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On the independence number of $(3,3)$-Ramsey graphs and the Folkman number $F_e(3,3;4)$

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Abstract

The graph G is called a $(3,3)$-Ramsey graph if in every coloring of the edges of G in two colors there is a monochromatic triangle. The minimum number of vertices of the $(3,3)$-Ramsey graphs without 4-cliques is denoted by $F_e(3,3;4)$. It is known that $20 \leq F_e(3,3;4) \leq 786$. In this paper we prove that if G is an n-vertex $(3,3)$-Ramsey graph without 4-cliques, then $\alpha(G) \leq n-16$, where $\alpha(G)$ denotes the independence number of G. Using the newly obtained bound on $\alpha(G)$ and complex computer calculations we obtain the new lower bound $F_e(3,3;4) \geq 21$.

1 Introduction

Only simple graphs are considered. Let $a_1,\dots,a_s$ be positive integers. The symbol $G \overset{e}{\rightarrow} (a_1,\dots,a_s)$ ($G \overset{v}{\rightarrow} (a_1,\dots,a_s)$) means that for every coloring of the edges (vertices) of the graph G in s colors there exist $i \in \{1,\dots,s\}$ such that there is a monochromatic a_i-clique of color i. If $G \overset{e}{\rightarrow} (3,3)$ we say that G is a $(3,3)$-Ramsey graph. The clique number and the independence number of a graph G are denoted by $\omega(G)$ and $\alpha(G)$, respectively. The classical Ramsey number $R(a_1,\dots,a_s)$ is the smallest integer n such that $K_n \overset{e}{\rightarrow} (a_1,\dots,a_s)$. All properties of the Ramsey numbers that we use in the paper can be found in [27].

Define:

$\mathcal{H}_e(a_1,\dots,a_s;q) = \{G : G \overset{e}{\rightarrow} (a_1,\dots,a_s) \text{ and } \omega(G) < q\}$,

$\mathcal{H}_e(a_1,\dots,a_s;q;n) = \{G : G \in \mathcal{H}_e(a_1,\dots,a_s;q) \text{ and } |\operatorname{V}(G)| = n\}$.

The edge Folkman numbers $F_e(a_1,\dots,a_s;q)$ are defined by $F_e(a_1,\dots,a_s;q) = \min\{|\operatorname{V}(G)| : G \in \mathcal{H}_e(a_1,\dots,a_s;q)\}$, i.e. $F_e(a_1,\dots,a_s;q)$ is the smallest positive

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integer n for which $\mathcal{H}_e(a_1, \dots, a_s; q; n) \neq \emptyset$. This notation is first defined in [19], where some important properties of the Folkman numbers are proved.

Folkman [8] proved in 1970 that $\mathcal{H}_e(a_1, a_2; q) \neq \emptyset \Leftrightarrow q \geq \max\{a_1, a_2\} + 1$. Therefore, $F_e(3, 3; q)$ exists if and only if $q \geq 4$.

From $R(3, 3) = 6$ it follows that $F_e(3, 3; q) = 6$ if $q \geq 7$. It is also known that

$$F_e(3,3;q) = \begin{cases} 8, & \textit{if } q = 6, \text{ [10]} \\ 15, & \textit{if } q = 5, \text{ [23] and [26].} \end{cases}$$

The exact value of the number $F_e(3, 3; 4)$ is not yet computed. For now it is known that

$$20 \leq F_e(3, 3; 4) \leq 786, [4][16].$$

Table 1 shows the main stages in the history of the bounds of $F_e(3, 3; 4)$.

year	lower/upper bounds	who/what
1967	any?	Erdős and Hajnal [7]
1970	exist	Folkman [8]
1972	11 –	Lin implicit in [17], implied by $F_e(3, 3; 5) \geq 10$
1975	– 10×10^{10}?	Erdős offers $100 for proof [6]
1983	13 –	implied by a result of Nenov [24]
1984	14 –	implied by a result of Nenov [25]
1986	– 8×10^{11}	Frankl and Rödl [9]
1988	– 3×10^9	Spencer [30]
1999	16 –	Piwakowski, Radziszowski and Urbański, implicit in [26]
2007	19 –	Radziszowski and Xu [28]
2008	– 9697	Lu [18]
2008	– 941	Dudek and Rödl [5]
2012	– 100?	Graham offers $100 for proof
2014	– 786	Lange, Radziszowski and Xu [16]
2017	20 –	Bikov and Nenov [4]

Table 1: History of the Folkman number $F_e(3, 3; 4)$ from [15]

More information about the numbers $F_e(3, 3; q)$ can be found in [11], [15], [16] and [29]. As seen in Table 1, the number $F_e(3, 3; 4)$ is very hard to bound and it is the most searched Folkman number. The reason for this is that we know very little about the graphs in $\mathcal{H}_e(3, 3; 4)$.

In this work we give an upper bound on the independence number of the graphs in $\mathcal{H}_e(3, 3; 4)$ by proving the following:

Theorem 1.1. *Let $G \in \mathcal{H}_e(3, 3; 4; n)$. Then*

$$\alpha(G) \leq n - 16.$$

With the help of computer calculations and Theorem 1.1 we improve the main result $F_e(3, 3; 4) \geq 20$ from [4] by proving:

Theorem 1.2. $F_e(3, 3; 4) \geq 21$.

2 Some necessary properties of the graphs in $\mathcal{H}_e(3,3;q)$

Many useful properties of the graphs in $\mathcal{H}_e(3,3;q)$ follow from the fact that homomorphism of graphs preserves the Ramsey properties. In our situation, this means:

Proposition 2.1. *Let* $G \xrightarrow{\phi} G'$ *be a graph homomorphism and* $G \overset{e}{\rightarrow} (3,3)$*. Then* $G' \overset{e}{\rightarrow} (3,3)$*.*

Proof. Suppose the opposite is true and consider a 2-coloring of $\mathrm{E}(G')$ without monochromatic triangles. Define a 2-coloring of $\mathrm{E}(G)$ in the following way: the edge $[u,v]$ is colored in the same color as the edge $[\phi(u),\phi(v)]$. Clearly, this coloring of $\mathrm{E}(G)$ does not contain monochromatic triangles. □

In the general case, it is true that $G \overset{e}{\rightarrow} (a_1,\dots,a_s) \Rightarrow G' \overset{e}{\rightarrow} (a_1,\dots,a_s)$, and $G \overset{v}{\rightarrow} (a_1,\dots,a_s) \Rightarrow G' \overset{v}{\rightarrow} (a_1,\dots,a_s)$, which is proved in the same way.

Now consider the canonical homomorphism $G \xrightarrow{\phi} K_{\chi(G)}$. If $G \overset{e}{\rightarrow} (3,3)$, then $K_{\chi(G)} \overset{e}{\rightarrow} (3,3)$, and therefore

Theorem 2.2. *[17]* $\min\{\chi(G) : G \in \mathcal{H}_e(3,3;q)\} \geq R(3,3) = 6$.

For $q \geq 5$, the inequality in Theorem 2.2 is tight. It is not known whether this inequality is tight in the case $q = 4$. Theorem 2.2 is a special case of a result of Lin [17] that $G \overset{e}{\rightarrow} (a_1,\dots,a_s) \Rightarrow \chi(G) \geq R(a_1,\dots,a_s)$.

Let $K_p + G$ denote the graph obtained by connecting every vertex of K_p by an edge to every vertex of G. We will need the following:

Proposition 2.3. *Let G be a graph such that* $G \overset{e}{\rightarrow} (3,3)$*, A be an independent set of vertices of G, and $H = G - A$. Then* $K_1 + H \overset{e}{\rightarrow} (3,3)$*.*

Proof. Consider the mapping $G \xrightarrow{\phi} K_1 + H$:

$$\phi(v) = \begin{cases} \mathrm{V}(K_1), & \textit{if } v \in A \\ v, & \textit{if } v \in \mathrm{V}(H). \end{cases}$$

It is clear that ϕ is a homomorphism, and according to Proposition 2.1, $K_1 + H \overset{e}{\rightarrow} (3,3)$. □

The usefulness of Proposition 2.3 lies in the fact that the graph G can be obtained by adding independent vertices to the smaller graph H, such that $K_1 + H \overset{e}{\rightarrow} (3,3)$. In the general case it is true that if $G \overset{e}{\rightarrow} (a_1,\dots,a_s)$, then $K_1 + H \overset{e}{\rightarrow} (a_1,\dots,a_s)$.

Remark 2.4. Another proof of Proposition 2.3 is given in the proof of Theorem 3 from [28]. However, the proposition is not explicitly formulated.

A topic of significant interest is homomorphisms in Proposition 2.1 which do not increase the clique number. They could be used to obtain non-trivial results. For example, in [12] a 20-vertex graph in $\mathcal{H}_e(3,3;5)$ is constructed. Using a homomorphism, in the same work a 16-vertex graph in $\mathcal{H}_e(3,3;5)$ is obtained from this graph.

Thus, in 1979 the bound $F_e(3,3;5) \le 16$ was proved, improving the previous result $F_e(3,3;5) \le 18$ from 1973 [14].

The graph G is vertex-critical (edge-critical) in $\mathcal{H}_e(3,3;4)$ if $G \in \mathcal{H}_e(3,3;4)$ and $G - v \notin \mathcal{H}_e(3,3;4), \forall v \in \mathrm{V}(G)$ $(G - e \notin \mathcal{H}_e(3,3;4), \forall e \in \mathrm{E}(G))$. Further, in Algorithm 5.3 we will need the following:

Theorem 2.5. *[2][3]* $\min\{\delta(G) :$ *G is a vertex-critical graph in* $\mathcal{H}_e(3,3;4)\} \ge 8$, *where $\delta(G)$ is the minimum degree of G.*

Remark 2.6. In [2] and [3], Theorem 2.5 is formulated for edge-critical graphs without isolated vertices. The proof is easily also true for vertex critical graphs.

It is not known whether the inequality in Theorem 2.5 is tight.

3 Auxiliary notation and propositions

Let $G \in \mathcal{H}_e(3,3;4)$, A be an independent set of vertices of G, and $H_1 = G - A$. By Proposition 2.3, $K_1 + H_1 \overset{e}{\to} (3,3)$. If A_1 is an independent set in H_1 and $H_2 = H_1 - A_1$, then $K_2 + H_2 \overset{e}{\to} (3,3)$. If A_2 is an independent set in H_2 and $H_3 = H_2 - A_2$, then $K_3 + H_3 \overset{e}{\to} (3,3)$, etc. This way, we obtain a sequence $G \supseteq H_1 \supseteq H_2 \supseteq H_3 \supseteq \dots$, in which $\omega(H_i) \le 3$ and $K_i + H_i \overset{e}{\to} (3,3)$. Further, in the proof of Theorem 1.2, we will use such a sequence of graphs. Because of this, the following notation is convenient:

$$\mathcal{L}(n;p) = \Big\{ G : |\mathrm{V}(G)| = n, \omega(G) < 4 \text{ and } K_p + G \overset{e}{\to} (3,3) \Big\},$$

$$\mathcal{L}(n;p;s) = \{ G \in \mathcal{L}(n;p) : \alpha(G) = s \}.$$

Obviously, $\mathcal{L}(n;0) = \mathcal{H}_e(3,3;4;n)$. The following is known.

Theorem 3.1. *[26]* $\mathcal{L}(n;1) = \emptyset$ *for* $n \le 13$, *and* $|\mathcal{L}(14;1)| = 153$.

In [4] we prove that $|\mathcal{L}(15;1)| = 2081234$ ([4], Remark 4.4 and Table 1). The graphs in $\mathcal{L}(16;1)$ are not known. In the proof of Theorem 1.2 we obtain 3 892 126 874 of the graphs in $\mathcal{L}(16;1)$, but our computations suggest that the real number is much higher. The graphs in $\mathcal{L}(15;1)$ will be used in the proofs of Theorems 1.1 and 1.2. Some properties of the graphs in $\mathcal{L}(14;1)$, $\mathcal{L}(15;1)$, and some of the graphs in $\mathcal{L}(16;1)$, are given in Tables 2, 3, and 6. We can provide the graphs obtained in this paper to researchers upon request.

Posa used the following implication to prove that $\mathcal{H}_e(3,3;5) \ne \emptyset$ (unpublished):

Proposition 3.2. *(Posa's implication; see acknowledgments in [14])*

$$G \overset{v}{\to} (3,3) \Rightarrow K_1 + G \overset{e}{\to} (3,3).$$

Also with the help of this implication, Irwing [14] obtained the bound $F_e(3,3;5) \le 18$. According to Proposition 3.2, if G is an n-vertex graph, $G \overset{v}{\to} (3,3)$ and $\omega(G) = 3$, then $G \in \mathcal{L}(n,1)$. In [26] the following is proved.

$\lvert E(G)\rvert$	#	$\delta(G)$	#	$\Delta(G)$	#	$\alpha(G)$	#
42	1	4	91	7	3	4	111
43	2	5	58	8	90	5	39
44	7	6	4	10	60	6	2
45	20					7	1
46	37						
47	45						
48	28						
49	11						
50	2						

Table 2: Some properties of the graphs in $\mathcal{L}(14;1)$ obtained in [26]

$\lvert E(G)\rvert$	#	$\delta(G)$	#	$\Delta(G)$	#	$\alpha(G)$	#
42	1	0	153	7	65	3	5
43	4	1	1 629	8	675 118	4	1 300 452
44	44	2	10 039	9	1 159 910	5	747 383
45	334	3	34 921	10	165 612	6	32 618
46	2 109	4	649 579	11	80 529	7	766
47	9 863	5	1 038 937			8	10
48	35 812	6	339 395				
49	101 468	7	6 581				
50	223 881						
51	378 614						
52	478 582						
53	436 693						
54	273 824						
55	110 592						
56	26 099						
57	3 150						
58	160						
59	4						

Table 3: Some properties of the graphs in $\mathcal{L}(15;1)$ obtained in [4]

Theorem 3.3. *[26] If $G \in \mathcal{L}(14,1)$, then $G \overset{v}{\rightarrow} (3,3)$.*

This result was used in [28] to obtain the bound $F_e(3,3;4) \geq 19$. There exist, however, graphs G in $\mathcal{L}(15,1)$ which do not have the property $G \overset{v}{\rightarrow} (3,3)$. There are 20 such graphs and they are obtained in [4] (see Remark 4.4 and Table 2). Furthermore, these graphs do not have the property $G \overset{v}{\rightarrow} (2,2,3)$. This is one of the reasons why the method in the proof of $F_e(3,3;4) \geq 19$ in [28] is inapplicable for proving $F_e(3,3;4) \geq n$ for $n \geq 20$.

By Proposition 2.3, if $G \in \mathcal{L}(n,0)$ and A is an independent set of vertices of G, then $G - A \in \mathcal{L}(n - |A|, 1)$. In [4] we formulate without proof the following generalization of this fact.

Proposition 3.4. *[4] Let $G \in \mathcal{L}(n;p)$, $A \subseteq \mathrm{V}(G)$ be an independent set of vertices of G and $H = G - A$. Then $H \in \mathcal{L}(n - |A|; p+1)$.*

Proof. Since $G \in \mathcal{L}(n;p)$, we have $K_p + G \overset{e}{\to} (3,3)$. According to Proposition 2.3, $K_1 + ((K_p + G) - A) \overset{e}{\to} (3,3)$. Since $(K_p + G) - A = K_p + (G - A) = K_p + H$ and $K_1 + (K_p + H) = K_{p+1} + H$, we obtain $K_{p+1} + H \overset{e}{\to} (3,3)$. Thus, $H \in \mathcal{L}(n - |A|; p + 1)$. □

We denote by $\mathcal{L}_{max}(n;p;s)$ the set of all maximal K_4-free graphs in $\mathcal{L}(n;p;s)$, i.e. the graphs $G \in \mathcal{L}(n;p;s)$ for which $\omega(G + e) = 4$ for every $e \in \mathrm{E}(\overline{G})$.

The graph G is called a $(+K_3)$-graph if $G + e$ contains a new 3-clique for every $e \in \mathrm{E}(\overline{G})$. Clearly, G is a $(+K_3)$-graph if and only if $N(u) \cap N(v) \neq \emptyset$ for every pair of non-adjacent vertices u and v of G, i.e. either G is a complete graph or the diameter of G is equal to 2. The set of all $(+K_3)$-graphs in $\mathcal{L}(n;p;s)$ is denoted by $\mathcal{L}_{+K_3}(n;p;s)$. Obviously, $\mathcal{L}_{max}(n;p;s) \subseteq \mathcal{L}_{+K_3}(n;p;s)$.

For convenience, we will also use the following notation:

$\mathcal{L}_{max}(n;p;\leq s) = \bigcup_{s' \leq s} \mathcal{L}_{max}(n;p;s')$,

$\mathcal{L}_{+K_3}(n;p;\leq s) = \bigcup_{s' \leq s} \mathcal{L}_{+K_3}(n;p;s')$.

It is easy to see that if G is a maximal K_4-free graph and A is an independent set of vertices in G, then $G - A$ is a $(+K_3)$-graph. Because of this, regarding the graphs in $\mathcal{L}_{max}(n;p;s)$, from Proposition 3.4 the following proposition follows easily.

Proposition 3.5. *[4] Let $G \in \mathcal{L}_{max}(n;p;s)$. Let $A \subseteq \mathrm{V}(G)$ be an independent set of vertices of G, $|A| = s$ and $H = G - A$. Then $H \in \mathcal{L}_{+K_3}(n - s; p + 1; \leq s)$.*

Furthermore, the bound $F_e(3,3;4) \geq 21$ will be proved with the help of Algorithms 5.1 and 5.3, which are based on Proposition 3.5.

Definition 3.6. The graph G is called a Sperner graph if $N_G(u) \subseteq N_G(v)$ for some pair of vertices $u, v \in \mathrm{V}(G)$.

Let $G \in \mathcal{L}(n;p;s)$ and $N_G(u) \subseteq N_G(v)$. Then $K_p + (G - u)$ is a homomorphic image of $K_p + G$ and by Proposition 2.1, $K_p + (G - u) \overset{e}{\to} (3,3)$, that is, $G - u \in \mathcal{L}(n-1;p;s')$, where $s' = s - 1$ or $s' = s$. Therefore, every Sperner graph $G \in \mathcal{L}(n;p;s)$ is obtained by adding one new vertex to some graph $H \in \mathcal{L}(n-1;p;s-1) \cup \mathcal{L}(n-1;p;s)$. In the special case when G is a Sperner graph and $G \in \mathcal{L}_{max}(n;p;s)$, from $N_G(u) \subseteq N_G(v)$ it follows that $N_G(u) = N_G(v)$ and $G - u \in \mathcal{L}_{max}(n-1;p;s-1) \cup \mathcal{L}_{max}(n-1;p;s)$. Hence the following is true.

Proposition 3.7. *Let $G \in \mathcal{L}_{max}(n;p;s)$ be a Sperner graph. Then G is obtained by duplicating a vertex in some graph $H \in \mathcal{L}_{max}(n-1;p;s-1) \cup \mathcal{L}_{max}(n-1;p;s)$.*

From Theorem 2.2 and $K_p + G \overset{e}{\to} (3,3)$, the following holds.

Proposition 3.8. *Let $G \in \mathcal{L}(n;p)$. Then $\chi(G) \geq 6 - p$.*

We will use this fact in Algorithm 5.1.

4 Proof of Theorem 1.1

Definition 4.1. For every graph H, denote by $\mathcal{M}(H)$ the set of all maximal K_3-free subsets of $\mathrm{V}(H)$. Let

$$\mathcal{M}(H) = \{M_1, \dots, M_k\}.$$

We denote by $\mathrm{B}(H)$ the graph which is obtained by adding to H new independent vertices $u_1, \dots, u_k$ and new edges incident to $u_1, \dots, u_k$ such that

$$N_{\mathrm{B}(H)}(u_i) = M_i,\ i = 1, \dots, k.$$

Lemma 4.2. *Let G be a graph with $\omega(G) = 3$, A be an independent set of vertices of G, and $H = G - A$. If $G \overset{e}{\to} (3,3)$, then $\mathrm{B}(H) \overset{e}{\to} (3,3)$.*

Proof. Let $\mathcal{M}(H) = \{M_1, \dots, M_k\}$ be the same as in Definition 4.1 and $A = \{v_1, \dots, v_s\}$. Let $v_i \in A$. Then $N_G(v_i) \subseteq M_j$ for some $j \in \{1, \dots, k\}$. Let j_i be the smallest index j such that $N_G(v_i) \subseteq M_j$. We define a supergraph $\widetilde{G}$ of G in the following way: for each $v_i \in A$ we add to $\mathrm{E}(G)$ the new edges $[v_i, u], u \in M_{j_i} \setminus N_G(v_i)$. Clearly, $\mathrm{V}(\widetilde{G}) = \mathrm{V}(G)$, A is an independent set of vertices of $\widetilde{G}$, $\widetilde{G} - A = H$ and

$$N_{\widetilde{G}}(v_i) \in \mathcal{M}(H),\ i = 1, \dots, s.$$

Since G is a subgraph of $\widetilde{G}$, it follows that

$$\widetilde{G} \overset{e}{\to} (3,3). \tag{4.1}$$

If $\{N_{\widetilde{G}}(v_1), \dots, N_{\widetilde{G}}(v_s)\}$ is a subset of $\mathcal{M}(H)$, then $\widetilde{G}$ is a subgraph of $\mathrm{B}(H)$ and from (4.1) it follows that $\mathrm{B}(H) \overset{e}{\to} (3,3)$.

Let $\{N_{\widetilde{G}}(v_1), \dots, N_{\widetilde{G}}(v_s)\}$ be a multiset and $N_{\widetilde{G}}(v_i) = N_{\widetilde{G}}(v_j)$ for some $i < j$. Let $\widetilde{G'} = \widetilde{G} - v_j$. Obviously, $\widetilde{G'}$ is a homomorphic image of G. Therefore, from Proposition 2.1 and (4.1) it follows that $\widetilde{G'} \overset{e}{\to} (3,3)$.

If in $\{N_{\widetilde{G'}}(v_i) \mid i = 1, \dots, s, i \neq j\}$ there is also a duplication, then in the same way we remove from $\widetilde{G'}$ one of the duplicating vertices and we obtain a smaller graph $\widetilde{G''}$ such that $\widetilde{G''} \overset{e}{\to} (3,3)$.

In the end, a graph $\widetilde{\widetilde{G}}$ is reached such that $H = \widetilde{\widetilde{G}} - A'$, where $A' \subseteq A$, $\{N_{\widetilde{\widetilde{G}}}(v) \mid v \in A'\}$ is a subset of $\mathcal{M}(H)$, and $\widetilde{\widetilde{G}} \overset{e}{\to} (3,3)$. Since $\widetilde{\widetilde{G}}$ is a subgraph of $\mathrm{B}(H)$, it follows that $\mathrm{B}(H) \overset{e}{\to} (3,3)$. □

Proof of **Theorem 1.1.** Suppose the opposite is true, that is, suppose $\alpha(G) \geq n-15$. Let $A = \{v_1, \dots, v_{n-15}\}$ be an independent set of vertices of G, and let $H = G - A$. Then from Lemma 4.2 it follows that $\mathrm{B}(H) \overset{e}{\to} (3,3)$.

According to Proposition 3.4 ($p = 0$), $H \in \mathcal{L}(15; 1)$. All 2 081 234 graphs in $\mathcal{L}(15; 1)$ were obtained in [4] (see Remark 4.4); see also Table 3 in this paper. With a computer we have checked that $\mathrm{B}(H) \overset{e}{\not\to} (3,3)$ for all $H \in \mathcal{L}(15, 1)$. This contradiction proves the theorem. □

Corollary 4.3. *[4]* $F_e(3, 3; 4) \geq 20$.

Proof. Suppose that G is a 19-vertex $(3, 3)$-Ramsey graph and $\omega(G) = 3$. From Theorem 1.1 it follows that $\alpha(G) \leq 3$. This contradicts the equality $R(4, 4) = 18$. □

The graphs $\mathrm{B}(H)$, $H \in \mathcal{L}(15; 1)$, in the proof of Theorem 1.1, have between 50 and 210 vertices. We used different SAT solvers, such as *MapleSAT* [20] and *zchaff* [31], to prove that these graphs are not $(3, 3)$-Ramsey graphs. The problem of determining whether a graph G is a $(3, 3)$-Ramsey graph can be transformed into a problem of satisfiability of a boolean formula ϕ_G in conjunctive normal form with $|\mathrm{E}(G)|$ variables. Let $e_1, \ldots, e_{|\mathrm{E}(G)|}$ be the edges of G and $x_1, \ldots, x_{|\mathrm{E}(G)|}$ be the corresponding boolean variables in ϕ_G. For every triangle in G formed by the edges $e_i e_j e_k$, we add two clauses to ϕ_G:

$$(x_i \vee x_j \vee x_k) \wedge (\overline{x_i} \vee \overline{x_j} \vee \overline{x_k}).$$

It is easy to see that $G \overset{e}{\rightarrow} (3, 3)$ if and only if ϕ_G is not satisfiable.

Even though the graphs $\mathrm{B}(H)$, $H \in \mathcal{L}(15; 1)$, have up to 210 vertices, SAT solvers are able to test the satisfiability of the resulting boolean formulas in a short amount of time. There exist smaller graphs G for which it is difficult to determine whether $G \overset{e}{\rightarrow} (3, 3)$. For example, Exoo conjectured that the 127-vertex graph G_{127}, used by Hill and Irwing [13] to prove the bound $R(4, 4, 4) \geq 128$, has the property $G_{127} \overset{e}{\rightarrow} (3, 3)$. This conjecture was studied in [28] and [15]. It is still unknown whether $G_{127} \overset{e}{\rightarrow} (3, 3)$.

5 Proof of Theorem 1.2

According to Proposition 3.7, all Sperner graphs in $\mathcal{L}_{max}(n; p; s)$ can be obtained easily by duplicating a vertex in graphs in $\mathcal{L}_{max}(n-1; p; s-1) \cup \mathcal{L}_{max}(n-1; p; s)$. By Proposition 3.5, the non-Sperner graphs in $\mathcal{L}_{max}(n; p; s)$ are obtained by adding s independent vertices to some graphs in $\mathcal{L}_{+K_3}(n-s; p+1; \leq s)$. This is realized with the help of the following algorithm:

Algorithm 5.1. *[4] Finding all non-Sperner graphs in $\mathcal{L}_{max}(n; p; s)$ for fixed n, p, and s.*

1. *The input of the algorithm is the set $\mathcal{A} = \mathcal{L}_{+K_3}(n-s; p+1; \leq s)$. The output will be the set $\mathcal{B}$ of all non-Sperner graphs in $\mathcal{L}_{max}(n; p; s)$. Initially, set $\mathcal{B} = \emptyset$.*

2. *For each graph $H \in \mathcal{A}$:*

   2.1 *Find the family $\mathcal{M}(H) = \{M_1, \ldots, M_t\}$ of all maximal K_3-free subsets of $\mathrm{V}(H)$.*

   2.2 *Find all s-element subsets $N = \{M_{i_1}, M_{i_2}, \ldots, M_{i_s}\}$ of $\mathcal{M}(H)$ that fulfill the conditions:*

(a) $M_{i_j} \neq N_H(v)$ *for every* $v \in \mathrm{V}(H)$ *and for every* $M_{i_j} \in N$.

(b) $K_2 \subseteq M_{i_j} \cap M_{i_k}$ *for every* $M_{i_j}, M_{i_k} \in N$.

(c) $\alpha(H - \bigcup_{M_{i_j} \in N'} M_{i_j}) \leq s - |N'|$ *for every* $N' \subseteq N$.

2.3 *For each of the* s*-element subsets* $N = \{M_{i_1}, M_{i_2}, \dots, M_{i_s}\}$ *of* $\mathcal{M}(H)$ *found in step* 2.2*, construct the graph* $G = G(N)$ *by adding new independent vertices* $v_1, v_2, \dots, v_s$ *to* $\mathrm{V}(H)$ *such that* $N_G(v_j) = M_{i_j}, j = 1, \dots, s$. *If* G *is not a Sperner graph and* $\omega(G + e) = 4, \forall e \in \mathrm{E}(\overline{G})$*, then add* G *to* $\mathcal{B}$.

3. *Remove the isomorphic copies of graphs from* $\mathcal{B}$.

4. *Remove from* $\mathcal{B}$ *all graphs with chromatic number less than* $6 - p$.

5. *Remove from* $\mathcal{B}$ *all graphs* G *for which* $K_p + G \overset{e}{\not\rightarrow} (3,3)$.

Theorem 5.2. *[4] After the execution of Algorithm 5.1, the obtained set* $\mathcal{B}$ *coincides with the set of all non-Sperner graphs in* $\mathcal{L}_{max}(n;p;s)$.

The correctness of Algorithm 5.1 is guaranteed by the proof of Theorem 5.2 in [4]. Here we will only note some details. The condition (a) has to be satisfied since $G = G(N)$ is not a Sperner graph. The condition (b) follows from the maximality of $G = G(N)$. Even if both conditions (a) and (b) are satisfied, additional checks in step 2.3 are still needed to make sure that only maximal non-Sperner graphs are added to $\mathcal{B}$. From condition (c) it follows that only graphs with independence number s are added to $\mathcal{B}$. If $N' = \emptyset$, then (c) clearly holds, since $\alpha(H) \leq s$. If $|N'| = 1$, then for each added vertex v_j it is guarantied that v_j does not form an independent set with s vertices of H. If $|N'| = 2$, then for every two added vertices v_j, v_k it is guarantied that v_j and v_k do not form an independent set with $(s-1)$ vertices of H, etc. The graphs in $\mathcal{B}$ must satisfy the condition in step 4 according to Proposition 3.8.

In ([4], Remark 2.10) we note that in the special case $n = 19, p = 0$, Algorithm 5.1 can be improved. All computations in [4] are done only with the help of Algorithm 5.1. Here we develop this idea for arbitrary n in the following way:

Algorithm 5.3. *Optimization of Algorithm 5.1 for finding all non-Sperner graphs* $G \in \mathcal{L}_{max}(n;0;s)$ *with* $\delta(G) \geq 8$.

1. *In step* 1 *we remove from the set* $\mathcal{A}$ *the graphs with minimum degree less than* $8 - s$.

2. *In step* 2.2 *we add the following conditions for the subset* N:

(d) $|M_{i_j}| \geq 8$ *for every* $M_{i_j} \in N$.

(e) *If* $N' \subseteq N$*, then* $d_H(v) \geq 8 - s + |N'|$ *for every* $v \notin \bigcup_{M_{i_j} \in N'} M_{i_j}$.

According to Theorem 2.5, the set $L_{max}(20;0;4)$ contains only graphs with minimum degree greater than or equal to 8. Therefore, at the end of the proof of Theorem 1.2 we can use Algorithm 5.3 to prove that $L_{max}(20;0;4) = \emptyset$. In this way, the computational time is reduced significantly.

Proof of **Theorem 1.2.** Suppose the opposite is true and let G be a 20-vertex maximal $(3,3)$-Ramsey graph with $\omega(G) = 3$. From Theorem 1.1 it follows that $\alpha(G) \leq 4$. Now, from $R(4,4) = 18$ it follows that $\alpha(G) = 4$. Therefore, it is enough to prove that $\mathcal{L}_{max}(20;0;4) = \emptyset$. First, we will successively obtain all graphs in the sets $\mathcal{L}_{+K_3}(8;3;\leq 4)$, $\mathcal{L}_{+K_3}(12;2;\leq 4)$, and $\mathcal{L}_{+K_3}(16;1;\leq 4)$, and then we will prove that $\mathcal{L}_{max}(20;0;4) = \emptyset$.

Obtaining all graphs in $\mathcal{L}_{+K_3}(8;3;\leq 4)$:

We use the *geng* tool included in the *nauty* package [22] to generate all non-isomorphic graphs of order 8. Among them we find all 1178 graphs in $\mathcal{L}_{+K_3}(8;3;\leq 4)$ (see Table 4).

$\lvert E(G)\rvert$	#	$\delta(G)$	#	$\Delta(G)$	#	$\alpha(G)$	#
10	1	1	15	3	2	2	3
11	3	2	552	4	108	3	705
12	28	3	560	5	610	4	470
13	114	4	49	6	387		
14	258	5	2	7	71		
15	328						
16	253						
17	127						
18	47						
19	14						
20	4						
21	1						

Table 4: Some properties of the graphs in $\mathcal{L}_{+K_3}(8;3;\leq 4)$

Obtaining all graphs in $\mathcal{L}_{+K_3}(12;2;\leq 4)$:

From $R(3,4) = 9$ it follows that $\mathcal{L}(12;2;\leq 2) = \emptyset$. All 1 449 166 12-vertex graphs G with $\omega(G) < 4$ and $\alpha(G) < 4$ are known and available [21]. Among them there are 321 graphs in $\mathcal{L}_{max}(12;2;3)$. We use *geng* to generate all non-isomorphic graphs of order 11. Among them we find all 372 graphs in $\mathcal{L}_{max}(11;2;\leq 4)$. According to Proposition 3.7, all Sperner graphs in $\mathcal{L}_{max}(12;2;4)$ are obtained by duplicating a vertex in some of the graphs in $\mathcal{L}_{max}(11;2;\leq 4)$. This way, we find all 1341 Sperner graphs in $\mathcal{L}_{max}(12;2;4)$. We execute Algorithm 5.1 ($n = 12, p = 2, s = 4$) with input set $\mathcal{A} = \mathcal{L}_{+K_3}(8;3;\leq 4)$ to output all 815 non-Sperner graphs in $\mathcal{L}_{max}(12;2;4)$. Thus, $|\mathcal{L}_{max}(12;2;\leq 4)| = 2477$. By removing edges from the graphs in $\mathcal{L}_{max}(12;2;\leq 4)$ we find all 539 410 034 graphs in $\mathcal{L}_{+K_3}(12;2;\leq 4)$ (see Table 5).

Obtaining all graphs in $\mathcal{L}_{+K_3}(16;1;\leq 4)$:

From $R(3,4) = 9$ it follows that $\mathcal{L}(16;1;\leq 2) = \emptyset$. There are only two 16-vertex graphs G such that $\omega(G) < 4$ and $\alpha(G) < 4$, [21]. We checked with a computer that none of them belongs to $\mathcal{L}(16;1)$, and therefore $\mathcal{L}(16;1;3) = \emptyset$. Thus, $\mathcal{L}(16;1;\leq 4) = \mathcal{L}(16;1;4)$ and $\mathcal{L}_{max}(16;1;\leq 4) = \mathcal{L}_{max}(16;1;4)$. All 5772 graphs in $\mathcal{L}_{max}(15;1;\leq 4)$ were obtained in part 1 of the proof of the Main Theorem in [4]. According to Proposition 3.7, all Sperner graphs in $\mathcal{L}_{max}(16;1;4)$ are obtained by duplicating a vertex in some of the graphs in $\mathcal{L}_{max}(15;1;\leq 4)$. In this way, we find

|E(G)|	#	δ(G)	#	Δ(G)	#	α(G)	#
23	5	2	3271422	5	449820	3	1217871
24	231	3	200573349	6	90348516	4	538192163
25	10970	4	317244496	7	326214208		
26	254789	5	18296860	8	113842493		
27	2675686	6	23902	9	8451810		
28	14355266	7	5	10	103082		
29	44690777			11	105		
30	88716906						
31	119843548						
32	115345475						
33	81922759						
34	44228481						
35	18667991						
36	6345554						
37	1795212						
38	437931						
39	95241						
40	18959						
41	3517						
42	617						
43	101						
44	16						
45	2						

Table 5: Some properties of the graphs in $\mathcal{L}_{+K_3}(12;2;\leq 4)$

all 21 749 Sperner graphs in $\mathcal{L}_{max}(16;1;4)$. We execute Algorithm 5.1 ($n = 16, p = 1, s = 4$) with input set $\mathcal{A} = \mathcal{L}_{+K_3}(12;2;\leq 4)$ to output all 1 676 267 non-Sperner graphs in $\mathcal{L}_{max}(16;1;4)$. Thus, $|\mathcal{L}_{max}(16;1;4)| = |\mathcal{L}_{max}(16;1;\leq 4)| = 1\,698\,016$. By removing edges from the graphs in $\mathcal{L}_{max}(16;1;\leq 4)$ we find all 3 892 126 874 graphs in $\mathcal{L}_{+K_3}(16;1;\leq 4)$ (see Table 6).

Proving that $\mathcal{L}_{max}(20;0;4) = \emptyset$:

We execute Algorithm 5.3 ($n = 20, p = 0, s = 4$) with input set $\mathcal{A} = \mathcal{L}_{+K_3}(16;1;\leq 4)$. After the completion of step 4, 19 803 568 graphs remain in the set $\mathcal{B}$ (see Table 7). None of these graphs satisfies the condition in step 5, and hence after step 5, $\mathcal{B} = \emptyset$. We obtained that there are no non-Sperner graphs in $\mathcal{L}_{max}(20;0;4)$ with minimum degree greater than or equal to 8. According to Corollary 4.3, all graphs in $\mathcal{L}_{max}(20;0;4)$ must be vertex critical. Therefore, there are no Sperner graphs in $\mathcal{L}_{max}(20;0;4)$, and by Theorem 2.5, no graphs with minimum degree less than 8. We proved that $\mathcal{L}_{max}(20;0;4) = \emptyset$, which finishes the proof. □

Some properties of the graphs in $\mathcal{L}_{+K_3}(8;3;\leq 4)$, $\mathcal{L}_{+K_3}(12;2;\leq 4)$, and $\mathcal{L}_{+K_3}(16;1;4)$ are given in Tables 4, 5 and 6. Properties of the 20-vertex graphs obtained after the completion of step 4 of Algorithm 5.3 ($n = 20, p = 0, s = 4$) are given in Table 7.

All computations were performed on a personal computer. The most time consuming part of the proof was obtaining all graphs in $\mathcal{L}_{+K_3}(16;1;4)$ by removing edges

$\lvert E(G)\rvert$	#	$\delta(G)$	#	$\Delta(G)$	#
48	1	3	2782333	7	426
49	41	4	248294425	8	269602932
50	1263	5	1961917314	9	3080309372
51	24897	6	1627736506	10	535664232
52	340818	7	51394620	11	6544240
53	3215961	8	1676	12	5672
54	20943254				
55	94567255				
56	295234663				
57	632937375				
58	926347803				
59	921306723				
60	619034510				
61	278204812				
62	82280578				
63	15662269				
64	1876177				
65	141052				
66	7088				
67	314				
68	18				
69	2				

Table 6: Some properties of the graphs in $\mathcal{L}_{+K_3}(16; 1; \leq 4) = \mathcal{L}_{+K_3}(16; 1; 4)$

$\lvert E(G)\rvert$	#	$\delta(G)$	#	$\Delta(G)$	#
86	317	8	19599716	9	35
87	8539	9	203852	10	6072772
88	94179			11	13316933
89	480821			12	411501
90	1574738			13	2327
91	3492540				
92	5122647				
93	4864736				
94	2923601				
95	1026658				
96	194534				
97	18960				
98	1272				
99	25				
100	1				

Table 7: Some properties of the 20-vertex graphs obtained after the completion of step 4 of Algorithm 5.3 ($n = 20, p = 0, s = 4$)

from the graphs in $\mathcal{L}_{max}(16; 1; 4)$, which took about 4 months. After that, executing Algorithm 5.3 ($n = 20$, $p = 0$, $s = 4$) with input the graphs in $\mathcal{L}_{+K_3}(16; 1; 4)$, was done in under 2 months.

In order to check the correctness of our computer programs implementing Algorithm 5.1, we reproduced the 153 graphs in $\mathcal{L}(14;1)$, which were first obtained in [26], in a different way. Among the graphs in $\mathcal{L}(14;1)$ there are 8 maximal graphs, all of which have independence number 4, i.e. $|\mathcal{L}_{max}(14;1)| = 8$ and $\mathcal{L}_{max}(14;1) = \mathcal{L}_{max}(14;1;4)$. Using *nauty* we obtained all 547 524 graphs in $\mathcal{L}_{+K_3}(10;2;\leq 4)$. By executing Algorithm 5.1 ($n = 14$, $p = 1$, $s = 4$) with input $\mathcal{A} = \mathcal{L}_{+K_3}(10;2;\leq 4)$ we found all 8 graphs in $\mathcal{L}_{max}(14;1;4)$. By removing edges from the graphs in $\mathcal{L}_{max}(14;1;4)$ we obtained the 153 graphs in $\mathcal{L}(14;1)$.

6 Concluding remarks

In this section we consider the possibilities for improving the inequality

$$F_e(3,3;4) \geq 21.$$

With the help of a computer in [26] the following surprising fact is proved:

Theorem 6.1. *[26]* $\min\{\alpha(G) : G \in \mathcal{H}_e(3,3;5;15)\} = 4$.

From Table 3 we see that in $\mathcal{H}_e(3,3;5)$ there are at least five 16-vertex graphs with independence number 3 (one of these graphs is given in Figure 1). The 18-vertex graph from [14] which proves $F_e(3,3;5) \leq 18$ also has independence number 3. Therefore, we have

Theorem 6.2. $\min\{\alpha(G) : G \in \mathcal{H}_e(3,3;5)\} = 3$.

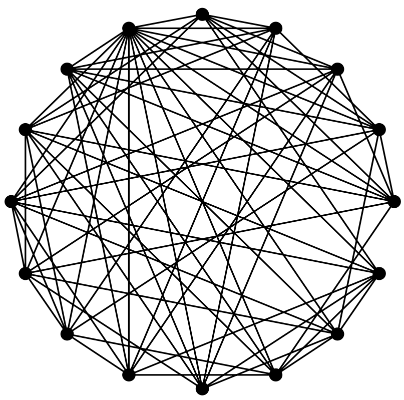

Figure 1: 16-vertex graph in $\mathcal{H}_e(3,3;5)$ with independence number 3

We believe the following conjecture is true:

Conjecture 6.3. $\min\{\alpha(G) : G \in \mathcal{H}_e(3,3;4)\} \geq 5$.

If $G \in \mathcal{H}_e(3,3;4;n)$, $n \geq 25$, according to the equality $R(4,5) = 25$ we have $\alpha(G) \geq 5$. All 24-vertex graphs with independence number 4 and clique number 3 are obtained in [1]. With the help of a computer we checked that none of these graphs belongs to $\mathcal{H}_e(3,3;4)$. In this way, we proved that if $G \in \mathcal{H}_e(3,3;4;n)$, $n \geq 24$, then $\alpha(G) \geq 5$. To prove the conjecture it remains to consider the cases $n = 21$, 22, and 23.

By similar reasoning as in the proof of Theorem 1.1, but with more calculations, potentially it could be proved that

$$\alpha(G) \leq n - 17.$$

From this inequality and Conjecture 6.3 it would follow that $F_e(3,3;4) \geq 22$.

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