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The r-Fubini–Lah numbers and polynomials

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Abstract

In this paper we introduce and study the r-Fubini–Lah numbers and polynomials, in connection with the enumeration of those partitions of a finite set, where both the blocks and the partition itself are ordered, and r distinguished elements belong to distinct ordered blocks.

1 Introduction

Bell numbers are well-known objects in enumerative combinatorics. The nth Bell number $B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$ $(n \geq 0)$, where the numbers $\left\{ {n \atop k} \right\}$ are Stirling numbers of the second kind, is the number of partitions of an n-element set into nonempty subsets.

If we count the number of ordered partitions of an n-element set, we obtain the nth Fubini number $F_n = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\}$ $(n \geq 0)$. These numbers appeared in several papers from different points of view. Their first mention is due to Cayley [3] in connection with the enumeration of certain trees. A number theoretical interpretation was given by James [8], who counted ordered factorizations of square-free integers. Using equivalent combinatorial definitions, Fubini numbers were also investigated by Gross [5], Good [4] and Tanny [17]. These authors proved, among others, a recurrence, a Dobiński type formula, and derived the exponential generating function of the sequence of Fubini numbers. We remark that Gross gave an additional geometric interpretation for Fubini numbers. As an extension, Tanny introduced Fubini polynomials $F_n(x) = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\} x^k$ $(n \geq 0)$, as well. We note that Kereskényi-Balogh and Nyul [9] gave a graph theoretical generalization of Fubini numbers and polynomials.

Another variant of the above numbers is their r-generalization. The nth r-Bell number $B_{n,r} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r$ $(n, r \geq 0)$, defined by Carlitz [2] and Mező [11], counts the number of those partitions of a set with $n + r$ elements, where r distinguished elements belong to distinct blocks. Here $\left\{ {n \atop k} \right\}_r$ denotes an r-Stirling number of the second kind (see [1, 2, 10]) with the parametrization, where $\left\{ {n \atop k} \right\}_r$ is the number

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of partitions of an $(n+r)$-element set into $k+r$ nonempty subsets, such that r distinguished elements belong to distinct blocks.

Again, if we are interested in the number of ordered partitions of an $(n+r)$-element set such that r distinguished elements belong to distinct blocks, then we arrive at the nth r-Fubini number $F_{n,r} = \sum_{k=0}^{n}(k+r)!\left\{{n \atop k}\right\}_r$ $(n, r \geq 0)$. These numbers were studied by Mező and Nyul [12], together with the r-Fubini polynomials $F_{n,r}(x) = \sum_{k=0}^{n}(k+r)!\left\{{n \atop k}\right\}_r x^k$ $(n, r \geq 0)$.

If not the partition itself, but the blocks are ordered, then the counting numbers are relatives of Lah numbers. By the definition due to Nyul and Rácz [14], the nth summed r-Lah number $L_{n,r} = \sum_{k=0}^{n} \left\lfloor {n \atop k} \right\rfloor_r$ $(n, r \geq 0)$ is the number of those partitions of an $(n+r)$-element set into ordered blocks, where r distinguished elements belong to distinct ordered blocks. Here $\left\lfloor {n \atop k} \right\rfloor_r$ denotes an r-Lah number which counts the number of partitions on an $(n+r)$-element set into $k+r$ ordered nonempty subsets such that r distinguished elements belong to distinct ordered blocks (for more details, see [13]). In [14], the related r-Lah polynomials $L_{n,r}(x) = \sum_{k=0}^{n} \left\lfloor {n \atop k} \right\rfloor_r x^k$ $(n, r \geq 0)$ are also investigated. We note that one can find a graph theoretical interpretation of summed r-Lah numbers and r-Lah polynomials in [15], while a further generalization, the r-Dowling–Lah numbers and polynomials were studied by Gyimesi [6].

In this paper we introduce the Fubini type variants of summed r-Lah numbers and r-Lah polynomials, which we call r-Fubini–Lah numbers and polynomials. We prove two recurrences and a Dobiński type formula for them, we determine their exponential generating functions, finally we give their connection with r-Fubini numbers and polynomials. Definitions of these notions will be given together with combinatorial interpretations, which will be used in the proofs when it is possible.

2 r-Fubini–Lah numbers and polynomials

Let n, r be non-negative integers, not both 0. Define the nth *r-Fubini–Lah number* $FL_{n,r}$ as the number of ordered partitions of an $(n+r)$-element set into ordered subsets such that r distinguished elements belong to distinct ordered blocks. Furthermore, let $FL_{0,0} = 1$. From this definition it immediately follows that

$$FL_{n,r} = \sum_{k=0}^{n}(k+r)!\left\lfloor {n \atop k} \right\rfloor_r .$$

In addition, for $n, r \geq 0$, we introduce the nth *r-Fubini–Lah polynomial* as

$$FL_{n,r}(x) = \sum_{k=0}^{n}(k+r)!\left\lfloor {n \atop k} \right\rfloor_r x^k.$$

We can give a combinatorial interpretation for these polynomials when n, r are not both 0. If c is a positive integer, then $FL_{n,r}(c)$ is the number of coloured ordered partitions of a set with $n+r$ elements into ordered blocks, such that r distinguished elements belong to distinct ordered subsets and ordered blocks containing no distinguished element are coloured with c colours.

It follows from the above definitions that the polynomial $FL_{n,r}(x)$ is of degree n with leading coefficient $(n+r)!$. Since $FL_{n,r}(1) = FL_{n,r}$, it will suffice to prove our results only for polynomials in most cases.

The 0-Fubini–Lah polynomials and numbers are simply $FL_{n,0}(x) = n!x(x+1)^{n-1}$ and $FL_{n,0} = n!2^{n-1}$ $(n \geq 1)$, which can be proved easily: Starting from a permutation of the n elements, we colour the ordered block of the first element with c colours, then we have $c+1$ possibilities for each of the other $n-1$ elements, since it can open a new ordered block and this ordered block is coloured with c colours, or it belongs to the ordered block of the previous element. Moreover, 1-Fubini–Lah polynomials and numbers clearly satisfy $xFL_{n,1}(x) = FL_{n+1,0}(x)$ and $FL_{n,1} = FL_{n+1,0}$ $(n \geq 0)$. In contrast to these, for $r \geq 2$, r-Fubini–Lah polynomials and numbers become much more interesting.

The r-Fubini–Lah polynomials and numbers satisfy the following formulas, which are recurrences simultaneously in n and r.

Theorem 1. *If $n \geq 0$ and $r \geq 1$, then*

$$FL_{n,r}(x) = r\sum_{j=0}^{n}\binom{n}{j}(n-j+1)!FL_{j,r-1}(x) + x\sum_{j=0}^{n-1}\binom{n}{j}(n-j)!FL_{j,r}(x),$$

$$FL_{n,r} = r\sum_{j=0}^{n}\binom{n}{j}(n-j+1)!FL_{j,r-1} + \sum_{j=0}^{n-1}\binom{n}{j}(n-j)!FL_{j,r}.$$

Proof. Let $c \geq 1$. We count the number of coloured ordered partitions of a set with $n+r$ elements into ordered subsets such that r distinguished elements belong to distinct ordered blocks, and we colour the ordered blocks containing no distinguished element with c colours.

Consider the first ordered block. If it contains a distinguished element, then we can choose it in r ways. Let j be the number of those non-distinguished elements, which do not belong to this ordered subset $(j = 0, \ldots, n)$. The number of coloured ordered partitions of these elements, together with the other $r-1$ distinguished ones, into ordered subsets, where the ordered blocks containing no distinguished elements are coloured with c colours, is $FL_{j,r-1}(c)$. The remaining $n-j$ non-distinguished elements will be placed into the first ordered block. In this ordered block we can permute the elements in $(n-j+1)!$ ways. Therefore, the number of possibilities is $r\binom{n}{j}(n-j+1)!FL_{j,r-1}(c)$ for a fixed j.

Similarly, if the first ordered block does not contain any distinguished element, then let j be the number of those non-distinguished elements, which do not belong to this ordered block $(j = 0, \ldots, n-1)$. The number of ordered partitions of these j elements together with the distinguished elements into ordered subsets, and colourings of the ordered blocks containing no distinguished element with c colours, is

$FL_{j,r}(c)$. The remaining $n-j$ non-distinguished elements will be placed into the first ordered block, which can be done in $(n-j)!$ ways, but this ordered block has to be coloured. This means that now the number of possibilities is $c\binom{n}{j}(n-j)!FL_{j,r}(c)$ for a fixed j. □

We can prove another recurrence for r-Fubini–Lah polynomials, where the only running parameter is n. Here the derivative of the polynomials also appears.

Theorem 2. *If $n, r \geq 0$, then*

$$FL_{n+1,r}(x) = ((r+1)x + n + 2r)\, FL_{n,r}(x) + (x^2+x)FL'_{n,r}(x).$$

Proof. We use [13, Theorem 3.1] and some special values of r-Lah numbers to obtain that

$$\begin{aligned}
FL_{n+1,r}(x) &= \sum_{k=0}^{n+1}(k+r)!\left\lfloor {n+1 \atop k} \right\rfloor_r x^k \\
&= r!\left\lfloor {n+1 \atop 0} \right\rfloor_r + \sum_{k=1}^{n}(k+r)!\left(\left\lfloor {n \atop k-1} \right\rfloor_r + (n+k+2r)\left\lfloor {n \atop k} \right\rfloor_r\right) x^k \\
&\quad +(n+r+1)!\left\lfloor {n+1 \atop n+1} \right\rfloor_r x^{n+1} \\
&= r!(2r)^{\overline{n+1}} + \sum_{k=0}^{n-1}(k+r+1)!\left\lfloor {n \atop k} \right\rfloor_r x^{k+1} + (n+2r)\sum_{k=1}^{n}(k+r)!\left\lfloor {n \atop k} \right\rfloor_r x^k \\
&\quad +\sum_{k=1}^{n} k(k+r)!\left\lfloor {n \atop k} \right\rfloor_r x^k + (n+r+1)!x^{n+1} \\
&= (n+2r)\sum_{k=0}^{n}(k+r)!\left\lfloor {n \atop k} \right\rfloor_r x^k + \sum_{k=0}^{n}(k+r+1)!\left\lfloor {n \atop k} \right\rfloor_r x^{k+1} \\
&\quad +x\sum_{k=1}^{n} k(k+r)!\left\lfloor {n \atop k} \right\rfloor_r x^{k-1} \\
&= (n+2r)FL_{n,r}(x) + (r+1)x\sum_{k=0}^{n}(k+r)!\left\lfloor {n \atop k} \right\rfloor_r x^k \\
&\quad +x^2\sum_{k=1}^{n} k(k+r)!\left\lfloor {n \atop k} \right\rfloor_r x^{k-1} + xFL'_{n,r}(x) \\
&= (n+2r)FL_{n,r}(x) + (r+1)xFL_{n,r}(x) + x^2FL'_{n,r}(x) + xFL'_{n,r}(x).
\end{aligned}$$

□

The following Dobiński type formula holds for r-Fubini–Lah polynomials and numbers.

Theorem 3. *If $n, r \geq 0$, then*

$$FL_{n,r}(x) = \frac{1}{(x+1)x^r} \sum_{j=0}^{\infty} (j+r)^{\overline{n}} j^{\underline{r}} \left(\frac{x}{x+1} \right)^j,$$

$$FL_{n,r} = \sum_{j=0}^{\infty} \frac{(j+r)^{\overline{n}} j^{\underline{r}}}{2^{j+1}}.$$

Proof. First we prove the identity for polynomials. Using [13, Theorem 3.2] and the binomial series, we have that

$$\begin{aligned}
\sum_{j=0}^{\infty} (j+r)^{\overline{n}} j^{\underline{r}} x^j &= \sum_{j=0}^{\infty} j^{\underline{r}} x^j \sum_{k=0}^{n} \left\lfloor {n \atop k} \right\rfloor_r (j-r)^{\underline{k}} \\
&= \sum_{k=0}^{n} \left\lfloor {n \atop k} \right\rfloor_r \sum_{j=0}^{\infty} j^{\underline{k+r}} x^j = \sum_{k=0}^{n} (k+r)! \left\lfloor {n \atop k} \right\rfloor_r \sum_{j=k+r}^{\infty} \binom{j}{k+r} x^j \\
&= \sum_{k=0}^{n} (k+r)! \left\lfloor {n \atop k} \right\rfloor_r \sum_{j=0}^{\infty} \binom{j+k+r}{k+r} x^{j+k+r} \\
&= \sum_{k=0}^{n} (k+r)! \left\lfloor {n \atop k} \right\rfloor_r x^{k+r} \sum_{j=0}^{\infty} \binom{-k-r-1}{j} (-x)^j \\
&= \sum_{k=0}^{n} (k+r)! \left\lfloor {n \atop k} \right\rfloor_r x^{k+r} (1-x)^{-k-r-1} \\
&= \frac{1}{1-x} \left(\frac{x}{1-x} \right)^r \sum_{k=0}^{n} (k+r)! \left\lfloor {n \atop k} \right\rfloor_r \left(\frac{x}{1-x} \right)^k \\
&= \frac{1}{1-x} \left(\frac{x}{1-x} \right)^r FL_{n,r} \left(\frac{x}{1-x} \right).
\end{aligned}$$

If we substitute $\frac{x}{x+1}$ for x, then we get

$$(x+1)x^r FL_{n,r}(x) = \sum_{j=0}^{\infty} (j+r)^{\overline{n}} j^{\underline{r}} \left(\frac{x}{x+1} \right)^j,$$

which completes the proof for polynomials.

Now we prove the theorem for r-Fubini–Lah numbers. Let ξ be a random variable with probability distribution

$$\mathsf{P}(\xi = j) = \frac{1}{2^{j+1}} \quad (j \geq 0).$$

We calculate the expected value

$$\mathsf{E}\left((\xi + r)^{\overline{n}} \xi^{\underline{r}} \right) = \sum_{j=0}^{\infty} (j+r)^{\overline{n}} j^{\underline{r}} \frac{1}{2^{j+1}},$$

which, using [13, Theorem 3.2] and the identity $\mathsf{E}\xi^{\underline{n}} = n!$ (see, e.g., [9]), also satisfies

$$\begin{aligned}\mathsf{E}\left((\xi+r)^{\overline{n}}\xi^{\underline{r}}\right) &= \mathsf{E}\left(\sum_{k=0}^{n} \left\lfloor {n \atop k} \right\rfloor_r (\xi-r)^{\underline{k}}\xi^{\underline{r}}\right)\\ &= \sum_{k=0}^{n} \left\lfloor {n \atop k} \right\rfloor_r \mathsf{E}\xi^{\underline{k+r}} = \sum_{k=0}^{n} \left\lfloor {n \atop k} \right\rfloor_r (k+r)! = FL_{n,r}.\end{aligned}$$

□

The next theorem gives the exponential generating functions of the sequences of r-Fubini–Lah polynomials and numbers.

Theorem 4. *For $r \geq 0$, the exponential generating function of $(FL_{n,r}(x))_{n=0}^{\infty}$ is*

$$\sum_{n=0}^{\infty} \frac{FL_{n,r}(x)}{n!} y^n = \frac{r!}{(1-y)^{r-1}(1-y-xy)^{r+1}},$$

while the exponential generating function of $(FL_{n,r})_{n=0}^{\infty}$ is

$$\sum_{n=0}^{\infty} \frac{FL_{n,r}}{n!} y^n = \frac{r!}{(1-y)^{r-1}(1-2y)^{r+1}}.$$

Proof 1. We begin with the proof for polynomials. Using [13, Theorem 3.10] and the binomial series, we can obtain that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{FL_{n,r}(x)}{n!} y^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} (k+r)! \left\lfloor {n \atop k} \right\rfloor_r x^k \frac{1}{n!} y^n = \sum_{k=0}^{\infty} (k+r)! x^k \sum_{n=k}^{\infty} \left\lfloor {n \atop k} \right\rfloor_r \frac{1}{n!} y^n\\ &= \sum_{k=0}^{\infty} (k+r)! x^k \frac{1}{k!} \left(\frac{y}{1-y}\right)^k \left(\frac{1}{1-y}\right)^{2r} = \frac{r!}{(1-y)^{2r}} \sum_{k=0}^{\infty} \binom{k+r}{k} \left(\frac{xy}{1-y}\right)^k\\ &= \frac{r!}{(1-y)^{2r}} \sum_{k=0}^{\infty} \binom{-r-1}{k} \left(\frac{-xy}{1-y}\right)^k = \frac{r!}{(1-y)^{2r}} \left(1-\frac{xy}{1-y}\right)^{-r-1}\\ &= \frac{r!(1-y)^{r+1}}{(1-y)^{2r}(1-y-xy)^{r+1}}.\end{aligned}$$

Now we prove the theorem for r-Fubini–Lah numbers by induction on r.

If $r = 0$, then using $FL_{n,0} = n!2^{n-1}$ $(n \geq 1)$, we get

$$\sum_{n=0}^{\infty} \frac{FL_{n,0}}{n!} y^n = 1 + \sum_{n=1}^{\infty} 2^{n-1} y^n = 1 + \frac{y}{1-2y} = \frac{1-y}{1-2y}.$$

Assume that $r \geq 1$, and the statement is true for $r-1$. Then for r, if $n \geq 0$, it follows from Theorem 1 that

$$2FL_{n,r} = r\sum_{j=0}^{n} \binom{n}{j} (n-j+1)! FL_{j,r-1} + \sum_{j=0}^{n} \binom{n}{j} (n-j)! FL_{j,r}.$$

Let $f_r(y)$ be the exponential generating function of $(FL_{n,r})_{n=0}^{\infty}$. Then the above formula implies that

$$2f_r(y) = rf_{r-1}(y)\frac{1}{(1-y)^2} + f_r(y)\frac{1}{1-y}.$$

Using the induction hypothesis for $f_{r-1}(y)$, after some calculation we arrive at the equality

$$f_r(y) = \frac{r!}{(1-y)^{r-1}(1-2y)^{r+1}}.$$

□

Proof 2. For r-Fubini–Lah numbers we can provide another proof, where we use [7, Theorem 2.1], the so-called r-compositional formula.

The exponential generating functions of the sequences

$$g_1(n) = (n+1)!, \quad g_2(n) = \begin{cases} 0 & \text{if } n = 0 \\ n! & \text{if } n \geq 1 \end{cases}, \quad h(n) = (n+r)!$$

are

$$G_1(y) = \frac{1}{(1-y)^2}, \quad G_2(y) = \frac{y}{1-y}, \quad H(y) = \frac{r!}{(1-y)^{r+1}},$$

respectively.

Then $FL_{0,r} = 1$, and for $n \geq 1$ we have

$$FL_{n,r} = \sum g_1(|Y_1|) \cdots g_1(|Y_r|)\, g_2(|Z_1|) \cdots g_2(|Z_k|)\, h(k),$$

where the sum is taken for all partitions $\{Y_1 \cup \{a_1\}, \ldots, Y_r \cup \{a_r\}, Z_1, \ldots, Z_k\}$ of the $(n+r)$-element set $\{a_1, \ldots, a_r, b_1, \ldots, b_n\}$ with distinguished elements $a_1, \ldots, a_r$.

Then, according to [7, Theorem 2.1], the exponential generating function of the sequence $(FL_{n,r})_{n=0}^{\infty}$ is

$$(G_1(y))^r\, H(G_2(y)) = \frac{1}{(1-y)^{2r}} \cdot \frac{r!}{\left(1 - \frac{y}{1-y}\right)^{r+1}} = \frac{r!(1-y)^{r+1}}{(1-y)^{2r}(1-2y)^{r+1}}.$$

□

The last identity shows that the sequence of r-Fubini–Lah polynomials is the r-Stirling transform of the first kind of the sequence of r-Fubini polynomials, and the same holds for numbers (for r-Stirling numbers of the first kind $\left[{n \atop k}\right]_r$, see [1, 2, 10]). We note that a very special case of this formula, namely for 0-Fubini–Lah numbers, appeared in [16].

Theorem 5. *If $n, r \geq 0$, then*

$$FL_{n,r}(x) = \sum_{j=0}^{n} \left[{n \atop j}\right]_r F_{j,r}(x),$$

$$FL_{n,r} = \sum_{j=0}^{n} \left[{n \atop j}\right]_r F_{j,r}.$$

Proof. Let $c \geq 1$. We count the number of coloured ordered partitions of a set with $n+r$ elements into ordered blocks such that r distinguished elements belong to distinct ordered subsets and we colour the ordered blocks containing no distinguished element with c colours.

First we arrange the elements into $j+r$ disjoint cycles ($j = 0, \ldots, n$), such that the distinguished elements belong to distinct cycles, this can be done in $\left[{n \atop j}\right]_r$ ways. After that, we partition these cycles in an ordered way such that the cycles containing a distinguished element belong to distinct blocks, and we colour the blocks containing no cycles with a distinguished element with c colours. The number of such coloured partitions is $F_{j,r}(c)$. If we multiply the cycles in each block, we get a coloured ordered partition into ordered blocks of the original $(n+r)$-element set. Therefore, the number of possibilities is $\left[{n \atop j}\right]_r F_{j,r}(c)$ for a fixed j. $\square$

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