

# Some further results in Ramsey graph construction

FRED ROWLEY\*

*West Pennant Hills, NSW  
Australia*

fred.rowley@ozemail.com.au

## Abstract

A construction described by the current author (2017) uses two linear ‘prototype’ graphs to build a compound graph with Ramsey properties inherited from the prototypes. The resulting graph is linear; and cyclic if both prototypes are cyclic. However, it will not generate a cyclic graph from a general linear prototype. Building on the properties of that construction, this paper proves that a general linear prototype graph of order  $m$  can be extended using a single new colour to produce a new cyclic graph of order  $3m - 1$  which is triangle-free in the new colour, and has the same clique-number as the prototype in every other colour. The paper then describes a cyclic Ramsey  $(3, 3, 4, 4; 173)$ -graph derived by constrained tree search—thus proving that  $R(3, 3, 4, 4) \geq 174$ . Using a quadrupling construction to produce a further cyclic graph, it is shown that  $R(3, 4, 5, 5) \geq 693$ .

A compound cyclic Ramsey  $(3, 7, 7; 622)$ -graph derived by a limited manual search is then described. Further construction steps produce a  $(8, 8, 8; 6131)$ -graph, showing that  $R_3(8) \geq 6132$ . The paper concludes by showing that  $R_4(7) \geq 81206$  and  $R_4(9) \geq 630566$ , implying corresponding improvements in the lower bounds for  $R_5(7)$  and  $R_5(9)$  and beyond.

## 1 Introduction

This paper addresses the properties of undirected loopless graphs with edge-colourings in an arbitrary number of colours, and the corresponding multicolour classical Ramsey numbers.

The construction described in [5] allows the creation of Ramsey graphs with specific properties by combining the distance sets of two linear graphs. We refer to

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\* Formerly of Lincoln College, Oxford, UK.

these graphs as the ‘prototypes’ for the construction. The constructed graph is linear, and cyclic if both prototypes are cyclic. However, it will not generate a cyclic graph from a general linear prototype—a property that is useful on occasions if subsequent constructions require cyclic prototypes. This paper describes a development of the previous construction (itself based on a construction by Giraud [2]), which does so in a significant special case. It also records some further computational findings and results in the construction of linear and cyclic graphs.

Notation is defined in Section 2.

In Section 3, starting from the construction in [5], it is proved that a general linear prototype graph can be extended using a single new colour to produce a new **cyclic** graph of order  $3m - 1$  which is triangle-free in the new colour, and has the same clique-number as the prototype in every other colour. This result is of some practical use in providing graphs with known characteristics for use as prototypes in further constructions. Its distance profile, and variations on it, provide a broad theme for the paper.

In Section 4, the distance sets of a cyclic Ramsey  $(3, 3, 4, 4; 173)$ -graph are listed—thus proving that  $R(3, 3, 4, 4) \geq 174$ . These sets were found by a non-exhaustive tree search, subject to a heuristically-derived constraint, which made search times manageable. The form of the constraint was inspired by the broad characteristics of colourings related to the construction described in Section 3. It is further deduced that  $R(3, 4, 5, 5) \geq 693$  using a cyclic ‘quadrupling’ technique related to that described in [8].

In Section 5, the existence of a Ramsey  $(8, 8, 8; 6131)$ -graph is established. We start from a prototype  $(3, 7, 7)$ -graph, use the quadrupling construction twice, and then delete vertices. Thus it is proved that  $R_3(8) \geq 6132$ .

In Section 6, the key tables from [5] are updated to reflect further work on obtaining linear prototype graphs. It is proved that  $R_4(7) \geq 81206$  and  $R_4(9) \geq 630566$ , noting corresponding improvements in the lower bounds for  $R_5(7)$  and  $R_5(9)$  and beyond. These results follow from the existence (in all cases) of cyclic prototype subgraphs of the graphs obtained by the most basic Mathon-Shearer ‘doubling’ construction.

Section 7 includes graph specifications for the  $(3, 7, 7; 622)$ -graph and  $(3, 4, 5, 5; 692)$ -graph referenced above.

## 2 Notation

In this paper,  $K_n$  denotes the complete graph with order  $n$ .

If  $U$  denotes a complete graph with  $m$  vertices  $\{u_0, \dots, u_{m-1}\}$ , then a  $(q)$ -colouring of  $U$  is a mapping of the edges  $(u_i, u_j)$  of  $U$  into the set of integers  $s$  where  $1 \leq s \leq q$ .

The *distance* between two vertices  $u_i, u_j$ , or, equivalently, the *length* of the edge  $(u_i, u_j)$  connecting them, is defined as  $|j - i|$ .

A colouring of  $U$  is *linear* if and only if the colour of any edge  $(u_i, u_j)$  depends only on the length of that edge. In such a case the colour of an edge of length  $l$  is written  $c(l)$ .

A colouring of  $U$  is *cyclic* if and only if (a) it is linear, and (b)  $c(l) = c(m - l)$  for all  $l$  such that  $1 \leq l \leq m - 1$ .

The *clique number* of graph  $U$  in colour  $s$  is the largest integer  $i$  such that  $U$  contains a subgraph which is a copy of  $K_i$  in that colour.

A *Ramsey graph*  $U(k_1, \dots, k_r; m)$ , with all  $k_s \geq 2$ , is a complete graph of order  $m$  with a colouring such that for each colour  $s$ , where  $1 \leq s \leq r$ , there exists no complete monochromatic subgraph  $K_q$  of  $U$  in the colour  $s$  for any  $q \geq k_s$ . Equivalently, the clique number of  $U$ , in any colour  $s$ , is strictly less than  $k_s$ . Such a graph  $U$  may conveniently be described as a  $(k_1, \dots, k_r; m)$ -graph.

The *Ramsey number*  $R(k_1, \dots, k_r)$  is the unique lowest integer  $m$  such that no  $U(k_1, \dots, k_r; m)$  exists.

### 3 Construction of Cyclic Graphs from Linear Graphs

**Theorem 3.1** (*Construction Theorem*)

*Given any linear Ramsey graph  $U(k_1, k_2, \dots, k_r; m)$ , it is possible to construct a cyclic Ramsey graph  $W(k_1, k_2, \dots, k_r, 3; 3m - 1)$ .*

The theorem depends on a relatively simple construction process which adds  $2m - 1$  vertices and their incident edges, including a single new colour, to the linear prototype graph  $U$ .

PROOF: We begin by considering the set of lengths of all the edges of  $U$ , consisting of the integers  $\{l \mid 1 \leq l \leq m - 1\}$ , which we call  $L$ . A linear colouring of  $L$  gives rise to a natural partition of that set into subsets  $L_s$  containing the lengths of edges of each colour  $s$ . That is, for  $1 \leq s \leq r$ :

$$L_s = \{l \mid c(l) = s\}.$$

It is a well-known result that any linear graph  $U$  contains a copy of  $K_{k_s}$  in colour  $s$  if and only if there exists a subset of the set  $L_s$  of order  $k_s - 1$  such that each of the members of the subset and all of their non-zero pairwise differences are contained in  $L_s$ . For if such a subset exists, one can construct a set of all the vertices  $u_i \in U$  having index-numbers  $i$  in the subset. Taking the union of that set of vertices with  $u_0$  gives us the vertices of a copy of  $K_{k_s}$  in  $U$ . The converse is essentially proved by reversing the process, having first selected (using linearity) a copy of  $K_{k_s}$  with a vertex set that includes  $u_0$ .

This result provides the basis for our proof.

Using the construction in [5] we first construct a linear  $(k_1, k_2, \dots, k_r, 3; 3m - 1)$ -graph  $V$  with vertices  $v_i$ , for  $0 \leq i \leq 3m - 2$ . The set of lengths of all the edges of  $V$  may



Therefore  $H$  must have at least one edge of length at least  $2m$ . If so, then we can partition the set of index-numbers of  $H$  into two non-empty subsets  $S_1 = \{j_1, \dots, j_p\}$  and  $S_2 = \{j_{p+1}, \dots, j_{k_s}\}$ . This partition is made on the basis that the length of an edge joining any member of  $S_1$  to any member of  $S_2$  is of length at least  $2m$ . It is a straightforward consequence of the colouring that there can be no more than two such subsets. We may assume without loss of generality that the index-numbers of these subsets are strictly increasing.

We now define a mapping from the vertices of  $W$  to the vertices of  $V$  as follows:

For  $1 \leq t \leq p$ , define  $w_{j_t} \rightarrow v_{j_p-j_t}$ . Thus if there is an edge in  $W$  joining  $w_{j_x}$  and  $w_{j_y}$ , both index-numbers being members of  $S_1$ , then the length of the edge in  $V$  that joins their images is the same as in  $W$ , and the colours of those two edges are both  $s$ .

For  $p + 1 \leq t \leq s$ , define  $w_{j_t} \rightarrow v_{((5m-2)-(j_t-j_p))}$ . Again, if there is an edge in  $W$  joining  $w_{j_x}$  and  $w_{j_y}$ , both index-numbers being members of  $S_2$ , then the length of the edge in  $V$  that joins their images is the same as in  $W$ , and the colours of those two edges are both  $s$ .

It now remains only to consider the case of an edge joining  $w_{j_x}$  and  $w_{j_y}$ , where  $j_x \in S_1$  and  $j_y \in S_2$ . We can see that  $2m \leq (j_y - j_x) \leq 3m - 2$ . Since  $W$  is cyclic, we note that the colour of this edge is the same as an edge of length  $(3m - 1) - (j_y - j_x)$  also in  $W$ .

In that case, the image in  $V$  of the edge is of length  $(5m - 2) - (j_y - j_p) - (j_p - j_x) = (5m - 2) - (j_y - j_x)$ . Therefore its colour is the same as the colour of an edge in  $V$  of length  $(3m - 1) - (j_y - j_x)$ . Therefore we that the colours of the edge in  $W$  and its image in  $V$  are again both  $s$ .

Thus the image of  $H$  in  $V$  is a copy of  $K_{k_s}$  in colour  $s$ , which is another contradiction. This completes the proof. □

This quite simple result has been of some practical use in providing cyclic graphs with known characteristics for use as prototypes in further constructions. The overall profile of the distances – with one colour concentrated in a mid-range, sometimes also featuring some outlying distances – is a theme of all the constructions featured in this paper. The existence of outliers provides scope for defining broader search spaces, as illustrated in Sections 4 and 5.

#### 4 New Lower Bounds for $R(3, 3, 4, 4)$ and $R(3, 4, 5, 5)$

The distance sets for a cyclic  $(3, 3, 4, 4; 173)$ -graph derived by the author are listed below. Colours for distances greater than 86 are implied by the symmetry. The implied lower bound of 174 for  $R(3, 3, 4, 4)$  exceeds the current best lower bound quoted in [4].

Colour 1:
2, 6, 9, 10, 17, 21, 24, 25, 28, 32, 39, 40, 55, 62, 75
Colour 2:
49, 56, 59, 63, 64, 66, 67, 69, 70, 71, 72, 73, 74, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86
Colour 3:
1, 5, 11, 12, 15, 19, 20, 22, 27, 29, 30, 34, 37, 38, 44, 48, 50, 51, 54, 58, 60, 61, 68
Colour 4:
3, 4, 7, 8, 13, 14, 16, 18, 23, 26, 31, 33, 35, 36, 41, 42, 43, 45, 46, 47, 52, 53, 57, 65

Table 1: Distance Sets for a Cyclic (3,3,4,4;173)-graph.

This graph was one of several discovered through a non-exhaustive constrained tree search. The constraint imposed was merely that colour 2 (one of the triangle-avoiding colours) cannot be used for a distance less than 48.

This simple heuristic constraint follows the observation that in generating large triangle-free linear graphs with defined Ramsey properties, it is often effective in reducing search times, for a cluster of distances in the mid-range to be constrained of a single common colour. The lesser and greater distances must then largely avoid that colour. Previous papers featuring distance-based searches (notably [1]) have demonstrated this feature.

Searches with lower exclusion thresholds have not yet yielded better lower bounds. Exhaustive searches have not been possible because of the resulting time constraints.

By application of a cyclic ‘quadrupling’ construction closely related to that featured in Corollary 3 in [8], a (3, 4, 5, 5; 692)-graph can be constructed, proving that  $R(3, 4, 5, 5) \geq 693$ .

## 5 A New Lower Bound for $R_3(8)$

The construction described in Section 3 can be usefully varied by modifying the definition of  $L''_{r+1}$  in particular cases so that the order of the resulting graph  $W$  is increased.

In the case where the prototype is the well-known (7,7; 202)-graph obtained by the basic Mathon-Shearer ‘doubling’ construction, an interesting result is obtained by defining:

$$L''_{r+1} = L''_3 = \{202, 203, \dots, 207, 219, 220, \dots, 403, 415, 416, \dots, 420\}.$$

It can be seen that this retains the general shape of a cluster of distances in the mid-range, as mentioned in Section 4, while leaving some gaps in order to include

outliers. By doing this it has so far proved possible to extend the order of the resulting cyclic  $(3, 7, 7)$ -graph to 622 using only a short manual search. The colours of distances from 1 to 201 are the same as those of the  $(7, 7; 202)$ -graph, as are the colours of distances from 421 to 621. The colours of distances from 208 to 218 are copies of those from 6 to 16 and the colours of distances 404 to 414 are a reflection of them.

Next, we broadly follow the process of Theorem 7 in [7]. By applying the quadrupling construction featured in Corollary 5 in [8] twice in succession, we obtain firstly a  $(5, 7, 8; 2488)$ -graph and then a  $(9, 8, 8; 9952)$ -graph. By inspection of the construction at each stage, we can see that the degree of the first vertex in these graphs in colour 3 is 1214 and 6131 respectively. (In fact, the degrees of all vertices are equal in any colour, but that is not necessary to the proof.) These degrees have been validated by computer testing. Consider the subgraph induced in the latter case on the vertices forming the neighbourhood of the first vertex in colour 3. Since  $k_1 = 9$  this must be an  $(8, 8, 8; 6131)$ -graph, which demonstrates that  $R_3(8) \geq 6132$ . This again exceeds the current best lower bound quoted in [4].

## 6 Further Results from an Earlier Construction

The tables below update the results of the previous paper [5], allowing for the inclusion of linear isomorphic images of the well-known  $(7, 7; 202)$ - and  $(9, 9; 562)$ -graphs obtained by the most basic form of the Mathon-Shearer ‘doubling’ construction. The reader should refer to [3] and [6] for the basic construction: the subgraphs can be obtained from all such graphs by a straightforward rearrangement of vertices.

r	k	3	4	5	6	7	8	9
1		2	3	4	5	6	7	8
2		5	17	41	101	202	281	562
3		14	127	414	1069	2217	3647	8423
4		45	633	3281	20201	81205	157361	630565
5		161	4175	33494	214769	893250	2045687	9458468
6		537	32005	341965	4060301			
7		1681	160023	2712974				

Table 2: Highest order of linear Ramsey graphs known to the author. Results in blue font extend the table shown in [5]. Orders shown in blue bold font for  $r = 4$  imply  $R_4(7) \geq 81206$  and  $R_4(9) \geq 630566$ .

r	k	3	4	5	6	7	8	9
1		3.000	5.000	7.000	9.000	11.000	13.000	15.000
2		3.000	5.745	9.000	<b>14.1774</b>	<b>20.0749</b>	<b>23.6854</b>	<b>33.5112</b>
3		3.000	<b>6.3247</b>	<b>9.3865</b>	12.881			
4		3.071						
5		3.172						
6		<b>3.1996</b>						
7		3.190						

Table 3: Factors  $g_k$  calculated from the data in Table 2.

Numbers revised since the publication of [5] are shown in blue. As before, bold text indicates numbers exceeding those shown in the Radziszowski Dynamic Survey [4].

The inclusion of these graphs allows the proof that  $R_4(7) \geq 81206$  and  $R_4(9) \geq 630566$  by straightforward application of the methods of [5], with correspondingly improved lower bounds for  $R_5(7)$  and  $R_5(9)$  consistent with Table 2. Obviously there are further implied improvements in  $R_r(7)$  and  $R_r(9)$  for higher values of  $r$ .

The factors  $g_k$  in Table 3 that are highlighted in bold indicate lower bounds for the limiting growth rate of  $R_r(k)$  as  $r$  increases, i.e.  $\lim_{r \rightarrow \infty} R_r(k)^{1/r} \geq g_k$  for relevant  $r, k$ . Factors for  $r = 1$  indicate what was possible using the Giraud construction (see [2]).

## 7 Graph Specifications

The specifications of the larger key graphs mentioned above are included below. The format is simply a list of colours for edge-lengths from 1 upwards.

### (3,4,5,5; 692)-graph

43224343434433443322231323244324434323434322333243  
 14234423223314232343243332331243132342331423132323  
 43143312431323124312431423332314331233144314431233  
 12231343134314331423143312231433144314431223113312  
 43124314331433122333231143144312231143123312432323  
 11231133442311431133323321432123113321234123114333  
 33214341234143214321231123213341334143414321444123  
 41434143314331232113212341234143214341233333411321





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