

Small 4-regular planar graphs that are not circle representable

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Abstract

A 4-regular planar graph G is said to be circle representable if there exists a collection of circles drawn on the plane such that the touching and crossing points correspond to the vertices of G , and the circular arcs between those points correspond to the edges of G . Lovász (1970) conjectured that every 4-regular planar graph has a circle representation, but an infinite family of counterexamples was given by Bekos and Raftopoulou (2015). We reduce the order of the smallest known counterexamples among simple graphs from 822 to 68 based on a multigraph counterexample of order 12.

1 Introduction

A *circle representation* of a 4-regular planar (multi)graph G is a collection of circles embedded in \mathbb{R}^2 such that each point of the plane belongs to at most two circles, the set of points belonging to exactly two circles is in bijective correspondence with the vertex set of G , and the circular arcs between those points correspond to the edge (multi)set of G . A point in the intersection of two circles is a *touching* point if it is the only point at which those circles intersect, and a *crossing* point if it is one of the two points at which they intersect. We say that a graph is *circle representable* if it admits a circle representation. Two examples of circle representable graphs are given in Figure 1.

Circle representations are closely related to the classical *coin representations* of graphs, which are a collection of interior-disjoint circles in \mathbb{R}^2 such that the circles are in bijective correspondence with the vertex set of a graph G , and two vertices are adjacent in G if and only if their corresponding circles touch. The remarkable

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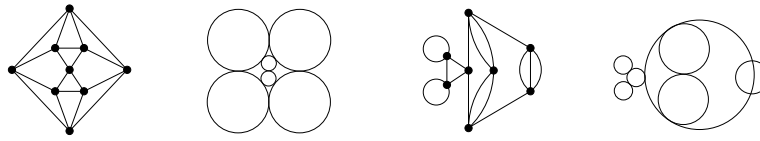


Figure 1: Circle representations of a simple graph and a multigraph.

fact that all simple planar graphs admit a coin representation, originally proved by Koebe [7], is known as the Circle Packing Theorem. Further representations of graphs involving circles and disks have since been studied in literature (see, e.g., [2–4, 6]).

In 1970, Lovász [5] conjectured an analogue of the Circle Packing Theorem for circle representations: that every simple 4-regular planar graph admits a circle representation. Bekos and Raftopoulou [1] showed that the conjecture holds for simple 3-connected 4-regular planar graphs as a consequence of the Circle Packing Theorem, and in the general case presented two infinite families of counterexamples, one of which consists of 2-connected graphs. To describe these, we introduce some terminology following [1].

Let a *gadget-subgraph* be the graph shown in Figure 2(b) where each of the shaded loops may be replaced with a plane graph with one vertex of degree 2 on the outer face and all other vertices of degree 4. We shall call such a graph a *mini-gadget*. The octahedral mini-gadget, obtained by replacing one edge of the octahedron graph with a path of length 2, is the smallest and yields the gadget-subgraph shown in Figure 2(c). Similarly, Figures 2(d) and 2(e) depict *bigadget-subgraphs* with abstract and octahedral *mini-bigadgets* respectively. The counterexamples in [1] are then constructed by taking a subdivision of the octahedron shown in Figure 2(a), and attaching one copy of a (bi)gadget-subgraph for each dotted line in the figure by identifying the degree 2 vertices in the gadget to those in the base. Using only bigadget-subgraphs produces a 2-connected counterexample.

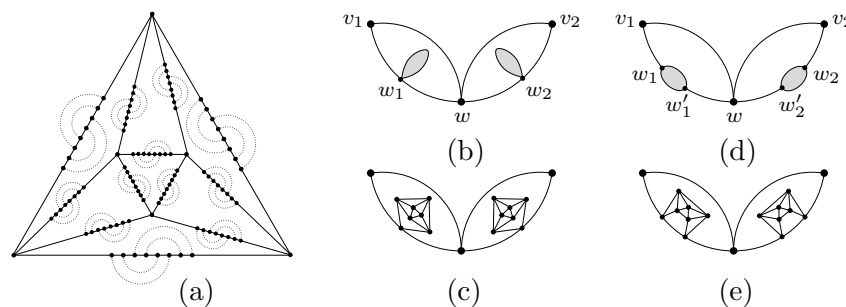


Figure 2: (a) The base octahedron, with dotted lines indicating placement of gadgets. Gadget-subgraphs with (b) abstract and (c) octahedral mini-gadgets are illustrated, as well as bigadget subgraphs with (d) abstract and (e) octahedral mini-bigadgets.

The graphs so constructed all have at least 822 vertices, so the authors asked for the smallest counterexample to Lovász’ conjecture. As a step toward answering this question, we construct two counterexamples on 68 vertices, again one of which is 2-connected. The structure of our counterexamples borrows from the configuration

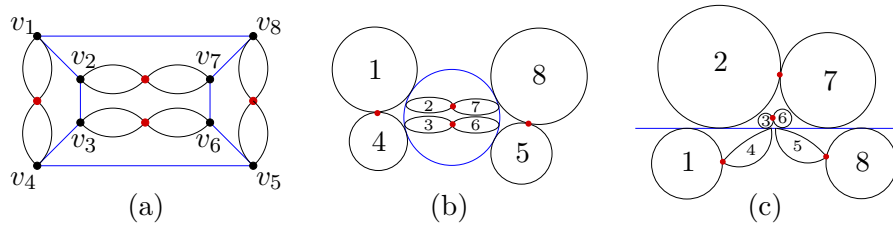


Figure 3: (a) The multigraph M . (b) A circle configuration that must be realisable if M were circle representable. (c) The induced configuration. Vertices of M that are adjacent to two digons are tracked in red, and the 8-cycle in blue. Each simple closed curve numbered i should be regarded as a circle C_i .

along one edge of Bekos and Raftopoulou’s base octahedron, and in particular will use the same gadgets and mini-gadgets. The key tools that we bring are the use of Möbius transformations (see [8] for some geometric intuition), and the introduction of multigraphs.

2 A base multigraph

Studying circle representations of multigraphs is actually somewhat easier than their simple counterparts because there are very few ways in which loops and digons can be represented. Capitalising on this, we construct the multigraph M shown in Figure 3(a) which consists of an 8-cycle $v_1v_2 \dots v_8v_1$ together with four pairs of neighbouring digons attached at each of (v_1, v_4) , (v_2, v_7) , (v_3, v_6) and (v_5, v_8) . We claim that M is not circle representable.

The idea of the proof is to first show that any circle representation of M must have four pairs of touching circles, this being the only way to realise the neighbouring digons, together with one additional circle representing the 8-cycle with adjacencies as shown in Figure 3(b). It then suffices to show that this configuration cannot be realised. We will handle this geometric aspect first.

To simplify the argument, we consider an equivalent configuration obtained by applying a Möbius transformation. Explicitly, this will be given by the inverse stereographic projection from the plane to S^2 , followed by a rotation of the sphere that takes a chosen point to the north pole, and then a stereographic projection back to the plane. By carefully choosing a point on the circle representing the 8-cycle to be sent to infinity, the transformation maps this circle to a line and yields an instance which we call the *induced configuration*. Without loss of generality, assume that the line in the image is the x -axis. The configuration then consists of eight circles $\{C_i\}_{i=1, \dots, 8}$ where each C_i has radius $r_i > 0$ and touches the x -axis at $(t_i, 0)$, numbered so that $t_1 < t_2 < \dots < t_8$, the circles in $\{C_1, C_4, C_5, C_8\}$ are disjoint from those in $\{C_2, C_3, C_6, C_7\}$, and in addition the pairs (C_1, C_4) , (C_2, C_7) , (C_3, C_6) and (C_5, C_8) are touching circles. This is illustrated in Figure 3(c), where the circle labelled i corresponds to C_i .

In fact, we can be even more restrictive. Suppose we have eight circles

$\{C_i(r_i, t_i)\}_{i=1, \dots, 8}$ arranged as per the induced configuration with the extra condition that (C_1, C_8) , (C_4, C_5) , (C_2, C_3) and (C_6, C_7) are also pairs of touching circles. We shall call this the *symmetric configuration*.

Lemma 2.1. *The induced configuration can be realised (by circles) only if the symmetric configuration can be realised.*

Proof. Let $\{C_i(r_i, t_i)\}_{i=1, \dots, 8}$ be in the induced configuration, with $C_2, C_3, C_6,$ and C_7 above the axis. Holding C_2 and C_6 fixed, replace C_3 and C_7 with two new circles, say C'_3 and C'_7 , that are both tangent to C_2, C_6 and the axis. Then $r'_3 > r_3$, which implies that $t_6 - t_3 = 2\sqrt{r_3 r_6} < 2\sqrt{r'_3 r_6} < t_6 - t'_3$, and hence $t'_3 < t_3 < t_4$. Here, we are using the fact that C_3 and C_6 are tangent circles that both touch the same line. Also, we know that $t_2 < t'_3$ by the choice of C'_3 being tangent to C_2 and C_6 , so this replacement preserves the order of the circles. Similarly, we have $r'_7 < r_7$ from which we deduce that $t_6 < t'_7 < t_7 < t_8$. Since the inequalities are strict at each step, no circle below the axis is tangent to any circle above the axis. The adjustments below the axis are similar. By possibly replacing C_1 with a larger circle with the same tangencies, we may assume that $r_1 > r_5$. Then, fixing C_1 and C_5 and replacing C_4 and C_8 produces the symmetric configuration. \square

The following lemma contains the key geometric properties satisfied by the systems of four circles above and below the axis in the symmetric configuration.

Lemma 2.2. *Suppose we have four circles $\{C_i(r_i, t_i)\}_{i=1,2,3,4}$ in the plane with radii $r_i > 0$ and which are tangent to the x-axis at points $(t_i, 0)$ respectively, and assume the circles are numbered so that $t_1 < t_2 < t_3 < t_4$. In addition, suppose that C_1 is tangent to C_2 and C_4 , and C_3 is tangent to C_2 and C_4 . Let $n = t_4 - t_1$, $m = t_3 - t_2$, $\ell = t_2 - t_1$ and $r = t_4 - t_3$ (see Figure 4). Then:*

- (i) $m \cdot n = \ell \cdot r$,
- (ii) m is determined by ℓ and r , where $m = f(\ell, r) := \frac{-(\ell+r) + \sqrt{(\ell+r)^2 + 4\ell r}}{2}$ for $\ell, r > 0$,
- (iii) $f(\ell, r)$ is increasing in both ℓ and r for $\ell, r > 0$.

Proof. The arrangement has been specified so that

$$(t_4 - t_1)^2 (t_3 - t_2)^2 = (4r_1 r_4)(4r_2 r_3) = (4r_1 r_2)(4r_3 r_4) = (t_2 - t_1)^2 (t_4 - t_3)^2,$$

as illustrated in Figure 4. This implies (i).

For (ii) we can substitute $n = \ell + m + r$ into (i) to obtain the equation $\ell r = m(\ell + m + r)$. Solving for m gives the expression claimed where, since we are dealing with lengths, the only positive root of the quadratic equation has been chosen.

Using the explicit expression for m as a function of ℓ and r , we can compute

$$\frac{\partial}{\partial \ell} f = \frac{1}{2} \left(\frac{\ell + 3r}{\sqrt{(\ell + r)^2 + 4\ell r}} - 1 \right)$$

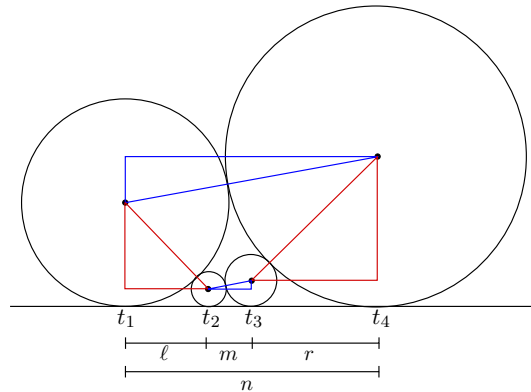


Figure 4: Configuration of circles in Lemma 2.2.

which is positive whenever ℓ and r are both positive. By symmetry, it is also true that $\frac{\partial}{\partial r} f$ is positive whenever ℓ and r are both positive, which completes the proof of (iii). \square

Lemma 2.3. *The symmetric and induced configurations cannot be realised.*

Proof. Let $\{C_i(r_i, t_i)\}_{i=1, \dots, 8}$ be in the symmetric configuration, and note that Lemma 2.2 applies to both $\{C_1, C_4, C_5, C_8\}$ and $\{C_2, C_3, C_6, C_7\}$. Let $\ell = t_3 - t_2$, $m = t_6 - t_3$, $r = t_7 - t_6$, $\ell' = t_4 - t_1$, $m' = t_5 - t_4$, and $r' = t_8 - t_5$. Then with f as defined in Lemma 2.2(ii), we can write $m = f(\ell, r)$ and $m' = f(\ell', r')$.

Since $t_1 < t_2 < t_3 < t_4$, we have that $\ell < \ell'$. As f is increasing in both variables by Lemma 2.2(iii), it follows that $m = f(\ell, r) < f(\ell', r)$. Similarly, from $t_5 < t_6 < t_7 < t_8$ we obtain the inequality $r < r'$, and hence $f(\ell', r) < f(\ell', r')$ where the right hand side is now just m' . Altogether, this means that $m < m'$. However, we also have $t_3 < t_4 < t_5 < t_6$ which implies that $m > m'$, giving a contradiction. This shows that the symmetric configuration cannot be realised by circles, so by Lemma 2.1, the induced configuration cannot be realised by circles either. \square

Theorem 2.4. *The multigraph M shown in Figure 3(a) is not circle representable.*

Proof. We first observe that M has a unique embedding on the sphere, since the cube graph is 3-connected. In addition, given a circle representation of a graph, one can obtain circle representations with any choice of outer face by applying a Möbius transformation that sends an interior point of that face to infinity. Thus, it is enough to show that the chosen embedding of M shown in Figure 3(a) does not have a circle representation.

Let $\{v_i\}_{i=1, \dots, 8}$ be the set of vertices in M incident to only one digon, numbered so that $S = v_1 \dots v_8 v_1$ is the cycle consisting of simple edges (see Figure 3(a)). Any pair of neighbouring digons sharing exactly one vertex must be realised by two circles that touch at that common vertex. This is because if one of the digons were produced by crossing circles, then the edges of the neighbouring digon are realised by arcs of the same two circles which is only possible if the two digons share both of their vertices. Hence, M must have one circle representing each digon. This means that one of the

circles on which v_i lies corresponds to a digon for each $i = 1, 2, \dots, 8$, so the edges of S incident to v_i lie on the same circle. Stringing this together, we find that all of the simple edges lie on the same circle, so S must be represented by a single circle, say C_0 . Furthermore, if a pair of vertices that do not form a 2-cut are joined by exactly two parallel edges, then the parallel edges must appear consecutively in the cyclic ordering at both of those vertices. From this we conclude that each v_i is a touching point.

Now suppose a circle representation of M exists. It must have one circle C_0 corresponding to S , and then 8 more circles C_1, \dots, C_8 labelled so that v_i is the tangent point of C_i with C_0 , and these points occur in the cyclic order around C_0 . In addition, (C_1, C_4) , (C_2, C_7) , (C_3, C_6) and (C_5, C_8) are pairs of touching circles. Then by applying a Möbius transformation that sends a point on the open arc of C_0 representing the edge v_1v_8 to infinity, we would obtain a realisation of the induced configuration, thereby contradicting Lemma 2.3. \square

3 Small simple counterexamples

From our base multigraph M , we proceed to construct simple counterexamples by subdividing one edge of each digon to obtain a simple graph, and then attaching gadgets at the degree 2 vertices to ensure 4-regularity. Taking eight copies of either the octahedral mini-gadget or the octahedral mini-bigadget as our gadgets gives the graphs of order 68 shown in Figure 5.

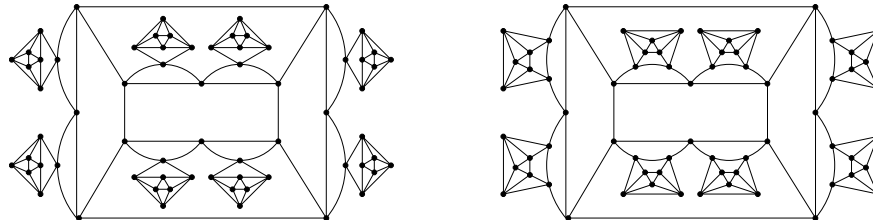


Figure 5: Simple graphs on 68 vertices that are not circle representable.

Lemma 3.1 ([1], Lemmas 5 and 12). *Let G be a 4-regular planar graph with at least one copy of the gadget-subgraph or bigadget-subgraph, labelled as in Figure 2. Then in any circle representation of G , each circle that contains an edge of a mini-(bi)gadget consists exclusively of edges belonging to that mini-(bi)gadget except for possibly, in the bigadget case, one circle containing the path $w_i v_i w'_i$ and another containing $w_i w'_i$ ($i = 1$ or 2).*

We note that in the statement of Lemma 12 in [1], the edge denoted (w_i, w'_i) should read (v_i, w) . Our next lemma plays a similar role to Corollaries 1 and 2 in [1], the difference being that we wish to remove mini-(bi)gadgets rather than entire (bi)gadget-subgraphs.

Lemma 3.2. *Let G be a 4-regular planar multigraph with a specified gadget-subgraph or bigadget-subgraph. Using the labelling of Figure 2, let G' be the multigraph obtained from G by removing the vertices of the two mini-(bi)gadgets associated to the chosen (bi)gadget-subgraph together with all incident edges, and adding a possibly parallel edge v_iw for $i = 1, 2$. If G has a circle representation, then so does G' .*

Proof. We shall remove one mini-(bi)gadget at a time. Suppose that the subcollection of circles containing the edges of our mini-(bi)gadget all consist exclusively of edges belonging to that mini-(bi)gadget. Then the mini-(bi)gadget can be removed by simply deleting those circles; the arcs representing its edges are gone, and at least one of the two circles whose touching or crossing point represented any given vertex has been deleted. Indeed, the only vertices for which one of its two defining circles remain are w_i if we have a mini-gadget, or w_i and w'_i in a mini-bigadget. This leaves an arc corresponding to an extra edge v_iw , so we precisely have a circle representation of G' .

By Lemma 3.1, the only other possibility is that among the circles representing the edges of a mini-bigadget, there are two that contain some edges belonging to the mini-bigadget and some that do not. In this case, deleting all arcs corresponding to edges of the mini-bigadget nearly gives the required circle representation of G' , except that the Jordan curve formed by the arcs representing $w_iv_iww'_i$ and $w_iw'_i$ is not a circle. To fix this, we transform the latter arc into that needed to complete the circle containing $w_iv_iww'_i$. This new arc existed in the original circle representation of G , representing some path within the now deleted mini-bigadget, so no extra intersections are produced. \square

Theorem 3.3. *The simple graphs shown in Figure 5 are not circle representable.*

Proof. Suppose we have a circle representation of either graph. Then applying Lemma 3.2 once for each (bi)gadget-subgraph would leave a circle representation of the base multigraph M from Figure 3(a). This contradicts Theorem 2.4. \square

One can generate more counterexamples on 68 vertices by using combinations of octahedral mini-gadgets and mini-bigadgets, and infinitely many larger counterexamples by attaching larger gadgets to the same base multigraph.

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