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A note on locally finite sublattices of free lattices

Brian T. Chan*

Department of Mathematics
University of British Columbia
Vancouver BC V6T 1Z2
Canada
bchan@math.ubc.ca

Abstract

The problem of determining which infinite lattices are (isomorphic to) sublattices of free lattices is in general unsolved and extremely difficult. The most notable classes of such lattices that are known are the projective lattices which were characterized by Freese and Nation in 1978. Despite this difficulty, general properties of infinite sublattices of free lattices have been established. In 1961, Galvin and Jónsson proved that there are no uncountable chains in a sublattice of a free lattice, and in 1995, Reinhold proved that all infinite sublattices of free lattices are staircase distributive and dually staircase distributive.

In this note, we use a property of antichains in free lattices due to Jónsson and Whitman to prove that all locally finite sublattices of free lattices are countable, thereby reducing the problem of determining which infinite lattices are sublattices of free lattices.

1 Introduction

Free lattices have been the subject of much investigation within lattice theory, with Whitman introducing *Whitman's condition* [8, 9] and Jónsson introducing *semidistributive lattices* to study properties of free lattices [4, 5]. An important problem within the theory of free lattices that has received a lot of attention over the years is the problem of determining, up to isomorphism, sublattices of free lattices [1]. The majority of what is known about sublattices of free lattices is based on what we know about finite sublattices of free lattices; extensions are known for finitely generated sublattices of free lattices and projective lattices [1]. Finite sublattices of free lattices can be characterized by using Whitman's condition and a property

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involving *join covers* of elements [1]. Later on, this characterization is strengthened to requiring only the semidistributive laws and Whitman's condition [6].

A few general properties of infinite sublattices of free lattices have been established. In 1961, Galvin and Jónsson used group actions on free lattices to prove that any chain in a free lattice is countable [3], and their proof also shows that every free lattice is a countable union of antichains [1]. Moreover, in 1995, Reinhold proved that all infinite sublattices of free lattices satisfy stronger forms of the semidistributive laws known as the *staircase distributive law* and the *dual staircase distributive law* by proving that all free lattices satisfy the $*$-*distributive laws*, an infinitary form of the staircase distributive laws [7].

In this note, we use a property of antichains in free lattices due to Jónsson and Whitman [4, 8] to prove that no uncountable sublattice of a free lattice can be locally finite. We also use the order-theoretic definition of the term *locally finite*. That is, if L is a lattice, then L is *locally finite* if for all $a, b \in L$ such that $a \leq b$, the interval $\{c \in L : a \leq c \leq b\}$ is finite. Moreover, if L is a lattice and $a \in L$, then write $\uparrow_L a = \{b \in L : a \leq b\}$ and write $\downarrow_L a = \{b \in L : a \geq b\}$. Lastly, we note that in the following section, we will use [1] as a reference.

2 Locally finite sublattices

In this section, we prove that all locally finite sublattices of free lattices are countable. Antichains in free lattices satisfy the following property.

Theorem 1 *(Jónsson, Whitman [4, 8]). Let A be an antichain in a free lattice with at least two elements and let w be an element of the free lattice such that the meet of any two elements of A is w. Then A is finite.*

Proof: Let $w = w_1 \wedge \cdots \wedge w_n$ canonically. For $a \in A$ let $S_a = \{w_i : a \leq w_i\}$. Note that since $w < a$, $S_a \neq \{w_1, \ldots, w_n\}$. By the dual of Theorem 1.19 of [1], $a \wedge b = w$ implies $S_a \cup S_b = \{w_1, \ldots, w_n\}$. If A is infinite there are distinct elements a and $b \in A$ with $S_a = S_b$. But this implies $a \wedge b > w$, a contradiction. □

More can be said about properties such as the one described above. The following related theorem also holds.

Theorem 2 *In a sublattice L of a free lattice if an element is covered by five distinct elements, then L is infinite.*

Proof: Suppose $a \in L$ is covered by five distinct elements, a_1, a_2, a_3, a_4, and a_5 in L. For $i > 1$, $a = a_1 \wedge a_i$. Since free lattices are semidistributive by Theorem 1.21 of [1], this implies $a = a_1 \wedge (a_2 \vee a_3 \vee a_4 \vee a_5)$. So $a_1 \not\leq a_1 \wedge (a_2 \vee a_3 \vee a_4 \vee a_5)$, and similarly no four of the a_i's join above the fifth. So L has breadth at least 5. Since finite sublattices of a free lattice have breadth at most 4 by Corollary 1.31 of [1], the result follows. □

We first prove the following theorem using Theorem 1.

Theorem 3 *Let L be a locally finite lattice such that each element has at most countably many upper and lower covers. Then $\uparrow_L a$ and $\downarrow_L a$ are both countable, for each $a \in L$.*

Proof: Let $S_0 = \{a\}$ and let $S_{i+1} = \{a\} \cup \{c \in L : c \succ b \text{ for some } b \in S_i\}$. Note $S_i \subseteq S_{i+1}$. Inductively each S_i is countable and so $S = \bigcup S_i$ is countable (since it is the countable union of countable sets) and $S \subseteq \uparrow_L a$. But if $b > a$, then, since L is locally finite, there is a finite chain of covers going from a to b. So by an easy induction, $b \in S$. □

Now, we prove the main theorem of this note.

Theorem 4 *If L is a locally finite sublattice of a free lattice, then L is countable.*

Proof: Suppose L is an uncountable, locally finite sublattice of a free lattice. Fix $a \in L$. If $x \in L$, then $a \wedge x \in \downarrow_L a$. Since $\downarrow_L a$ is countable by by Theorem 3, there must be an element $c \in \downarrow_L a$ such that $x \wedge a = c$ for uncountably many x's. But then $\uparrow_L c$ is uncountable, contrary to Theorem 3. □

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