k + 3) = (K_m \circ \overline{K}_n)(6) \oplus (K_m \circ \overline{K}_n)(6) \oplus \cdots \oplus (K_m \circ \overline{K}_n)(6) \oplus (K_m \circ \overline{K}_n)(3)$ and the required decomposition follows by Lemmas 4.10 and 4.11.

If $\lambda \equiv 2 \pmod 6$, then the graph $(K_m \circ \overline{K}_n)(6k + 2) = (K_m \circ \overline{K}_n)(6) \oplus (K_m \circ \overline{K}_n)(6) \oplus \cdots \oplus (K_m \circ \overline{K}_n)(6) \oplus (K_m \circ \overline{K}_n)(2)$ and the required decomposition follows by Lemmas 4.6 and 4.11.

If $\lambda \equiv 4 \pmod 6$, then the graph $(K_m \circ \overline{K}_n)(6k+4) = (K_m \circ \overline{K}_n)(6) \oplus (K_m \circ \overline{K}_n)(6) \oplus \cdots \oplus (K_m \circ \overline{K}_n)(6) \oplus (K_m \circ \overline{K}_n)(2) \oplus (K_m \circ \overline{K}_n)(2)$ and the required decomposition follows by Lemmas 4.6 and 4.11.

If $\lambda \equiv 1 \pmod 6$, then the graph $(K_m \circ \overline{K}_n)(6k+1) = (K_m \circ \overline{K}_n)(6) \oplus (K_m \circ \overline{K}_n)(6) \oplus \cdots \oplus (K_m \circ \overline{K}_n)(6) \oplus K_m \circ \overline{K}_n$ and the required decomposition follows by Lemma 4.11 and the proof of Theorem 1.1 when $\lambda = 1$ (see Section 3).

If $\lambda \equiv 5 \pmod 6$, then the graph $(K_m \circ \overline{K}_n)(6k+5) = (K_m \circ \overline{K}_n)(6) \oplus (K_m \circ \overline{K}_n)(6) \oplus \cdots \oplus (K_m \circ \overline{K}_n)(6) \oplus \underbrace{K_m \circ \overline{K}_n \oplus K_m \circ \overline{K}_n \oplus \cdots \oplus K_m \circ \overline{K}_n}_{5 \text{ copies}}$ and now, again the required decomposition follows by Lemma 4.11 and the proof of Theorem 1.1 when $\lambda = 1$ (see Section 3). $\square$

Acknowledgements

The authors would like to thank the referees for their careful reading and suggestions which improved the presentation of the paper. Also we would like to thank the Editor for bringing the reference [20] to our notice. The second author would like to thank the Kalasalingam Academy of Research and Education, Tamil Nadu, India, for financial support through University Research Fellowship.

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(Received 5 Aug 2019; revised 23 May 2020)