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Pancyclicity of 4-connected claw-free bull-free graphs

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Abstract

A graph G is said to be pancyclic if G contains cycles of lengths from 3 to $|V(G)|$. The bull $B(i,j)$ is obtained by associating one endpoint of each of the path P_{i+1} and P_{j+1} with distinct vertices of a triangle. In [M. Ferrara et al., *Discrete Math.* 313 (2013), 460–467], it was shown that every 4-connected $\{K_{1,3}, B(i,j)\}$-free graph with $i+j=6$ is pancyclic. In this paper we show that every 4-connected $\{K_{1,3}, B(i,j)\}$-free graph with $i+j=7$ is either pancyclic or it is the line graph of the Petersen graph.

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1 Introduction

We use [1] for terminology and notation not defined here, and we only consider finite simple graphs. Let G be a graph. If $v \in V(G)$ and $S \subseteq V(G)$, we say that $G[S]$ is the subgraph induced in G by S, $N(v)$ is the neighborhood of v in G, $d(v) = |N(v)|$, and $N(S) = \bigcup_{v \in S} N(v)$. The path with n vertices is denoted by P_n. Given a family $\mathcal{F}$ of graphs, G is said to be $\mathcal{F}$-free if G contains no member of $\mathcal{F}$ as an induced subgraph. If $\mathcal{F}= \{K_{1,3}\}$, then G is said to be claw-free. A graph G is hamiltonian if it contains a spanning cycle and pancyclic if it contains cycles of lengths from 3 to $|V(G)|$. In 1984, Matthews and Sumner [6] conjectured that every 4-connected claw-free graph is hamiltonian. This conjecture is still open and it has also fostered a large body of research into other structural properties of cycles for claw-free graphs. In this paper we are specifically interested in the pancyclicity of claw-free net-free graphs.

Let Ł denote the graph obtained by connecting two disjoint triangles with a single edge, and let $N(i, j, k)$ denote the net obtained by identifying each vertex of a triangle K_3 with an endpoint of three disjoint paths $P_{i+1}, P_{j+1}, P_{k+1}$, respectively. We refer to $N(i, j, 0)$ as the generalized bull, and denote it by $B(i, j)$.

Theorem 1.1 *(Gould, Łuczak, Pfender [4]) Let X and Y be connected graphs on at least three vertices. If neither X nor Y is P_3 and Y is not $K_{1,3}$, then every 3-connected $\{X, Y\}$-free graph G is pancyclic if and only if $X = K_{1,3}$ and Y is a subgraph of one of the graphs in the family*

$$\mathcal{F} = \{P_7, Ł, N(4, 0, 0), N(3, 1, 0), N(2, 2, 0), N(2, 1, 1)\}.$$

Motivated by the Matthews-Sumner Conjecture and Theorem 1.1, Ron Gould came up with the following problem at the 2010 SIAM Discrete Math Meeting in Austin, TX.

Problem 1.2 *Characterize the pairs of forbidden subgraphs that imply a 4-connected graph is pancyclic.*

Theorem 1.3 *(Ferrara, Morris, Wenger [3]) Every 4-connected $\{K_{1,3}, P_{10}\}$-free graph is either pancyclic or is the line graph of the Petersen graph.*

Theorem 1.4 *(Lai, Zhan, Zhang, and Zhou[5]) Every 4-connected $\{K_{1,3}, N(8, 0, 0)\}$-free graph is either pancyclic or is the line graph of the Petersen graph.*

Theorem 1.5 *(Ferrara, Gehrke, Gould, Magnant, and Powell [2]) Every 4-connected $\{K_{1,3}, B(i, j)\}$-free graph, where $i + j = 6$, is pancyclic.*

The result of this paper is as follows.

Theorem 1.6 *Every 4-connected $\{K_{1,3}, B(i,j)\}$-free graph with $i + j = 7$ is either pancyclic or is the line graph of the Petersen graph.*

The line graph of the Petersen graph is 4-connected $\{K_{1,3}, B(i,j)\}$-free if $i+j = 7$, but is not $\{K_{1,3}, B(i,j)\}$-free if $i + j = 6$, and it contains no cycle of length 4. So Theorem 1.6 implies Theorem 1.5.

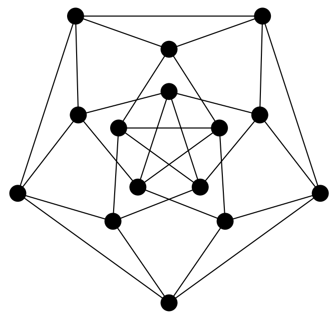

Figure 1. The line graph of the Petersen graph is the unique 4-connected $\{K_{1,3}, B(i,j)\}$-free graph with $i + j = 7$ that is not pancyclic.

In Section 2, we will show that every 4-connected $\{K_{1,3}, B(i,j)\}$-free graph with $i+j = 7$ contains cycles of all lengths from 9 to $|V(G)|$ by showing that if G contains a t-cycle ($t \geq 10$), then G also contains a $(t-1)$-cycle. The existence of a 3-cycle follows immediately from the fact that G is claw-free. For t-cycles with $4 \leq t \leq 5$, we use arguments based on the induced graphs $N(8,0,0)$ or P_{10}. For t-cycles with $6 \leq t \leq 8$, we use similar arguments based on the induced graphs P_{10}. The proof of the existence of short cycles ($4 \leq t \leq 8$) will be given in Section 3.

2 Long Cycles

Before we proceed, we introduce some additional notation. For the remainder of the paper, we will let $G[\{x,y,z\} \cup \{x_1,\ldots,x_i\} \cup \{y_1,\ldots,y_j\} \cup \{z_1,\ldots,z_k\}]$ denote a copy of $N(i,j,k)$ with central triangle xyz and appended paths $xx_1 \ldots x_i, yy_1 \ldots y_j$, and $zz_1 \ldots z_k$. A copy of the bull $B(i,j)$ is denoted $G[\{x,y,z\} \cup \{x_1,\ldots,x_i\} \cup \{y_1,\ldots,y_j\}]$ where xyz is the central triangle with appended paths $xx_1 \ldots x_i$ and $yy_1 \ldots y_j$. The following result allows us to establish the hamiltonicity of the graphs under consideration.

Lemma 2.1 *(Ferrara, Gehrke, Gould, Magnant, and Powell [2]) Let G be a 4-connected $K_{1,3}$-free graph containing a cycle C of length $t \geq 4$. If C has a chord or if there is a vertex $w \in V(G) - V(C)$ with at least 4 neighbors on C, then G contains another cycle C' of length $t - 1$.*

Lemma 2.2 *Let G be a 4-connected $\{K_{1,3}, B(i,j)\}$-free graph of order n with $i+j = 7$ and $i, j \neq 0$ and let C be a cycle of length $t \geq 10$ in G. Then G contains another cycle C' of length $t - 1$.*

Proof. Assume that G contains no $(t-1)$-cycles. By Lemma 2.1, C is chordless, and if $w \in V(G) - V(C)$ with $N(w) \cap V(C) \neq \emptyset$, then $|N(w) \cap V(C)| \leq 3$. Let $C = v_1 v_2 \ldots v_t v_1$.

Claim 1. Let $x \in V(G) - V(C)$. If $N(x) \cap V(C) \neq \emptyset$, then $|N(x) \cap V(C)| = 3$. Moreover, these three neighbors of x are consecutive on C.

By contradiction, we assume that $|N(x) \cap V(C)| \neq 3$. Then $|N(x) \cap V(C)| \leq 2$. Since $N(x) \cap V(C) \neq \emptyset$, we assume that $xv_i \in E(G)$. As $v_{i+1}v_{i-1} \notin E(G)$, we have either $v_{i+1}x \in E(G)$ or $v_{i-1}x \in E(G)$. Without loss of generality, we assume that $xv_{i-1} \in E(G)$. As $|N(x) \cap V(C)| \leq 2$, $xw \notin E(G)$ for $w \in V(C) - \{v_i, v_{i-1}\}$. As $t \geq 10$, the subgraph induced by $\{x, v_i, v_{i-1}\} \cup (V(C) - \{v_i, v_{i-1}\})$ contains a $B(i,j)(i+j=7)$, a contradiction. Claim 1 holds.

By Claim 1, every vertex with a neighbor on C has exactly three neighbors on C which are consecutive. For $1 \leq i \leq t$, let $V_i = N(v_{i-1}) \cap N(v_i) \cap N(v_{i+1})$ where indices are taken modulo t. If there is a vertex $w \notin V(C) \cup \bigcup_{i=1}^{t} V_i$ that has a neighbor w_i in some V_i, then $\{w_i, v_{i-1}, v_{i+1}, w\}$ induces a claw. Thus the sets $\{V_1, V_2, \ldots, V_t\}$ is a partition of $V(G) \backslash V(C)$. If there is an edge joining V_i and V_j when $|i-j| > 2 (\text{mod } t)$, we assume that $w_i \in V_i$, $w_j \in V_j$ and $w_i w_j \in E(G)$. Since $G[\{w_i, w_j, v_{i-1}, v_{i+1}\}] \neq K_{1,3}$, we have either $w_j v_{i+1} \in E(G)$ or $w_j v_{i-1} \in E(G)$. Thus $|N(w_j) \cap V(C)| \geq 4$, a contradiction. If there is an edge $w_i w_{i+2}$ between V_i and V_{i+2}, then $v_1 v_2 \ldots v_{i-1} w_i w_{i+2} v_{i+3} \ldots v_t v_1$ is a cycle of length $t-1$, a contradiction. If there are two nonconsecutive values $i < j$ such that $V_i = \emptyset$ and $V_j = \emptyset$, then $\{v_i, v_j\}$ is a cut set, a contradiction. Therefore, the set $\{i | V_i = \emptyset, i = 1, 2, \ldots, t\}$ has at most two elements. If the set has two elements, the indices are adjacent. Without loss of generality, we assume that for $i \in \{1, 2, \ldots, t-3\}$, $V_i \neq \emptyset$. Let $w_i \in V_i$. By Claim 1, $w_1, w_2, \ldots, w_{t-3}$ are distinct vertices. Let $C_3 = v_1 v_2 w_1 v_1$ be the 3-cycle. Then we can get the 4-cycle C_4 by inserting w_2 into C_3 as $C_4 = v_1 w_2 v_2 w_1 v_1$. Inserting v_3 into C_4, we can get the 5-cycle $C_5 = v_1 w_2 v_3 v_2 w_1 v_1$. Using this method, we can get all cycles of lengths from 3 to $2t-5$. As $t \geq 10$, G has a $(t-1)$-cycle, a contradiction. □

Theorem 2.3 *(Lai et al. [7]) Every 3-connected $\{K_{1,3}, B(i,j)\}$-free graph with $i + j \leq 8$ is hamiltonian.*

By Lemma 2.2 and Theorem 2.3, G contains cycles of length $|V(G)|$ through 9.

3 Short Cycles

In this section we will prove that if G is a 4-connected $\{K_{1,3}, B(i,j)\}$-free graph with $i + j = 7$ and if G is not the line graph of the Petersen graph, then G has t-cycles, where $4 \leq t \leq 8$. Suppose that $P_n = v_1 v_2 \ldots v_n$ is an induced path in G. Since G is claw-free, the following property follows.

(CF1) If $x \in V(G)\backslash V(P_n)$ is adjacent to v_i for $i \in \{2, 3, \ldots, n-1\}$, then x is adjacent to either v_{i+1} or v_{i-1}.

(CF2) If $x \in V(G)\backslash V(P_n)$, then $|N(x) \cap V(P_n)| \leq 4$. Furthermore, if $|N(x) \cap V(P_n)| = 4$, then $N(x) \cap V(P_n) = \{v_i, v_{i+1}, v_j, v_{j+1}\}$ for some $1 \leq i < j < n$.

Lemma 3.1 *If G is a 4-connected $\{K_{1,3}, B(i,j)\}$-free graph with $i + j = 7$, then G is the line graph of the Petersen graph or G has a 4-cycle.*

Proof. Suppose that G is a 4-connected $\{K_{1,3}, B(i,j)\}$-free graph with $i+j = 7$ and that G does not have 4-cycles. Since G is claw-free, the neighborhood of any vertex is either connected or two cliques. Since G is 4-connected, the minimum degree of G is at least 4. If the neighborhood of a vertex is connected, then it contains a path of length 3, yielding a 4-cycle. Thus the neighborhood of any vertex is two cliques. If a vertex has degree at least 5, then one of the cliques has at least three vertices, yielding a 4-cycle. Thus

(A1) G is 4-regular and, for any $v \in V(G)$, $G[N(v) \cup \{v\}]$ are two triangles identified at v.

Since G is $B(i,j)$-free with $i+j = 7$, by Theorem 1.4, we have $i, j \geq 1$. We prove the lemma by considering the following three cases.

Case 1. $B(i,j) = B(6,1)$.

Since G is a 4-connected $K_{1,3}$-free graph and G does not have 4-cycles, by Theorem 1.5, G has an induced subgraph $B(6,0)$. Let $B(6,0)$ be the graph obtained from $P_8 = v_1v_2 \ldots v_8$ by adding a vertex v and joining v to v_1 and v_2. By (A1), let $a_1, a_2 \in V(G) - V(B(6,0))$ be the other two adjacent neighbors of v, and let $b_1, b_2 \in V(G) - V(B(6,0))$ be the other two adjacent neighbors of v_1.

Let $x \in \{a_1, a_2, b_1, b_2\}$. Since G does not have 4-cycles, $N(x) \cap \{v_2, v_3\} = \emptyset$. Furthermore, as $G[\{v, v_1, v_2\} \cup \{v_3, \ldots, v_8\} \cup \{x\}] \neq B(6,1)$, $N(x) \cap \{v_4, v_5, \ldots, v_8\} \neq \emptyset$. If $N(a_1) \cap V(B(6,0)) = \{v, v_6, v_7\}$, then $v_5, v_6, v_7, v_8 \notin N(a_2)$, since G has no 4-cycles. By (CF1), $v_4 \notin N(a_2)$, a contradiction. Therefore $N(x) \cap \{v_4, v_5, \ldots, v_8\} \neq \{v, v_6, v_7\}$, and $N(x) \cap \{v_4, v_5, \ldots, v_8\} \in \{\{v_4, v_5\}, \{v_5, v_6\}, \{v_7, v_8\}, \{v_8\}\}$. Without loss of generality, we may assume that $N(a_1) \cap V(B(6,0)) = \{v, v_4, v_5\}$, $N(a_2) \cap V(B(6,0)) = \{v, v_7, v_8\}$, $N(b_1) \cap V(B(6,0)) = \{v_1, v_5, v_6\}$ and $N(b_2) \cap V(B(6,0)) = \{v_1, v_8\}$.

Let $c_1 \in N(b_2) \cap N(v_8)$. Since G does not have 4-cycles, $v_6, v_7, v_2 \notin N(c_1)$. Since $G[\{c_1, b_2, v_8\} \cup \{v_1, v_2, v_3, v_4, v_5, v_6\} \cup \{a_2\}] \neq B(6,1)$, we have $N(c_1) \cap V(B(6,0)) = \{v_8, v_3, v_4\}$. By (A1), there is $c_2 \in N(v_6) \cap N(v_7)$. If $N(c_2) \cap V(B(6,0)) = \{v_6, v_7\}$, then $G[\{c_2, v_6, v_7\} \cup \{v_5, v_4, v_3, v_2, v_1, b_2\} \cup \{a_2\}]$ is a $B(6,1)$, a contradiction. So $N(c_2) \cap V(B(6,0)) = \{v_2, v_3, v_6, v_7\}$. Then G is the line graph of the Petersen graph.

Case 2. $B(i,j) = B(5,2)$.

Since G is a 4-connected $K_{1,3}$-free graph and G does not have 4-cycles, by Theorem 1.5, G has an induced subgraph $B(5,1)$. Let $B(5,1)$ be the graph obtained from

$P_8 = v_1v_2 \dots v_8$ by adding a vertex v and joining v to v_2 and v_3. By (A1), let a_1, a_2 be two adjacent neighbors of v_1 and $a_3 \in N(v_1) \cap N(v_2)$. Then $v, v_3 \notin N(\{a_1, a_2, a_3\})$.

Suppose that $N(a_3) \cap V(B(5,1)) = \{v_1, v_2\}$. Let $b_1, b_2 \in V(G) - V(B(5,1))$ be two adjacent neighbors of a_3. Let $x \in \{a_1, a_2\}$ and $y \in \{b_1, b_2\}$. Then $N(x) \cap \{v_4, v_5, v_6, v_7, v_8\} \neq \emptyset$ and $N(y) \cap \{v_4, v_5, v_6, v_7, v_8\} \neq \emptyset$ (otherwise, $G[\{v, v_2, v_3\} \cup \{v_4, v_5, v_6, v_7, v_8\} \cup \{s, t\}]$ is a $B(5,2)$, where $s = v_1$ if $t \in \{a_1, a_2\}$, or $s = a_3$ if $t \in \{b_1, b_2\}$, a contradiction). Furthermore, $v_4 \in N(\{a_1, a_2, b_1, b_2\})$ (otherwise, by symmetry of b_1, b_2 and a_1, a_2, we have $N(a_1) \cap V(B(5,1)) = \{v_1, v_5, v_6\}$, $N(a_2) \cap V(B(5,1)) = \{v_1, v_8\}$, $N(b_1) \cap V(B(5,1)) = \{v_5, v_6\}$, and $N(b_2) \cap V(B(5,1)) = \{v_8\}$. Thus $a_1v_5b_1v_6a_1$ is a 4-cycle in G, a contradiction). Without loss of generality, we assume that $b_1v_4 \in E(G)$. By (CF1), $b_1v_5 \in E(G)$. Notice that G has no 4-cycles. By symmetry of a_1 and a_2, we may assume that $N(a_1) \cap V(B(5,1)) = \{v_1, v_5, v_6\}$ and $N(a_2) \cap V(B(5,1)) = \{v_1, v_8\}$. Thus $N(b_2) \cap V(B(5,1)) = \{v_7, v_8\}$. Thus $G[\{v, v_2, v_3\} \cup \{v_4, v_5, v_6, v_7, b_2\} \cup \{v_1, a_2\}]$ is a $B(5,2)$, a contradiction. Therefore, $N(a_3) \cap V(B(5,1)) \neq \{v_1, v_2\}$.

Assume that $v_4 \notin N(\{a_1, a_2\})$. Then, without loss of generality, we assume that $N(a_1) \cap V(B(5,1)) = \{v_1, v_5, v_6\}$ and $N(a_2) \cap V(B(5,1)) = \{v_1, v_8\}$. Thus $N(a_3) \cap V(B(5,1)) = \{v_1, v_2\}$, a contradiction. So $v_4 \in N(\{a_1, a_2\})$. We assume that $v_4 \in N(a_1)$. Then $N(a_1) \cap V(B(5,1)) = \{v_1, v_4, v_5\}$. Thus $N(a_2) \cap V(B(5,1)) = \{v_1, v_8\}$ and $N(a_3) \cap V(B(5,1)) = \{v_1, v_2, v_6, v_7\}$.

Since $d(v) = 4$, let $N(v) = \{v_2, v_3, b_1, b_2\}$. Then $b_1b_2 \in E(G)$, and $N(b_i) \cap \{v_3, v_4\} = \emptyset (i = 1, 2)$. Thus $N(b_i) \cap \{v_5, v_6, v_7, v_8\} \neq \emptyset$ (otherwise, $a_2b_i \notin E(G)$ as $b_iv_8 \notin E(G)$. Thus $G[\{a_3, v_6, v_7\} \cup \{v_5, v_4, v_3, v, b_i\} \cup \{v_8, a_2\}]$ is a $B(5,2)$, a contradiction). Since G has no 4-cycles, we may assume that $N(b_1) \cap V(B(5,1)) = \{v, v_5, v_6\}$ and $N(b_2) \cap V(B(5,1)) = \{v, v_8\}$. Since $G[\{v_8, v_7, b_2, a_2\}] \neq K_{1,3}$, $a_2b_2 \in E(G)$. Let $N(v_8) = \{b_2, v_7, a_2, x\}$. Then $xv_3, xv_4 \in E(G)$(Otherwise, $\{x, v_3, v_4\}$ is a 3-cut in G). By (A1), $xv_7 \in E(G)$. Therefore, $V(G) = V(B(5,1)) \cup \{a_1, a_2, a_3, b_1, b_2, x\}$ and G is the line graph of the Petersen graph.

Case 3. $B(i,j) = B(4,3)$.

By Theorem 1.3, G has an induced subgraph $P_{10} = v_1v_2 \dots v_{10}$. By (A1), suppose that $a_1 \in N(v_5) \cap N(v_6)$, $a_2 \in N(v_4) \cap N(v_5)$ and $a_3 \in N(v_6) \cap N(v_7)$. Since G does not have 4-cycles, a_1, a_2, a_3 are all distinct non-adjacent vertices.

Consider $N(a_1)$. Since G does not have 4-cycles, $N(a_1) \cap \{v_3, v_4, v_7, v_8\} = \emptyset$. Since G is $B(4,3)$-free, we have either $N(a_1) \cap \{v_1, v_2\} \neq \emptyset$ or $N(a_1) \cap \{v_9, v_{10}\} \neq \emptyset$. Without loss of generality, we assume that $N(a_1) \cap \{v_1, v_2\} \neq \emptyset$. By (CF2), $N(a_1) \cap \{v_9, v_{10}\} = \emptyset$. Since $G[\{a_1, v_5, v_6\} \cup \{v_7, v_8, v_9, v_{10}\} \cup \{v_4, v_3, v_2\}]$ is not a $B(4,3)$, $a_1v_2 \in E(G)$. By (CF1), $N(a_1) = \{v_1, v_2, v_5, v_6\}$.

Consider $N(a_2)$. Since G has no 4-cycles, $N(a_2) \cap \{v_1, v_2, v_3, v_6, v_7\} = \emptyset$. Since G is $B(4,3)$-free, $N(a_2) \cap \{v_8, v_9\} \neq \emptyset$. By (CF1), $a_2v_9 \in E(G)$. If $a_2v_8 \notin E(G)$, then $a_2v_{10} \in E(G)$. Thus $G[\{a_2, v_9, v_{10}\} \cup \{v_4, v_3, v_2, v_1\} \cup \{v_8, v_7, v_6\}]$ is a $B(4,3)$, a contradiction. So $a_2v_8 \in E(G)$. Therefore, $N(a_2) = \{v_8, v_9, v_4, v_5\}$.

Consider $N(a_3)$. Since G has no 4-cycles and $v_6 \in N(a_1) \cap N(a_3)$, it follows that $N(a_3) \cap \{v_1, v_2, v_8, v_9, v_4, v_5, a_1, a_2\} = \emptyset$. By (CF1), $v_3a_3 \notin E(G)$. Since $G[\{a_3, v_6, v_7\} \cup \{v_5, v_4, v_3, v_2\} \cup \{v_8, v_9, v_{10}]$ is not a $B(4, 3)$, $a_3v_{10} \in E(G)$, and so $N(a_3) \cap (V(P_{10}) \cup \{a_1, a_2\}) = \{v_6, v_7, v_{10}\}$. Therefore, $G[\{a_2, v_8, v_9\} \cup \{v_4, v_3, v_2, v_1\} \cup \{v_{10}, a_3, v_6\}]$ is a $B(4, 3)$, a contradiction. □

Lemma 3.2 *If G is a 4-connected $\{K_{1,3}, B(i, j)\}$-free graph with $i + j = 7$, then G has a 5-cycle.*

Proof. Suppose that G is a 4-connected $\{K_{1,3}, B(i, j)\}$-free graph with $i + j = 7$ and that G does not have 5-cycles. By Theorem 1.4, $i, j \geq 1$. By Theorem 1.3, G has an induced subgraph $P_{10} = v_1v_2 \dots v_{10}$.

(B1) If $N(v_i) \cap N(v_j) \neq \emptyset (1 \leq i < j \leq 10)$, then $j - i \notin \{2, 3\}$.

Let $x \in N(v_i) \cap N(v_j)$. Since G does not have 5-cycles, $j - i \neq 3$. If $j - i = 2$, then $w \in N(v_{i+1}) - \{x, v_i, v_{i+2}\}$. By (CF1), we have either $v_iw \in E(G)$ or $v_{i+2}w \in E(G)$. Thus the 4-cycle $xv_iv_{i+1}v_{i+2}x$ can be extended to 5-cycle $xv_iwv_{i+1}v_{i+2}x$ or $xv_iv_{i+1}wv_{i+2}x$, a contradiction. (B1) holds.

Case 1. $B(i, j) = B(6, 1)$

Assume that v_3 and v_4 have more than one common neighbor. Let a_1 and a_2 be two common neighbors of v_3 and v_4. By (B1), for $i = 1, 2$, $N(a_i) \cap \{v_1, v_2, v_5, v_6, v_7\} = \emptyset$. Since G is $B(6, 1)$-free, $N(a_i) \cap \{v_8, v_9, v_{10}\} \neq \emptyset$. Since G has no 5-cycle, $N(a_1) \cap N(a_2) \cap \{v_8, v_9, v_{10}\} = \emptyset$. Thus, by symmetry and (CF1), we have $v_8a_2, v_9a_2 \in E(G)$ and $a_1v_{10} \in E(G)$. Therefore, $a_1v_3a_2v_9v_{10}a_1$ is a 5-cycle, a contradiction. So v_3 and v_4 have at most one common neighbor. Similarly, v_2 and v_3 have at most one common neighbor. Therefore, $d(v_3) = 4$, and v_3 and v_4 have exactly one common neighbor. Similarly, $d(v_8) = 4$, and v_7 and v_8 have exactly one common neighbor.

Let $N(v_3) = \{v_2, v_4, a_1, a_2\}$ and $N(v_8) = \{v_7, v_9, b_1, b_2\}$. By (CF1), we assume that $a_1 \in N(v_3) \cap N(v_4)$, $a_2 \in N(v_2) \cap N(v_3)$, $b_1 \in N(v_7) \cap N(v_8)$, and $b_2 \in N(v_8) \cap N(v_9)$. Since G is $B(6, 1)$-free, by (B1), $N(a_1) \cap V(P_{10}) \subseteq \{v_3, v_4, v_8, v_9, v_{10}\}$ and $N(a_2) \cap V(P_{10}) \subseteq \{v_2, v_3, v_7, v_8, v_9, v_{10}\}$. Since G has no 5-cycles, $N(a_1) \cap V(P_{10}) = \{v_3, v_4, v_{10}\}$ and $N(a_2) \cap V(P_{10}) = \{v_2, v_3, v_7, v_8\}$. Similarly, $N(b_1) \cap V(P_{10}) = \{v_7, v_8, v_1\}$ and $N(b_2) \cap V(P_{10}) = \{v_8, v_9, v_3, v_4\}$. Thus, $a_2v_7v_8b_2v_3a_2$ is a 5-cycle in G, a contradiction.

Case 2. $B(i, j) = B(5, 2)$

Assume that v_4 and v_5 have more than one common neighbor. Let a_1 and a_2 be two common neighbors of v_4 and v_5. By (B1), $N(a_i) \cap \{v_1, v_2, v_3, v_6, v_7, v_8\} = \emptyset$ for $i = 1, 2$. Since G is $B(5, 2)$-free, $N(a_i) \cap \{v_9, v_{10}\} \neq \emptyset$. By (CF1), $v_{10}a_1, v_{10}a_2 \in E(G)$. Thus $a_1v_{10}a_2v_5v_4a_1$ is a 5-cycle, a contradiction. So v_4 and v_5 have at most one common neighbor. Similarly, v_3 and v_4 have at most one common neighbor. Thus, $d(v_4) = 4$, and v_4 and v_5 have exactly one common neighbor.

Let $N(v_4) = \{v_3, v_5, a_1, a_2\}$. By (CF1), we assume that $a_1 \in N(v_4) \cap N(v_5)$ and $a_2 \in N(v_3) \cap N(v_4)$. Since G is $B(5, 2)$-free, by (B1), $N(a_1) \cap V(P_{10}) \subseteq \{v_4, v_5, v_9, v_{10}\}$

and $N(a_2) \cap V(P_{10}) \subseteq \{v_3, v_4, v_8, v_9, v_{10}\}$. Since G has no 5-cycles, $N(a_1) \cap N(a_2) \cap \{v_8, v_9, v_{10}\} = \emptyset$. By (CF1), $N(a_1) \cap V(P_{10}) = \{v_4, v_5, v_{10}\}$ and $N(a_2) \cap V(P_{10}) = \{v_3, v_4, v_8, v_9\}$. Thus, $a_2v_9v_{10}a_1v_4a_2$ is a 5-cycle in G, a contradiction.

Case 3. $B(i, j) = B(4, 3)$

Assume that v_5 and v_6 have a common neighbor. Let a_1 be a common neighbor of v_5 and v_6. By (B1), $N(a_1) \cap \{v_2, v_3, v_4, v_7, v_8, v_9\} = \emptyset$. Since G is $B(4, 3)$-free, $a_1v_1, a_1v_{10} \in E(G)$, contrary to (CF2). Thus v_5 and v_6 have no common neighbors. Let $a_1, a_2 \in N(v_5) - \{v_4, v_6\}$; then $a_1a_2, a_1v_4, a_2v_4 \in E(G)$. By (B1), $N(a_i) \cap \{v_1, v_2, v_3, v_6, v_7, v_8\} = \emptyset$ for $i = 1, 2$. Since G is $B(4, 3)$-free, $v_9a_1, v_9a_2 \in E(G)$. Thus $a_1v_9a_2v_4v_5a_1$ is a 5-cycle, a contradiction. □

Lemma 3.3 *If G is a 4-connected $\{K_{1,3}, B(i, j)\}$-free graph with $i + j = 7$, then G has a 6-cycle.*

Proof. Suppose that G is a 4-connected $\{K_{1,3}, B(i, j)\}$-free graph with $i + j = 7$ and that G does not have 6-cycles. By Theorem 1.4, $i, j \geq 1$. By Theorem 1.3, G has an induced subgraph $P_{10} = v_1v_2 \dots v_{10}$.

(C1) If $N(v_i) \cap N(v_j) \neq \emptyset$ $(1 \leq i < j \leq 10)$, then $j - i \notin \{2, 3, 4\}$.

Let $x \in N(v_i) \cap N(v_j)$. Since G does not have 6-cycles, $j - i \neq 4$. If $j - i = 3$, let $w \in N(v_{i+1}) - \{x, v_i, v_{i+2}\}$. By (CF1), we have either $v_iw \in E(G)$ or $v_{i+2}w \in E(G)$. Thus the 5-cycle $xv_iv_{i+1}v_{i+2}v_{i+3}x$ can be extended to a 6-cycle $xv_iwv_{i+1}v_{i+2}v_{i+3}x$ or $xv_iv_{i+1}wv_{i+2}v_{i+3}x$, a contradiction. So $j - i \neq 3$.

Assume that $j - i = 2$. Let $N(v_{i+1}) - \{x, v_i, v_{i+2}\} = \{w_1, \dots, w_t\}$. By (CF1), either $w_sv_i \in E(G)$ or $w_sv_{i+2} \in E(G)$ for $s = 1, \dots, t$. Assume that $t \geq 2$. If $w_1v_i, w_2v_{i+2} \in E(G)$, then $xv_iw_1v_{i+1}w_2v_{i+2}x$ is a 6-cycle in G, a contradiction. So we may assume that $w_1v_i, w_2v_i \in E(G)$ and $w_1v_{i+2}, w_2v_{i+2} \notin E(G)$. Since G is claw-free, $w_1w_2 \in E(G)$. Thus $xv_iw_1w_2v_{i+1}v_{i+2}x$ is a 6-cycle in G, a contradiction. So $t = 1$. As G is 4-connected, $N(v_{i+1}) = \{w_1, v_i, v_{i+2}, x\}$. Consider $T = N(x) - \{v_i, v_{i+1}, v_{i+2}, w_1\}$. If $T \neq \emptyset$, let $y \in T$. Then either $yv_i \in E(G)$ or $yv_{i+2} \in E(G)$. Thus $G[\{x, v_i, v_{i+1}, w_1, v_{i+2}, y\}]$ must contain a 6-cycle, a contradiction. So $T = \emptyset$ and $N(x) = \{v_i, v_{i+1}, v_{i+2}, w_1\}$. Therefore, $\{w_1, v_i, v_{i+2}\}$ is a 3-cut in G, a contradiction. So $j - i \neq 2$. (C1) holds.

Case 1. $B(i, j) = B(4, 3)$.

Assume that v_5 and v_6 have a common neighbor. Let $a_1 \in N(v_5) \cap N(v_6)$. By (C1), $N(a_1) \cap V(P_{10}) = \{v_5, v_6\}$. Thus $G[\{a_1, v_5, v_6\} \cup \{v_1, v_2, v_3, v_4\} \cup \{v_7, v_8, v_9\}]$ is a $B(4, 3)$, a contradiction. So v_5 and v_6 do not have common neighbors. Let $a_1 \in N(v_4) \cap N(v_5)$. By (C1), $N(a_1) \cap \{v_1, v_2, v_3, v_6, v_7, v_8, v_9\} = \emptyset$. Thus $G[\{a_1, v_4, v_5\} \cup \{v_6, v_7, v_8, v_9\} \cup \{v_1, v_2, v_3\}]$ is a $B(4, 3)$, a contradiction.

Case 2. $B(i, j) = B(5, 2)$.

Let $x \in N(v_4) \cap N(v_5)$. By (C1), $N(x) \cap \{v_1, v_2, v_3, v_6, v_7, v_8, v_9\} = \emptyset$. As G is $B(5, 2)$-free, $xv_{10} \in E(G)$. Similarly, $yv_9 \in E(G)$ for any $y \in N(v_3) \cap N(v_4)$.

Assume that v_4 and v_5 have more than one common neighbor. Let $a_1, a_2 \in N(v_4) \cap N(v_5)$. Then $a_1a_2, v_{10}a_1, v_{10}a_2 \in E(G)$. As G has no 6-cycles, $N(a_1) \cup N(a_2) - \{a_1, a_2\} = \{v_4, v_5, v_{10}\}$, and so $\{v_4, v_5, v_{10}\}$ is a 3-cut in G, a contradiction. So v_4 and v_5 have at most one common neighbor. Similarly, v_3 and v_4 have at most one common neighbor.

Consider $N(v_4)$, and let $\{v_3, v_5, a_1, a_2\} \subseteq N(v_4)$. Then we may assume that $a_1 \in N(v_4) \cap N(v_5)$ and $a_2 \in N(v_3) \cap N(v_4)$. Then $a_1v_{10}, a_2v_9 \in E(G)$. Thus $a_1v_{10}v_9a_2v_4v_5a_1$ is a 6-cycle, a contradiction.

Case 3. $B(i, j) = B(6, 1)$.

Let $x \in N(v_3) \cap N(v_4)$. By (C1), $N(x) \cap \{v_1, v_2, v_5, v_6, v_7, v_8\} = \emptyset$. As G is $B(6, 1)$-free, $N(x) \cap \{v_9, v_{10}\} \neq \emptyset$. By (CF1), $xv_{10} \in E(G)$. Similarly, $yv_9 \in E(G)$ for any $y \in N(v_2) \cap N(v_3)$.

Assume that v_3 and v_4 have more than one common neighbor. Let $a_1, a_2 \in N(v_3) \cap N(v_4)$. Then $a_1a_2, v_{10}a_1, v_{10}a_2 \in E(G)$. As G has no 6-cycles, $N(a_1) \cup N(a_2) - \{a_1, a_2\} = \{v_3, v_4, v_{10}\}$, and so $\{v_3, v_4, v_{10}\}$ is a 3-cut in G, a contradiction. So v_3 and v_4 have at most one common neighbor. Similarly, v_2 and v_3 have at most one common neighbor.

Consider $N(v_3)$, and let $\{v_2, v_4, a_1, a_2\} \subseteq N(v_3)$. Then we may assume that $a_1 \in N(v_3) \cap N(v_4)$ and $a_2 \in N(v_2) \cap N(v_3)$. Then $a_1v_{10}, a_2v_9 \in E(G)$. Thus $a_1v_{10}v_9a_2v_3v_4a_1$ is a 6-cycle, a contradiction. □

Lemma 3.4 *If G is a 4-connected $\{K_{1,3}, B(i, j)\}$-free graph with $i + j = 7$, then G has a 7-cycle.*

Proof. Suppose that G is a 4-connected $\{K_{1,3}, B(i, j)\}$-free graph with $i + j = 7$ and that G does not have 7-cycles. By Theorem 1.4, $i, j \geq 1$. By Theorem 1.3, G has an induced subgraph $P_{10} = v_1v_2 \dots v_{10}$.

(D1) If $N(v_i) \cap N(v_j) \neq \emptyset (1 \leq i < j \leq 10)$, then $j - i \neq \{3, 4, 5\}$.

(D2) For $1 \leq i \leq 8$, $|N(v_i) \cap N(v_{i+2})| \leq 1$.

(D3) For $1 \leq i \leq 7$, if $N(v_i) \cap N(v_{i+2}) \neq \emptyset$, then $N(v_{i+1}) \cap N(v_{i+3}) = \emptyset$.

Let $x \in N(v_i) \cap N(v_j)$. Since G does not have 7-cycles, $j - i \neq 5$. If $j - i = 4$, let $w \in N(v_{i+1}) - \{v_i, v_{i+2}\}$. By (CF1), we have either $wv_i \in E(G)$ or $wv_{i+2} \in E(G)$. Thus the 6-cycle $xv_i \dots v_jx$ can be extended to a 7-cycle $xv_iwv_{i+1} \dots v_jx$ or $xv_iv_{i+1}wv_{i+2} \dots v_jx$, a contradiction. So $j - i \neq 4$. Assume that $j = i + 3$. Let $T = N(v_{i+1}) \cup N(v_{i+2}) - \{x, v_i, v_{i+3}\}$. Since G is 4-connected, $|T| \geq 1$. If $|T| \geq 2$, let $y_1, y_2 \in T$. By (CF1) and the fact that G is claw-free, $G[\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, x, y_1, y_2\}]$ must contain a 7-cycle, a contradiction. So $|T| = 1$. Assume that $T = \{y\}$. Since G is 4-connected, $N(v_{i+1}) = \{v_i, v_{i+2}, y, x\}$ and $N(v_{i+2}) = \{v_{i+1}, v_{i+3}, y, x\}$. Since G is claw-free and G does not have 7-cycles, $N(x) \subseteq \{v_i, v_{i+1}, v_{i+2}, v_{i+3}, y\}$, and so $\{v_i, v_{i+3}, y\}$ is a 3-cut of G, a contradiction. Therefore, $j - i \neq 3$. (D1) follows.

Suppose that $x, y \in N(v_i) \cap N(v_{i+2})$. By (D1) and (CF1), $x, y \in N(v_{i+1})$ and $xy \in E(G)$. Then G has the 5-cycle $xv_iv_{i+1}v_{i+2}yx$. Since G is claw-free and G does not have 7-cycles, $|(N(\{x, y, v_{i+1}\}) - \{v_i, v_{i+2}, x, y, v_{i+1}\}| \leq 1$ and then $N(\{x, y, v_{i+1}\}) - \{x, y, v_{i+1}\}$ is a 2-cut or 3-cut, a contradiction. So (D2) follows.

Suppose that $x \in N(v_i) \cap N(v_{i+2})$ and $y \in N(v_{i+1}) \cap N(v_{i+3})$. By (D1) and (CF1), $xv_{i+1}, yv_{i+2} \in E(G)$. Since G is claw-free and G does not have 7-cycles, $N(\{x, y, v_{i+1}, v_{i+2}\}) - \{x, y, v_i, v_{i+1}, v_{i+2}, v_{i+3}\} = \emptyset$, which implies that $\{v_i, v_{i+3}\}$ is a 2-cut of G, a contradiction. So (D3) follows.

Case 1. $B(i, j) = B(4, 3)$.

Assume that v_5 and v_6 have more than one common neighbor. Let $a_1, a_2 \in N(v_5) \cap N(v_6)$. For $i = 1, 2$, by (D1), $N(a_i) \cap V(P_{10}) \subseteq \{v_4, v_5, v_6, v_7\}$. Since G is $B(4, 3)$-free, $N(a_i) \cap \{v_4, v_7\} \neq \emptyset$, contradicting (D2) or (D3). So v_5 and v_6 have at most one common neighbor. Similarly, v_4 and v_5 have at most one common neighbor, and v_6 and v_7 have at most one common neighbor. Thus $d(v_5) = d(v_6) = 4$. Let $N(v_5) = \{v_4, v_6, a_1, a_2\}$ and $N(v_6) = \{v_5, v_7, a_1, a_3\}$. By (D1), $N(a_1) \cap V(P_{10}) = \{v_5, v_6\}$, and $G[\{a_1, v_5, v_6\} \cup \{v_7, v_8, v_9, v_{10}\} \cup \{v_4, v_3, v_2\}]$ is a $B(4, 3)$, a contradiction.

Case 2. $B(i, j) = B(5, 2)$.

Assume that v_4 and v_5 have more than one common neighbor. Let $a_1, a_2 \in N(v_4) \cap N(v_5)$. For $i = 1, 2$, by (D1), $N(a_i) \cap \{v_1, v_2, v_7, v_8, v_9, v_{10}\} = \emptyset$. Since G is $B(5, 2)$-free, $N(a_i) \cap \{v_3, v_6\} \neq \emptyset$, contradicting (D2) or (D3). So v_4 and v_5 have at most one common neighbor. Similarly, v_3 and v_4 have at most one common neighbor. Thus $d(v_4) = 4$. Let $N(v_4) = \{v_3, v_5, a_1, a_2\}$. Without loss of generality, we assume that $a_1 \in N(v_4) \cap N(v_5)$, $a_2 \in N(v_3) \cap N(v_4)$. Similarly, let $N(v_7) = \{v_6, v_8, b_1, b_2\}$, where $b_1 \in N(v_6) \cap N(v_7), b_2 \in N(v_7) \cap N(v_8)$.

By (D1), $N(a_1) \cap \{v_1, v_2, v_7, v_8, v_9, v_{10}\} = \emptyset$. Since G is $B(5, 2)$-free, $N(a_1) \cap \{v_3, v_6\} \neq \emptyset$. Similarly, $N(a_2) \cap \{v_2, v_5\} \neq \emptyset$. By (D2) and (D3), we have $a_1v_6, a_2v_2 \in E(G)$. Similarly, $b_1v_5, b_2v_9 \in E(G)$, contradicting (D3).

Case 3. $B(i, j) = B(6, 1)$.

Assume that v_3 and v_4 do not have common neighbors. Since G is 4-connected, let $a_1, a_2 \in N(v_2) \cap N(v_3)$ and $b_1, b_2 \in N(v_4) \cap N(v_5)$. Then $a_1a_2, b_1b_2 \in E(G)$, $v_4 \notin N(a_1) \cup N(a_2)$ and $v_3 \notin N(b_1) \cup N(b_2)$. Since G has no 7-cycles, $a_ib_j \notin E(G)$ for $i, j \in \{1, 2\}$. For $i = 1, 2$, by (D1), $N(a_i) \cap \{v_5, v_6, v_7, v_8\} = \emptyset$ and $N(b_i) \cap \{v_1, v_2, v_7, v_8, v_9, v_{10}\} = \emptyset$. By (D2), we may assume that $v_1a_1, v_6b_1 \notin E(G)$. Since G is $B(6, 1)$-free, we have $a_1v_9 \in E(G)$. Thus $G[\{a_1, v_2, v_3\} \cup \{v_1\} \cup \{v_9, v_8, v_7, v_6, v_5, b_1\}]$ is a $B(6, 1)$, a contradiction. So v_3 and v_4 have a common neighbor. Similarly, v_7 and v_8 have a common neighbor.

Claim 1. Assume that v_3 and v_4 have exactly one common neighbor. Let $a_1 \in N(v_3) \cap N(v_4)$, $a_2 \in N(v_2) \cap N(v_3)$ and $a_3 \in N(v_4) \cap N(v_5)$. Then

(i) $N(a_1) \cap \{v_1, v_2, v_6, v_7, v_8, v_9\} = \emptyset$. Therefore, either $N(a_1) \cap V(P_{10}) = \{v_3, v_4, v_5\}$ or $N(a_1) \cap V(P_{10}) = \{v_3, v_4, v_{10}\}$.

(ii) $N(a_2) \cap V(P_{10}) = \{v_1, v_2, v_3\}$.

By (D1), $N(a_1)\cap\{v_1, v_6, v_7, v_8, v_9\} = \emptyset$. Assume that $a_1v_2 \in E(G)$. By (D1), (D2) and (D3), $N(a_2) \cap \{v_1, v_4, v_5, v_6, v_7, v_8\} = \emptyset$. Since G is $B(6, 1)$-free, $a_2v_9 \in E(G)$. By (CF1), $a_2v_{10} \in E(G)$. Since G has no 7-cycles, $a_1v_5, a_1v_{10} \notin E(G)$. If there is $y \in N(a_1) - \{a_2, v_2, v_3, v_4\}$, then $yv_2 \in E(G)$ or $yv_4 \in E(G)$. If $yv_4 \in E(G)$, since v_3 and v_4 have exactly one common neighbor, by (CF1), $yv_5 \in E(G)$. This implies a 7-cycle $yv_5a_3v_4v_3v_2a_1y$, a contradiction. So $yv_4 \notin E(G)$ and $yv_2 \in E(G)$. Since G has no 7-cycles, $yv_1 \notin E(G)$ and so $yv_3 \in E(G)$. By (D1), (D2) and (D3), $N(y) \cap \{v_4, v_5, v_6, v_7, v_8\} = \emptyset$. As G is $B(6, 1)$-free, $yv_9 \in E(G)$. By (CF1), $yv_{10} \in E(G)$. Thus $yv_9v_{10}a_2v_2v_3a_1y$ is a 7-cycle in G, a contradiction. So $N(a_1) \subseteq \{a_2, v_2, v_3, v_4\}$. By the symmetry of a_1 and v_3, $N(v_3) \subseteq \{a_1, a_2, v_2, v_4\}$, and so $\{a_2, v_2, v_4\}$ is a 3-cut of G, a contradiction. Claim 1(i) holds.

Assume that $a_2v_1 \notin E(G)$. Since G is $B(6, 1)$-free, $a_2v_9 \in E(G)$, and so $a_2v_{10} \in E(G)$. Thus $N(a_2) \cap V(P_{10}) = \{v_2, v_3, v_9, v_{10}\}$. Since G has no 7-cycles, $v_{10} \notin N(a_1)$. By Claim 1(i), $N(a_1) \cap V(P_{10}) = \{v_3, v_4, v_5\}$. By (D3), $a_3v_6 \notin E(G)$. By (D1) and (D2), $N(a_3) \cap V(P_{10}) = \{v_4, v_5\}$. Since G has no 7-cycles, $a_2a_3 \notin E(G)$. Thus $G[\{a_2, v_2, v_3\} \cup \{v_9, v_8, v_7, v_6, v_5, a_3\} \cup \{v_1\}]$ is a $B(6, 1)$, a contradiction. So $a_2v_1 \in E(G)$. By (CF2), $N(a_2) \cap V(P_{10}) = \{v_1, v_2, v_3\}$. So Claim 1(ii) holds.

Claim 2. Assume that v_3 and v_4 have more than one common neighbor. Let $a_1, a_2 \in N(v_3) \cap N(v_4)$. Then, for $i = 1, 2$, $N(a_i) \cap \{v_1, v_2, v_6, v_7, v_8, v_9\} = \emptyset$. Therefore, by symmetry, $N(a_1) \cap V(P_{10}) = \{v_3, v_4, v_5\}$ and $N(a_2) \cap V(P_{10}) = \{v_3, v_4, v_{10}\}$.

By (D1), $N(a_i) \cap \{v_1, v_6, v_7, v_8, v_9\} = \emptyset$. Without loss of generality, we assume that $a_1v_2 \in E(G)$. By (D2) and (D3), $a_2v_2, a_2v_5 \notin E(G)$. Since G is $B(6, 1)$-free, $a_2v_{10} \in E(G)$. Since $G[\{v_4, a_1, a_2, v_5\}]$ is not a claw, $a_1a_2 \in E(G)$. Since G is 4-connected, there is a vertex $y \in (N(\{a_1, v_3\}) - \{a_1, v_3\}) - \{v_2, a_2, v_4\}$.

If $ya_1 \in E(G)$, by considering $G[\{a_1, y, v_2, v_4\}]$, we have $N(y)\cap\{v_2, v_4\} \neq \emptyset$. As G has no 7-cycles, $N(y)\cap\{v_1, v_5, v_6, v_7, v_8, v_9, v_{10}\} = \emptyset$. If $yv_4 \notin E(G)$, then $yv_2 \in E(G)$ and $yv_3 \in E(G)$ by (CF1), and so $G[\{y, v_2, v_3\}\cup\{v_4, v_5, v_6, v_7, v_8, v_9\}\cup\{v_1\}] = B(6, 1)$, a contradiction. If $yv_4 \in E(G)$, then $yv_2 \notin E(G)$ by (D2) and $yv_3 \in E(G)$ by (CF1), therefore $G[\{y, v_3, v_4\} \cup \{v_5, v_6, v_7, v_8, v_9, v_{10}\} \cup \{v_2\}] = B(6, 1)$, a contradiction. This implies $yv_3 \in E(G)$. By considering $G[\{v_3, a_2, y, v_2\}]$, we have $yv_2 \in E(G)$. By (D2), $yv_4 \notin E(G)$. As G has no 7-cycles, $N(y) \cap \{v_1, v_5, v_6, v_7, v_8, v_9, v_{10}\} = \emptyset$. Thus $G[\{y, v_2, v_3\}\cup\{v_4, v_5, v_6, v_7, v_7, v_9\}\cup\{v_1\}]$ is a $B(6, 1)$, a contradiction. Claim 2 holds.

Claim 3. Suppose that $a_1 \in N(v_3) \cap N(v_4)$ and $b_1 \in N(v_7) \cap N(v_8)$. If $N(a_1) \cap V(P_{10}) = \{v_3, v_4, v_{10}\}$, then $N(b_1) \cap V(P_{10}) \neq \{v_1, v_7, v_8\}$.

Assume that $N(b_1) \cap V(P_{10}) = \{v_1, v_7, v_8\}$. If there is $y \in N(v_5) \cap N(v_6)$, since G does not have 7-cycles, $ya_1, yb_1 \notin E(G)$. By (D1), $N(y) \cap V(P_{10}) \subseteq \{v_4, v_5, v_6, v_7\}$. If $N(y)\cap V(P_{10}) = \{v_5, v_6\}$, then G has a $B(6, 1) = G[\{a_1, v_3, v_4\}\cup\{v_{10}, v_9, v_8, v_7, v_6, y\}\cup \{v_2\}]$, a contradiction. By (D1), suppose that $N(y) \cap V(P_{10}) = \{v_4, v_5, v_6\}$. Let $y' \in N(v_5) - \{v_4, v_6, y\}$. By (D2) and (D3) and the same discussion as y, $y' \notin N(v_6)$. So $y' \in N(v_4) \cap N(v_5)$. By (D1) and (D3), $N(y') \cap V(P_{10}) = \{v_4, v_5\}$. Since G has no 7-cycles, $y'a_1, y'b_1 \notin E(G)$. Thus $G[\{y', v_4, v_5\} \cup \{v_3, v_2, v_1, b_1, v_8, v_9\} \cup \{v_6\}] = B(6, 1)$, a contradiction. So $N(v_5) \cap N(v_6) = \emptyset$. Therefore, there are $a_2, a_3 \in N(v_4) \cap N(v_5)$. By (D1), $N(a_i) \cap V(P_{10}) \subseteq \{v_3, v_4, v_5\}(i = 2, 3)$. Since G does not have 7-cycles,

$a_2b_1, a_3b_1 \notin E(G)$. By (D2), one of a_2 and a_3 has $N(a_i) \cap V(P_{10}) = \{v_4, v_5\}$, resulting a $B(6,1) = G[\{a_i, v_4, v_5\} \cup \{v_3, v_2, v_1, b_1, v_8, v_9\} \cup \{v_6\}]$ again, a contradiction. Claim 3 holds.

By Claims 2 and 3, since G is $B(6,1)$-free, either v_3 and v_4 have exactly one common neighbor, or v_7 and v_8 have exactly one common neighbor.

Claim 4. v_3 and v_4 have exactly one common neighbor, and v_7 and v_8 have exactly one common neighbor.

By symmetry, we assume that v_3 and v_4 have exactly one common neighbor, and v_7 and v_8 have two or more common neighbors. Let $a_1 \in N(v_3) \cap N(v_4)$, $a_3 \in N(v_4) \cap N(v_5)$, and $b_1, b_2 \in N(v_7) \cap N(v_8)$. By Claim 2, we assume that $N(b_1) \cap V(P_{10}) = \{v_1, v_7, v_8\}$, and $N(b_2) \cap V(P_{10}) = \{v_7, v_8, v_9\}$. By Claims 1(i) and 3, we have $N(a_1) \cap V(P_{10}) = \{v_3, v_4, v_5\}$.By (D1), (D2) and (D3), $N(a_3) \cap V(P_{10}) = \{v_4, v_5\}$. Since G has no 7-cycles, $a_3b_1, a_3b_2 \notin E(G)$. Thus G has a $B(6,1) = G[\{a_3, v_4, v_5\} \cup \{v_3, v_2, v_1, b_1, v_8, v_9\} \cup \{v_6\}]$, a contradiction. Claim 4 holds.

By Claim 4, let $a_1 \in N(v_3) \cap N(v_4)$, $a_2 \in N(v_2) \cap N(v_3)$ and $a_3 \in N(v_4) \cap N(v_5)$, and let $b_1 \in N(v_7) \cap N(v_8)$, $b_2 \in N(v_8) \cap N(v_9)$ and $b_3 \in N(v_6) \cap N(v_7)$. By Claim 1(ii), $N(a_2) \cap V(P_{10}) = \{v_1, v_2, v_3\}$ and $N(b_2) \cap V(P_{10}) = \{v_8, v_9, v_{10}\}$.

Claim 5. $N(a_1) \cap V(P_{10}) = \{v_3, v_4, v_5\}$ and $N(b_1) \cap V(P_{10}) = \{v_6, v_7, v_8\}$.

Assume that $N(a_1) \cap V(P_{10}) \neq \{v_3, v_4, v_5\}$. By Claim 1(i), $N(a_1) \cap V(P_{10}) = \{v_3, v_4, v_{10}\}$. By Claims 1(i) and 3, $N(b_1) \cap V(P_{10}) = \{v_6, v_7, v_8\}$. By (D1), (D2) and (D3), $N(b_3) \cap V(P_{10}) = \{v_6, v_7\}$. Since G has no 7-cycles, $a_1b_3 \notin E(G)$. Thus $G[\{b_3, v_6, v_7\} \cup \{v_8, v_9, v_{10}, a_1, v_3, v_2\} \cup \{v_5\}]$ is a $B(6,1)$, a contradiction. So $N(a_1) \cap V(P_{10}) = \{v_3, v_4, v_5\}$. By symmetry, $N(b_1) \cap V(P_{10}) = \{v_6, v_7, v_8\}$. Claim 5 holds.

Now we finish the proof of Case 3. Since G does not have 7-cycles, $|N(a_1) \cup N(v_4) - \{a_1, v_4, v_3, v_5, a_3\}| \leq 1$. Since G is 4-connected, $|N(a_1) \cup N(v_4) - \{a_1, v_4, v_3, v_5, a_3\}| = 1$. Let $a_4 \in N(a_1) \cup N(v_4) - \{a_1, v_4, v_3, v_5, a_3\}$. Since G has no 7-cycles, $a_4v_2, a_4v_6 \notin E(G)$. Thus $a_4v_4 \in E(G)$ (if $a_1a_4 \in E(G)$, then either $a_4v_3 \in E(G)$ or $a_4v_5 \in E(G)$. By (CF1), $a_4v_4 \in E(G)$). By Claim 4, $N(a_4) \cap V(P_{10}) = \{v_4, v_5\}$. Since G is claw-free, $G[\{a_1, a_3, a_4, v_4, v_5\}]$ is a K_5, and so $N(a_1) = \{v_3, v_4, v_5, a_3, a_4\}$ and $N(v_4) = \{v_3, v_5, a_1, a_3, a_4\}$. Similarly there is $b_4 \in N(b_1) \cup N(v_7) - \{v_6, v_8, b_3\}$ with $N(b_4) \cap V(P_{10}) = \{v_6, v_7\}$, and $N(b_1) = \{v_6, v_7, v_8, b_3, b_4\}$ and $N(v_7) = \{v_6, v_8, b_1, b_3, b_4\}$. Since G has no 7-cycles, $a_ib_j \notin E(G)$ for $i, j = 1, 2, 3, 4$.

Let $N(v_1) - \{a_2, v_2\} = \{c_1, c_2, \ldots, c_s\} (s \geq 2)$, and let $i \in \{1, \ldots, s\}$. Then $N(c_i) \cap \{a_1, v_4, b_1, v_7\} = \emptyset$. Since G has no 7-cycles, $N(c_i) \cap \{v_5, v_6, a_3, a_4\} = \emptyset$. If $c_iv_8 \in E(G)$, then, by (CF1), $c_iv_9 \in E(G)$. By Claim 1(ii), $c_iv_{10} \in E(G)$, and so $\{v_1, v_8, v_9, v_{10}\} \subseteq N(c_i) \cap V(P_{10})$, contrary to (CF2). So $c_iv_8 \notin E(G)$. If $c_iv_9 \in E(G)$, then $c_iv_{10} \in E(G)$. Since $G[\{c_i, v_9, v_{10}\} \cup \{v_8, v_7, v_6, v_5, v_4, v_3\} \cup \{v_1\}]$ is not a $B(6,1)$, we have $c_iv_3 \in E(G)$, contrary to (D2). So $c_iv_9 \notin E(G)$. If $c_iv_{10} \in E(G)$, by symmetry, $c_iv_2, c_iv_3 \notin E(G)$. Thus $G[\{a_3, v_4, v_5\} \cup \{v_6, v_7, v_8, v_9, v_{10}, c_i\} \cup \{v_3\}]$ is a $B(6,1)$, a contradiction. So $c_iv_{10} \notin E(G)$. If $c_ib_3 \in E(G)$, as G has no 7-cycles, $c_iv_2, c_iv_3 \notin E(G)$, so $G[\{b_3, v_6, v_7\} \cup \{c_i, v_1, v_2, v_3, v_4, a_3\} \cup \{v_8\}]$ is a $B(6,1)$. So $c_ib_3 \notin E(G)$. If $c_iv_2 \notin E(G)$, then $c_iv_3 \notin E(G)$, so $G[\{b_3, v_6, v_7\} \cup \{v_5, v_4, v_3, v_2, v_1, c_i\} \cup \{v_8\}]$

is a $B(6,1)$. This shows that $c_iv_2 \in E(G)$. By (D2), $c_iv_3 \notin E(G)$. Therefore, $N(c_i) \cap V(P_{10}) = \{v_1, v_2\}$, and $G[\{c_1, c_2, \dots, c_s\}]$ is a K_s. Since G has no 7-cycles, $s = 2$.

Consider $N(a_2)$ and $N(v_2)$. Since G has no 7-cycles, we have $N(v_2) = \{v_1, v_3, c_1, c_2, a_2\}$ and $N(a_2) \subseteq \{c_1, c_2, v_1, v_2, v_3\}$. Thus $\{c_1, c_2, v_3\}$ is a 3-cut in G, a contradiction. □

Lemma 3.5 *If G is a 4-connected $\{K_{1,3}, B(i,j)\}$-free graph with $i + j = 7$, then G has an 8-cycle.*

Proof. Suppose that G is a 4-connected $\{K_{1,3}, B(i,j)\}$-free graph with $i + j = 7$ and that G does not have 8-cycles. By Theorem 1.4, $i, j \geq 1$. By Theorem 1.3, G has an induced subgraph $P_{10} = v_1v_2 \dots v_{10}$.

(E1) If $N(v_i) \cap N(v_j) \neq \emptyset$ $(1 \leq i < j \leq 10)$, then $j - i \notin \{4, 5, 6\}$. Therefore, for some $x \notin V(P_{10})$, if $\{v_i, v_{i+2}\} \subseteq N(x) \cap V(P_{10})$ $(2 \leq i \leq 7)$, then $xv_{i+1} \in E(G)$, and if $\{v_i, v_{i+3}\} \subseteq N(x) \cap V(P_{10})$ $(2 \leq i \leq 6)$, then $xv_{i+1}, xv_{i+2} \in E(G)$.

(E2) Let $x \in N(v_i) \cap N(v_{i+2}) - \{v_{i+1}\}$ $(1 \leq i \leq 7)$. Then $N(v_{i+1}) \cap N(v_{i+3}) \subseteq \{x\}$. Therefore, there do not exist $x, y \in V(G) - V(P_{10})$ such that $(N(x) \cup N(y)) \cap V(P_{10}) = \{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$ and $\min(|N(x) \cap V(P_{10})|, |N(y) \cap V(P_{10})|) \geq 3$, where $1 \leq i \leq 7$.

(E3) Assume that $a_1, a_2 \in N(v_i) \cap N(v_{i+1}) \cap N(v_{i+2})$ $(2 \leq i \leq 7)$, and let $T = N(\{a_1, a_2, v_{i+1}\}) - \{a_1, a_2, v_{i+1}, v_i, v_{i+2}\}$.
(i) For $y \in T$, $yv_{i+1} \in E(G)$.
(ii) Let $y \in T$ and $w \in N(y) \cap \{v_i, v_{i+2}\}$, $G[\{a_1, a_2, y, v_{i+1}, w\}]$ is a complete graph.
(iii) $|T| = 2$, and for any $y \in T$, $|N(y) \cap \{v_i, v_{i+2}\}| = 1$. If $T = \{y_1, y_2\}$, then $N(a_1) = \{a_2, v_{i+1}, y_1, y_2, v_i, v_{i+2}\}$, $N(a_2) = \{a_1, v_{i+1}, y_1, y_2, v_i, v_{i+2}\}$, $N(v_{i+1}) = \{a_1, a_2, y_1, y_2, v_i, v_{i+2}\}$.

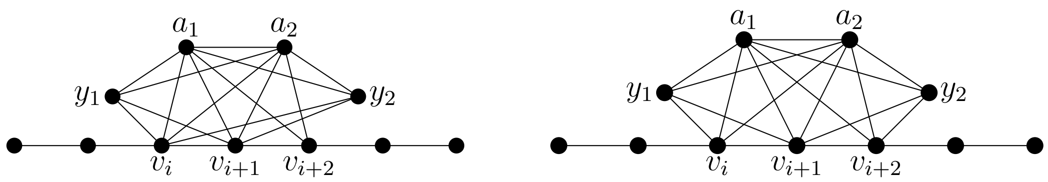

Figure 2. Graph for (E3)

(E4) Assume that $N(x) \cap V(P_{10}) = \{v_i, v_{i+1}, v_{i+2}\}$, and $y \in N(x) - \{v_i, v_{i+1}, v_{i+2}\}$. Then $yv_{i+3} \notin E(G)$ if $i \leq 7$ and $yv_{i-1} \notin E(G)$ if $i \geq 2$. Therefore, for $2 \leq i \leq 7$, $yv_{i+1} \in E(G)$, and $N(\{x, v_{i+1}\}) = N(v_{i+1}) = N(x)$.

Let $x \in N(v_i) \cap N(v_j)$. Since G has no 8-cycles, $j - i \neq 6$. If $j - i = 5$, then let $w \in N(v_{i+1}) - \{v_i, v_{i+2}, x\}$. By (CF1), either $wv_i \in E(G)$ or $wv_{i+2} \in E(G)$. Thus the 7-cycle $xv_i \ldots v_j x$ can be extended to an 8-cycle $xv_i w v_{i+1} \ldots v_j x$ or $xv_i v_{i+1} w v_{i+2} \ldots v_j x$. So $j - i \neq 5$. Assume that $j - i = 4$. Consider the set $S = (N(\{v_{i+1}, v_{i+2}, v_{i+3}\}) - \{v_{i+1}, v_{i+2}, v_{i+3}\}) - \{x, v_i, v_{i+4}\}$. Then $|S| \geq 1$. If $|S| = 1$, let $S = \{y\}$. Since G is 4-connected, we have $x \in N(v_{i+l})$ for $l = 1, 2, 3$, therefore $|N(x) \cap V(P_{10})| \geq 5$, contradicting (CF2). So $|S| \geq 2$. Let $w_1, w_2 \in S$. Then, by (CF1) and G is claw-free, the 6-cycle $xv_i v_{i+1} \ldots v_j x$ can be extended to an 8-cycle by inserting w_1 and w_2, a contradiction. So $j - i \neq 4$. (E1) holds.

Assume that $y \in N(v_{i+1}) \cap N(v_{i+3})$ and $y \neq x$. Let $S = (N(\{x, y, v_{i+1}, v_{i+2}\}) - \{x, y, v_{i+1}, v_{i+2}\}) - \{v_i, v_{i+3}\}$. Since G is 4-connected, $|S| \geq 2$. Let $w_1, w_2 \in S$. If $w_1, w_2 \in N(x) \cup N(y) \cup (N(v_i) \cap N(v_{i+1})) \cup (N(v_{i+2}) \cap N(v_{i+3}))$, then we can insert w_1 and w_2 into the 6-cycle $v_i v_{i+1} y v_{i+3} v_{i+2} x v_i$ to have an 8-cycle. Otherwise, by (CF1), we may assume that $w_1 \in N(v_{i+1}) \cap N(v_{i+2})$. Since $w_1 v_i, w_1 v_{i+3}, w_1 x, w_1 y \notin E(G)$, $xy, xv_{i+3}, yv_i \in E(G)$. Then we can insert w_1 and w_2 into either $v_i v_{i+1} v_{i+2} v_{i+3} y x v_i$, $y v_{i+1} v_{i+2} v_{i+3} x v_i y$, or $x v_{i+2} v_{i+1} v_i y v_{i+3} x$ to have an 8-cycle, a contradiction. (E2) holds.

By (E2), $a_1 v_{i-1}, a_1 v_{i+3}, a_2 v_{i-1}, a_2 v_{i+3} \notin E(G)$. Thus $a_1 a_2 \in E(G)$. Since G is 4-connected, $|T| \geq 2$. Let $y \in T$ and assume that $yv_{i+1} \notin E(G)$. Without of loss of generality, we assume that $ya_2 \in E(G)$. Since G is claw-free, we have either $yv_i \in E(G)$ or $yv_{i+2} \in E(G)$. We assume that $yv_{i+2} \in E(G)$. By (CF1), $yv_{i+3} \in E(G)$. Since $|T| \geq 2$, let $z \in T - \{y\}$. If $z \in N(a_1)$, then we can insert z into the cyle $v_i a_1 v_{i+2} v_{i+3} y a_2 v_{i+1} v_i$ to have an 8-cycle; if $z \in N(v_{i+1})$, we can insert z into the cycle $v_i v_{i+1} v_{i+2} v_{i+3} y a_2 a_1 v_i$ to have an 8-cycle. We may assume that $z \in N(a_2) - (N(a_1) \cup N(v_{i+1}))$. If $zv_i \in E(G)$, then we have an 8-cycle $v_i z a_2 y v_{i+3} v_{i+2} v_{i+1} a_1 v_i$; if $zv_i \notin E(G)$, then $zv_{i+2} \in E(G)$. Since G is claw-free, $yz \in E(G)$. Then we have an 8-cycle $v_i v_{i+1} v_{i+2} v_{i+3} y z a_2 a_1 v_i$, a contradiction. So $yv_{i+1} \in E(G)$. By (CF1), we assume that $yv_i \in E(G)$. By (E2), $yv_{i-1} \notin E(G)$. Since G is claw-free, $ya_1, ya_2 \in E(G)$. Thus $G[\{a_1, a_2, y, v_{i+1}, v_i\}]$ is a complete graph, so (E3)(ii) holds. Notice that G has no 8-cycles and is claw-free, $|T| = 2$, and $N(a_1) = \{a_2, v_{i+1}, y_1, y_2, v_i, v_{i+2}\}$, $N(a_2) = \{a_1, v_{i+1}, y_1, y_2, v_i, v_{i+2}\}$, and $N(v_{i+1}) = \{a_1, a_2, y_1, y_2, v_i, v_{i+2}\}$. Let $T = \{y_1, y_2\}$. If $y_1 v_i, y_1 v_{i+2} \in E(G)$, then $N(y_1) = \{y_2, v_i, v_{i+1}, v_{i+2}, a_1, a_2\}$ and so $\{y_2, v_i, v_{i+2}\}$ is a 3-cut in G, a contradiction. So $|N(y_1) \cap \{v_i, v_{i+2}\}| = 1$. Similarly, $|N(y_2) \cap \{v_i, v_{i+2}\}| = 1$. (E3) holds.

Assume that $yv_{i+3} \in E(G)$. By (E2), $yv_{i+1} \notin E(G)$. Since $d(v_{i+1}) \geq 4$, let $z \in N(v_{i+1}) - \{v_i, v_{i+2}, x\}$. Then we have either $zv_i \in E(G)$ or $zv_{i+2} \in E(G)$. Let $C = xv_i z v_{i+1} v_{i+2} v_{i+3} y x$ if $zv_i \in E(G)$, or $C = x v_i v_{i+1} z v_{i+2} v_{i+3} y x$ if $zv_{i+2} \in E(G)$. Then C is a 7-cycle in G. Notice that G has no 8-cycles, $N(\{x, v_{i+1}, v_{i+2}\} - \{x, v_{i+1}, v_{i+2}\} \subseteq \{y, z, v_i, v_{i+3}\}$. Thus $\{y, z, v_i, v_{i+3}\}$ is a 4-cut in G. Therefore, $N(y) - \{x, z, v_i, v_{i+1}, v_{i+2}, v_{i+3}\} \neq \emptyset$. Since C is a 7-cycle in G and G does not have 8-cycles, $xv_{i+3} \in E(G)$, a contradiction. So $yv_{i+3} \notin E(G)$. Similarly, $yv_{i-1} \notin E(G)$. Since G is claw-free, by (CF1), $yv_{i+1} \in E(G)$. So (E4) holds.

We will prove the lemma by considering the following three cases.

Case 1. $B(i,j) = B(4,3)$.

Assume that v_5 and v_6 have more than one common neighbor. Let $a_1, a_2 \in N(v_5) \cap N(v_6)$. By (E1), $N(a_i) \cap \{v_1, v_2, v_9, v_{10}\} = \emptyset$. If $v_3a_1 \in E(G)$, by (E1), $v_4a_1 \in E(G)$. By (E1) and (E2), $a_2v_3, a_2v_4, a_2v_7, a_2v_8 \notin E(G)$. So $G[\{a_2, v_5, v_6\} \cup \{\{v_7, v_8, v_9, v_{10}\} \cup \{v_4, v_3, v_2\}]$ is a $B(4,3)$, a contradiction. So $v_3a_1 \notin E(G)$. Similarly, $a_2v_3, a_1v_8, a_2v_8 \notin E(G)$. Since G is $B(4,3)$-free, $a_i \cap \{v_4, v_7\} \neq \emptyset$. By (E2), we may assume that $N(a_1) \cap V(P_{10}) = N(a_2) \cap V(P_{10}) = \{v_4, v_5, v_6\}$. By (E3), let $T = N(\{a_1, a_2, v_5\}) - \{a_1, a_2, v_5, v_4, v_6\} = \{y_1, y_2\}$. Then $|N(y_1) \cap \{v_4, v_6\}| = 1$. By symmetry, we assume that $y_1v_4 \in E(G)$. By (E1) and (E2), $N(y_1) \cap \{v_1, v_3, v_7, v_8, v_9\} = \emptyset$. By (CF1), $y_1v_2 \notin E(G)$. So $G[\{y_1, v_4, v_5\} \cup \{v_6, v_7, v_8, v_9\} \cup \{v_3, v_2, v_1\}]$ is a $B(4,3)$, a contradiction.

Assume that v_5 and v_6 have one common neighbor. Let $a_1 \in N(v_5) \cap N(v_6)$, $a_2 \in N(v_4) \cap N(v_5)$ and $a_3 \in N(v_6) \cap N(v_7)$. Then $a_2v_6, a_3v_5 \notin E(G)$. By (E1) and (CF1), $N(a_2) \cap V(P_{10}) \subseteq \{v_2, v_3, v_4, v_5\}$ and $N(a_1) \cap V(P_{10}) \subseteq \{v_3, v_4, v_5, v_6, v_7, v_8\}$. If $v_3 \in N(a_1)$, then by (CF1), $v_4 \in N(a_1)$. By (CF2), $N(a_1) \cap V(P_{10}) = \{v_3, v_4, v_5, v_6\}$, and so $G[\{a_1, v_5, v_6\} \cup \{v_7, v_8, v_9, v_{10}\} \cup \{v_3, v_2, v_1\}]$ is a $B(4,3)$, a contradiction. So $a_1v_3 \notin E(G)$. Similarly, $a_1v_8 \notin E(G)$. Notice that $N(a_1) \cap \{v_4, v_7\} \neq \emptyset$. By symmetry, we assume that $a_1v_4 \in E(G)$. Consider $N(a_2)$. By (E2), $a_2v_3 \notin E(G)$. By (CF1), $v_2a_2 \notin E(G)$. Thus $N(a_2) \cap V(P_{10}) = \{v_4, v_5\}$, and so $G[\{a_2, v_4, v_5\} \cup \{v_6, v_7, v_8, v_9\} \cup \{v_3, v_2, v_1\}]$ is a $B(4,3)$, a contradiction. So v_5 and v_6 do not have common neighbors.

Let $a_1, a_2 \in N(v_4) \cap N(v_5)$. By (E1), $N(a_i) \cap \{v_1, v_8, v_9, v_{10}\} = \emptyset$. Since $v_6 \notin N(a_1) \cup N(a_2)$, by (CF1), $v_7 \notin N(a_1) \cup N(a_2)$. Thus $a_1a_2 \in E(G)$. If $v_2a_1 \in E(G)$, by (CF1), $v_3a_1 \in E(G)$. By (E2), $a_2v_2, a_2v_3 \notin E(G)$. Thus $G[\{a_2, v_4, v_5\} \cup \{v_6, v_7, v_8, v_9\} \cup \{v_1, v_2, v_3\}]$ is a $B(4,3)$, a contradiction. So $a_1v_2 \notin E(G)$. Similarly, $a_2v_2 \notin E(G)$. Since G is $B(4,3)$-free, $N(a_1) \cap V(P_{10}) = N(a_2) \cap V(P_{10}) = \{v_3, v_4, v_5\}$. By (E3), let $S = (N(\{a_1, a_2, v_4\}) - \{a_1, a_2, v_4\}) - \{v_3, v_5\} = \{y_1, y_2\}$. Then $y_1v_4, y_2v_4 \in E(G)$. For $i = 1, 2$, if $N(y_i) \cap \{v_3, v_4, v_5\} = \{v_4, v_5\}$, then, by (E1) and (E2), $G[\{y_i, v_4, v_5\} \cup \{v_6, v_7, v_8, v_9\} \cup \{v_3, v_2, v_1\}] = B(4,3)$, a contradiction. So $N(y_i) \cap \{v_3, v_4, v_5\} = \{v_3, v_4\}$. By (E1), (E2) and (E3), $N(y_i) \cap V(P_{10}) = \{v_3, v_4\}$, $N(a_1) = \{a_2, v_3, v_4, v_5, y_1, y_2\}$, $N(a_2) = \{a_1, v_3, v_4, v_5, y_1, y_2\}$, and $N(v_4) = \{a_1, a_2, v_3, v_5, y_1, y_2\}$. Since v_5 and v_6 do not have common neighbors, $N(v_5) = \{a_1, a_2, v_4, v_6\}$. Similarly, let $b_1, b_2 \in N(v_6) \cap N(v_7)$. Let $T = (N(b_1) \cup N(b_2) \cup N(v_7) - \{b_1, b_2, v_7\}) - \{v_6, v_8\} = \{z_1, z_2\}$. Then $N(z_1) \cap V(P_{10}) = N(z_2) \cap V(P_{10}) = \{v_7, v_8\}$, $G[\{b_1, b_2, z_1, z_2, v_7, v_8\}]$ is a K_6, and $N(v_6) = \{b_1, b_2, v_5, v_7\}$, $N(v_7) = \{b_1, b_2, z_1, z_2, v_6, v_8\}$, $N(b_1) = \{b_2, z_1, z_2, v_6, v_7, v_8\}$ and $N(b_2) = \{b_1, z_1, z_2, v_6, v_7, v_8\}$ (see Figure 3).

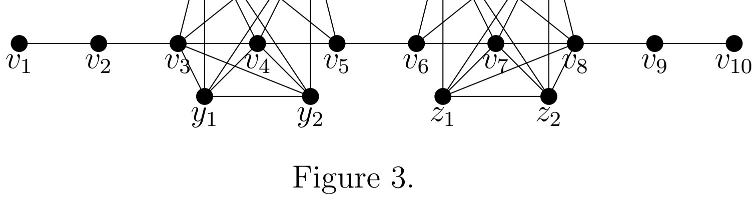

Figure 3.

Now let us consider $N(v_1)$. Let $x \in N(v_1) - \{v_2\}$. Then $N(x) \cap \{a_1, a_2, b_1, b_2, v_4, v_5, v_6, v_7\} = \emptyset$. Since G has no 8-cycles, $xy_1, xy_2 \notin E(G)$. If $x \notin N(v_2)$, then, by (CF1), $xv_3 \notin E(G)$. Since $G[\{y_1, v_3, v_4\} \cup \{v_5, v_6, v_7, v_8\} \cup \{v_2, v_1, x\}] \neq B(4, 3)$, we have $xv_8 \in E(G)$. Similarly, $xz_1, xz_2 \in E(G)$. This would result in the 8-cycle $v_6v_7v_8xz_2z_1b_1b_2v_6$. So $x \in N(v_2)$, and $N(x) \cap V(P_{10}) \subseteq \{v_1, v_2, v_3, v_9, v_{10}\}$ and $xz_1, xz_2 \notin E(G)$.

Let $W = N(\{v_1, v_2, v_3\}) - \{v_1, v_2, v_3, a_1, a_2, v_4, y_1, y_2\}$, $W_1 = \{x \mid x \in N(v_1) \cap N(v_2) \cap N(v_3)\}$, $W_2 = \{x \mid x \in N(v_1) \cap N(v_2) - N(v_3)\}$ and $W_3 = \{x \mid x \in N(v_2) \cap N(v_3) - N(v_1)\}$. Then $N(v_2) = W_1 \cup W_2 \cup W_3 \cup \{v_1, v_3\}$, $N(v_1) = W_1 \cup W_2 \cup \{v_2\}$. Also, $G[W_2 \cup \{v_1, v_2\}]$, $G[W_1 \cup W_3 \cup \{v_2, v_3\}]$ are complete subgraphs in G, and $N(W_1) - W_1 = W_2 \cup W_3 \cup \{v_1, v_2, v_3\}$. Thus $W_1 \cup W_2 \cup \{v_2\}$ is a cut in G. For $i = 1, 2, 3$, let $w_i = |W_i|$. Since G is 4-connected, we have $w_1 + w_2 \geq 3$.

Since G is 4-connected, $|N(W_2) - (W_2 \cup \{v_1, v_2\})| \geq 2$. Consider $W_2' = N(W_2) - (W_1 \cup W_3 \cup \{v_1, v_2, v_3\})$. If $W_2' = \emptyset$, then $W_3 \cup \{v_3\}$ is a cut in G, and so $w_3 \geq 3$. Thus $w_1 + w_2 + w_3 \geq 6$. Therefore, $G[W \cup \{v_1, v_2, v_3\}]$ must contain an 8-cycle, a contradiction. So $W_2' \neq \emptyset$. Let $d \in W_2'$ and $c \in W_2$ with $cd \in E(G)$. Then $dv_2, dv_3 \notin E(G)$. Clearly, $N(d) \cap \{v_4, v_5, v_6, v_7\} = \emptyset$. Since G has no 8-cycles, $cy_1, dy_1 \notin E(G)$. Since $G[\{y_1, v_3, v_4\} \cup \{v_5, v_6, v_7, v_8\} \cup \{v_2, c, d\}] \neq B(4, 3)$, $dv_8 \in E(G)$. Similarly, $dz_1, dz_2 \in E(G)$. Thus $v_6b_1b_2v_8dz_2z_1v_7v_6$ is an 8-cycle in G, a contradiction.

Case 2. $B(i, j) = B(5, 2)$.

Assume that v_5 and v_6 do not have common neighbors. Let $a_1, a_2 \in N(v_4) \cap N(v_5)$. By (E1), $N(a_i) \cap \{v_1, v_8, v_9, v_{10}\} = \emptyset$. Since $v_6 \notin N(a_1) \cup N(a_2)$, by (CF1), $v_7 \notin N(a_1) \cup N(a_2)$. If $v_2a_1 \in E(G)$, by (CF1), $v_3a_1 \in E(G)$. Then $G[\{a_1, v_4, v_5\} \cup \{v_6, v_7, v_8, v_9, v_{10}\} \cup \{v_2, v_1\}]$ is a $B(5, 2)$, a contradiction. So $a_1v_2 \notin E(G)$. Similarly, $a_2v_2 \notin E(G)$. Since G is $B(5, 2)$-free, $N(a_1) \cap V(P_{10}) = N(a_2) \cap V(P_{10}) = \{v_3, v_4, v_5\}$. By (E3), let $S = (N(\{a_1, a_2, v_4\}) - \{a_1, a_2, v_4\}) - \{v_3, v_5\} = \{y_1, y_2\}$, $N(a_1) = \{v_3, v_4, v_5, y_1, y_2, a_2\}$, $N(a_2) = \{v_3, v_4, v_5, y_1, y_2, a_1\}$, and $N(v_4) = \{v_3, v_5, a_1, a_2, y_1, y_2\}$. Also, $|N(y_1) \cap \{v_3, v_5\}| = 1$. Notice that G has no 8-cycles. If $y_1v_3 \in E(G)$, then, by (E1), (E2), $N(y_1) \cap V(P_{10}) = \{v_3, v_4\}$, and so $G[\{y_1, v_3, v_4\} \cup \{v_5, v_6, v_7, v_8, v_9\} \cup \{v_2, v_1\}] = B(5, 2)$; if $y_1v_5 \in E(G)$, then, by (E1), (E2), $N(y_1) \cap V(P_{10}) = \{v_4, v_5\}$, and so $G[\{y_1, v_4, v_5\} \cup \{v_6, v_7, v_8, v_9, v_{10}\} \cup \{v_3, v_2\}] = B(5, 2)$, a contradiction.

Assume that v_5 and v_6 have one common neighbor. Let $a_1 \in N(v_5) \cap N(v_6)$, $a_2 \in N(v_4) \cap N(v_5)$ and $a_3 \in N(v_6) \cap N(v_7)$. Then $a_2v_6 \notin E(G)$. By (E1) and (CF1), $N(a_2) \cap V(P_{10}) \subseteq \{v_2, v_3, v_4, v_5\}$. Since $G[\{a_2, v_4, v_5\} \cup \{v_6, v_7, v_8, v_9, v_{10}\} \cup \{v_1, v_2\}] = B(5, 2)$ if $a_2v_2 \in E(G)$ and $G[\{a_2, v_4, v_5\} \cup \{v_6, v_7, v_8, v_9, v_{10}\} \cup \{v_2, v_3\}] = B(5, 2)$ if $a_2v_3 \notin E(G)$, we have $N(a_2) \cap V(P_{10}) = \{v_3, v_4, v_5\}$. Consider $S = N(\{a_2, v_4\}) - \{a_2, v_3, v_4, v_5\}$. Let $y \in S$. By (E4), $yv_4 \in E(G)$. We want to prove that $y \in N(v_3) \cap N(v_4) \cap N(v_5)$. Otherwise, we have $yv_4, yv_3 \in E(G)$, but $yv_5 \notin E(G)$. By (E1) and (E2), $N(y) \cap \{v_2, v_7, v_8, v_9, v_{10}\} = \emptyset$, and so $yv_6 \notin E(G)$ by (CF1). Since G is $B(5, 2)$-free, $v_1y \in E(G)$. Let $w \in N(v_2)$. Thus we have an 8-cycle $v_1wv_2v_3a_2v_5v_4yv_1$ or $v_1v_2wv_3a_2v_5v_4yv_1$, a contradiction. So, for any $y \in S$, $y \in N(v_3) \cap N(v_4) \cap N(v_5)$. Therefore, $\{v_3, v_5\}$ is a 2-cut in G, a contradiction.

Therefore, v_5 and v_6 have more than one common neighbor. Let $a_1, a_2 \in N(v_5) \cap N(v_6)$. For $i = 1, 2$, by (E1), $N(a_i) \cap V(P_{10}) \subseteq \{v_3, v_4, v_5, v_6, v_7, v_8\}$. If $v_3 \in N(a_i)$, then by (CF1), $v_4 \in N(a_i)$. Thus $G[\{a_i, v_3, v_4\} \cup \{v_6, v_7, v_8, v_9, v_{10}\} \cup \{v_1, v_2\}]$ is a $B(5, 2)$, a contradiction. So $v_3a_i \notin E(G)$. Similarly, $v_8a_i \notin E(G)$. So, for $i \in \{1, 2\}$, $N(a_i) \cap V(P_{10}) \subseteq \{v_4, v_5, v_6, v_7\}$.

Claim 2.1. Both $|N(a_1) \cap V(P_{10})| \leq 3$ and $|N(a_2) \cap V(P_{10})| \leq 3$.

Assume that $N(a_1) \cap V(P_{10}) = \{v_4, v_5, v_6, v_7\}$. By (E2), $N(a_2) \cap V(P_{10}) = \{v_5, v_6\}$. Let $a_3 \in N(v_4)$. Then $a_3v_5 \notin E(G)$ (otherwise, by (E1) and (E2), $N(a_3) \cap \{v_3, v_6, v_7, v_8, v_9, v_{10}\} = \emptyset$. Since G is $B(5, 2)$-free, $v_2a_3 \in E(G)$. This would result in the 8-cycle $v_2a_3v_5v_6v_7a_1v_4v_3v_2$, a contradiction). By (CF1), $a_3v_3 \in E(G)$. By (E1) and (E2), $N(a_3) \cap \{v_5, v_6, v_7, v_8, v_9, v_{10}\} = \emptyset$. Since G is $B(5, 2)$-free, $N(a_3) \cap \{v_1, v_2\} \neq \emptyset$. Let $x \in N(v_1) - \{v_2\}$. By (E1), $N(x) \cap \{v_5, v_6, v_7\} = \emptyset$. Since G has no 8-cycles, $N(x) \cap \{v_8, a_1, a_2\} = \emptyset$. Thus $N(x) \cap V(P_{10}) \subseteq \{v_1, v_2, v_3, v_4, v_9, v_{10}\}$.

We claim that $v_1a_3 \in E(G)$. Otherwise, $N(a_3) \cap V(P_{10}) = \{v_2, v_3, v_4\}$. Consider $N(v_1) = \{v_2, c_1, c_2, \ldots, c_t\} (t \geq 3)$. By (E1), $c_iv_3 \notin E(G)$. By (CF1), $c_iv_4 \notin E(G)$. Since $G[\{a_2, v_5, v_6\} \cup \{v_4, v_3, v_2, v_1, c_i\} \cup \{v_7, v_8\}] \neq B(5, 2)$, $c_iv_2 \in E(G)$. Thus $N(c_i) \cap V(P_{10}) = \{v_1, v_2\}$ and so $G[N(v_1) \cup \{v_1\}]$ is a complete subgraph in G. Since G is 4-connected, there is a vertex z such that $zc_i \in E(G)$ but $zv_2 \notin E(G)$ for some c_i. Since G has no 8-cycles, $N(z) \cap \{a_1, a_2, v_3, v_4, v_5, v_6, v_7, v_8\} = \emptyset$. So $G[\{a_2, v_5, v_6\} \cup \{v_4, v_3, v_2, c_i, z\} \cup \{v_7, v_8\}] = B(5, 2)$, a contradiction. So $v_1a_3 \in E(G)$.

Let $N(v_1) = \{v_2, a_3, d_1, \ldots, d_s\} (s \geq 2)$. Since G has no 8-cycles, $N(d_i) \cap \{v_5, v_6, v_7, v_8\} = \emptyset$. Since $G[\{a_2, v_5, v_6\} \cup \{v_4, v_3, v_2, v_1, d_i\} \cup \{v_7, v_8\}] \neq B(5, 2)$, we have $N(d_i) \cap \{v_2, v_3, v_4\} \neq \emptyset$. If $d_iv_4 \in E(G)$, as $d_iv_5 \notin E(G)$, we have $d_iv_3 \in E(G)$. By (E2), $a_3v_2, d_iv_2 \notin E(G)$. Thus the 6-cycle $v_1d_iv_4a_3v_3v_2v_1$ can be extended to an 8-cycle by considering the two neighbors of v_2 which are not in $V(P_{10})$, a contradiction. So $d_iv_4 \notin E(G)$. By (CF1), $d_iv_2 \in E(G)$. By (E2), $d_iv_3 \notin E(G)$. Thus $G[N(v_1)]$ is a complete subgraph in G. The 7-cycle $v_1d_1d_2v_2v_3v_4a_3v_1$ can be extended to an 8-cycle by considering a neighbors of v_3 which are not in $\{v_2, v_4, a_3\}$, a contradiction.

Claim 2.2. $|N(a_1) \cap V(P_{10})| = 2$ and $|N(a_2) \cap V(P_{10})| = 2$.

Assume that $N(a_1) \cap V(P_{10}) = \{v_4, v_5, v_6\}$. By (E2), $a_2v_7 \notin E(G)$. Thus $N(a_2) \cap V(P_{10}) \subseteq \{v_4, v_5, v_6\}$. Consider $N(v_7)$. Let $y \in N(v_7) - (V(P_{10}) \cup \{a_1, a_2\})$. Assume that $yv_6 \in E(G)$. By (E1) and (E2), $N(y) \cap \{v_5, v_3, v_2, v_1, v_{10}\} = \emptyset$. Thus $yv_4 \notin E(G)$. Since $G[\{y, v_6, v_7\} \cup \{v_5, v_4, v_3, v_2, v_1\} \cup \{v_8, v_9\}]$ is not a $B(5, 2)$, $N(y) \cap \{v_8, v_9\} \neq \emptyset$. If $yv_9 \in E(G)$, then $G[\{y, v_6, v_7\} \cup \{v_5, v_4, v_3, v_2, v_1\} \cup \{v_9, v_{10}\}]$ is a $B(5, 2)$, a contradiction. So $N(y) \cap V(P_{10}) = \{v_6, v_7, v_8\}$. Let $S = N(y) \cup N(v_7) - \{v_6, v_8\}$, and let $w \in S$. By (E4), $wv_7 \in E(G)$. Then $w \in N(v_6) \cap N(v_7) \cap N(v_8)$ (Otherwise, we have $wv_6 \notin E(G)$ by considering the method we just used above for $y \in N(v_7)$. By (CF1), $wv_8 \in E(G)$. Since G has no 8-cycles, by (E1) and (E2), $N(w) \cap \{v_1, v_2, v_3, v_4, v_5, v_9\} = \emptyset$. Since G is $B(5, 2)$-free, $wv_{10} \in E(G)$. Thus the 7-cycle $v_6yv_8v_9v_{10}wv_7v_6$ can be extended to an 8-cycle by considering a neighbor of v_9, a contradiction). Hence, $\{v_6, v_8\}$ is a 2-cut in G, a contradiction. So, for any $y \in N(v_7)$, $yv_6 \notin E(G)$.

Let $a_3, a_4 \in N(v_7) - \{v_6, v_8\}$. Then, for $i = 3, 4$, $a_iv_8 \in E(G)$, and $N(a_i) \cap$

$\{v_1, v_2, v_3, v_4, v_5\} = \emptyset$ by (E1) and (E2). Since G is $B(5,2)$-free, $N(a_i) \cap \{v_9, v_{10}\} \neq \emptyset$ $(i = 3, 4)$. Assume that $a_3v_9 \notin E(G)$. Then $a_3v_{10} \in E(G)$. By (E2), $a_4v_9 \notin E(G)$, and so $a_4v_{10} \in E(G)$. Thus the 6-cycle $v_7v_8v_9v_{10}a_3a_4v_7$ can be extended to an 8-cycle by considering two neighbors of v_9, a contradiction. So $a_3v_9 \in E(G)$. Similarly, $a_4v_9 \in E(G)$. By (E1) and (E2), $N(a_i) \cap V(P_{10}) = \{v_7, v_8, v_9\}$. Then $a_3a_4 \in E(G)$.

Let $S = N(\{a_3, a_4, v_8\}) - \{a_3, a_4, v_8, v_7, v_9\}$. Since G is 4-connected, let $S = \{c_1, c_2, \ldots, c_t\}(t \geq 2)$. For $i = 1, 2, \ldots, t$, by (E4), $c_iv_8 \in E(G)$. By (CF1), we have either $c_iv_7 \in E(G)$ or $c_iv_9 \in E(G)$, and so $t = 2$. Furthermore, $c_iv_7 \notin E(G)$ (otherwise, $N(c_i) \cap V(P_{10}) = \{v_7, v_8, v_9\}$ and so $\{c_1, c_2, v_7, v_9\} - \{c_i\}$ is a 3-cut in G, a contradiction). Thus $G[\{a_3, a_4, v_8, v_9, c_1, c_2\}]$ is a complete subgraph in G, $N(a_3) = \{v_7, v_8, v_9, a_4, c_1, c_2\}$, $N(a_4) = \{v_7, v_8, v_9, a_3, c_1, c_2\}$, $N(v_8) = \{v_7, v_9, a_3, a_4, c_1, c_2\}$. Since G has no 8-cycles, by (E1) and (E2), $N(c_i) \cap \{v_2, v_3, v_4, v_5, v_6, v_7, v_{10}\} = \emptyset$ $(i = 1, 2)$.

For $i = 1, 2$, consider $C_i = N(c_i) - \{v_8, v_9, a_3, a_4, c_1, c_2\}$. Since G is 4-connected, $C_i \neq \emptyset$. Let $d_i \in C_i$. Since G has no 8-cycles, $C_1 \cap C_2 = \emptyset$, and there are no edges between C_1 and C_2. Thus $d_1d_2 \notin E(G)$. Let $e_i \in N(d_i) - \{c_i\}$. Since G has no 8-cycles, e_1 and e_2 are different vertices, $e_1e_2 \notin E(G)$, $N(e_1) \cap N(e_2) = \emptyset$, $N(d_i) \cap \{v_3, v_4, \ldots, v_9\} = \emptyset$ and $N(e_i) \cap \{v_4, v_5, \ldots, v_9\} = \emptyset$. Since $G[\{c_i, v_8, v_9\} \cup \{v_7, v_6, v_5, v_4, v_3\} \cup \{d_i, e_i\}]$ is not a $B(5,2)$, $e_iv_3 \in E(G)$, a contradiction. So Claim 2.2 holds.

By Claim 2.2, we have $N(a_1) \cap V(P_{10}) = N(a_2) \cap V(P_{10}) = \{v_5, v_6\}$. Actually, for any $x \in N(v_5) \cap N(v_6)$, $N(x) \cap V(P_{10}) = \{v_5, v_6\}$. Let $y \in N(v_4)$. Assume that $yv_5 \in E(G)$. Then $yv_6 \notin E(G)$ by Claim 2.2. By (E1) and (E2), $N(y) \cap \{v_1, v_8, v_9, v_{10}\} = \emptyset$. Thus $yv_7 \notin E(G)$. Since $G[\{y, v_4, v_5\} \cup \{v_6, \ldots, v_{10}\} \cup \{v_2, v_3\}] \neq B(5,2)$, $N(y) \cap \{v_2, v_3\} \neq \emptyset$. Notice that $G[\{y, v_4, v_5\} \cup \{v_6, \ldots, v_{10}\} \cup \{v_1, v_2\}]$ would be a $B(5,2)$ if $yv_2 \in E(G)$. So $yv_2 \notin E(G)$ and then $yv_3 \in E(G)$. Consider $S = N(y) \cup N(v_4) - \{v_3, v_5\}$, and let $z \in S$. By (E4), $z \in N(v_4)$. Next we want to prove that $z \in N(v_3) \cap N(v_4) \cap N(v_5)$. Otherwise, we have $zv_5 \notin E(G)$ and $zv_3 \in E(G)$. By (E1) and (E2), $N(z) \cap \{v_2, v_6, v_7, v_8, v_9, v_{10}\} = \emptyset$. If $zv_1 \in E(G)$, then the 7-cycle $v_1v_2v_3yv_5v_4zv_1$ can be extended to an 8-cycle by considering a neighbor of v_2. This tells us that $zv_1 \notin E(G)$. Thus $G[\{z, v_3, v_4\} \cup \{v_5, \ldots, v_9\} \cup \{v_1, v_2\}]$ is a $B(5,2)$, a contradiction. Thus $z \in N(v_3) \cap N(v_4) \cap N(v_5)$, and so $\{v_3, v_5\}$ is a 2-cut in G, a contradiction. So, for any $y \in N(v_4)$, $yv_5 \notin E(G)$.

Let $N(v_4) - \{v_3, v_5\} = \{c_1, c_2, \ldots, c_t\}(t \geq 2)$. Then $c_iv_5 \notin E(G)$, $c_iv_3 \in E(G)$ for $i = 1, 2, \ldots, t$, and $c_ic_j \in E(G)$ for $1 \leq i < j \leq t$. By (E1) and (E2), $N(c_i) \cap \{v_7, v_8, v_9, v_{10}\} = \emptyset$. By (CF1), $c_iv_6 \notin E(G)$. If $c_iv_1 \in E(G)$ for some i, then the cycle $v_1c_ic_{i+1} \ldots c_tc_1 \ldots c_{i-1}v_4v_3v_2v_1$ can be extended to an 8-cycle by considering neighbors of v_2. So, for $i = 1, 2, \ldots, t$, $c_iv_1 \notin E(G)$. Thus $c_iv_2 \in E(G)$ since $G[\{c_i, v_3, v_4\} \cup \{v_5, \ldots, v_{10}\} \cup \{v_1, v_2\}] \neq B(5,2)$. Similarly, $|N(v_7) \cap N(v_8) \cap N(v_9)| \geq 2$. Let $d_1, d_2 \in N(v_7) \cap N(v_8) \cap N(v_9)$. Then $d_1d_2 \in E(G)$.

Consider $S = N(\{c_1, c_2, \ldots, c_t, v_3\}) - \{c_1, c_2, \ldots, c_t, v_2, v_3, v_4\}$, and let $w \in S$. Then $wv_4 \notin E(G)$. By (E4), $wv_3 \in E(G)$. By (CF1), $wv_2 \in E(G)$. By (E1) and (E2), $N(w) \cap \{v_1, v_5, v_6, v_7, v_8, v_9\} = \emptyset$. Let $V_1 = N(v_1) - \{v_2\} = \{e_1, e_2, \ldots, e_s\}$ $(s \geq$

3). Since G has no 8-cycles, $N(e_i) \cap \{c_1, \dots, c_t, w, v_3, v_4, \dots, v_7\} = \emptyset$. Considering $G[\{w, v_2, v_3\} \cup \{v_4, v_5, v_6, v_7, v_8\} \cup \{v_1, e_i\}]$, we have $N(e_i) \cap \{v_2, v_8\} \neq \emptyset$. Since G has no 8-cycles, $|\{e_i \mid e_iv_8 \in E(G)\}| \leq 1$. (Otherwise, assume that $e_1v_8, e_2v_8 \in E(G)$. By (CF1), $e_1v_9, e_2v_9 \in E(G)$. Thus $v_7v_8e_1v_1e_2v_9d_2d_1v_7$ is an 8-cycle in G, a contradiction.) So we assume that, for $i = 2, 3, \dots, s$, we have $e_iv_8 \notin E(G)$, and so $e_iv_2 \in E(G)$.

Let $V_2 = N(\{e_2, \dots, e_s\}) - \{e_1, e_2, \dots, e_s, v_1, v_2\}$. Since G is 4-connected, $|V_2| \geq 2$. Furthermore, there are two vertices in V_2 adjacent to two different vertices in $\{e_2, \dots, e_s\}$. Without loss of generality, we assume that $f_2, f_3 \in V_2$ such that $e_2f_2, e_3f_3 \in E(G)$. Then $f_2v_1, f_3v_1 \notin E(G)$. For $i = 2, 3$, if $f_iv_2 \in E(G)$, then $f_iv_3 \in E(G)$. Thus $v_1e_if_iv_3v_4c_2c_1v_2v_1$ is an 8-cycle in G, a contradiction. So $f_2v_2, f_3v_2 \notin E(G)$. Since G has no 8-cycles, $N(f_i) \cap \{w, v_3, v_4, v_5, v_6, v_7\} = \emptyset (i = 2, 3)$. Notice that $G[\{w, v_2, v_3\} \cup \{v_4, v_5, v_6, v_7, v_8\} \cup \{e_i, f_i\} \neq B(5, 2)$ for $i = 2, 3$, $f_iv_8 \in E(G)$ and so $f_iv_9 \in E(G)$. This would result in an 8-cycle $v_1e_2f_2v_8v_9f_3e_3v_2v_1$, a contradiction. This finishes the proof of Case 2.

Case 3. $B(i, j) = B(6, 1)$.

Claim 3.1. Let $x \in (N(v_3) - \{v_2, v_4\}) - N(v_4)$, and let $y \in (N(v_8) - \{v_7, v_9\}) - N(v_7)$. Then $N(x) \cap V(P_{10}) = \{v_1, v_2, v_3\}$ and $N(y) \cap V(P_{10}) = \{v_8, v_9, v_{10}\}$.

Since $xv_4 \notin E(G)$, by (CF1), $xv_2 \in E(G)$. By (E1), $N(x) \cap \{v_6, v_7, v_8, v_9\} = \emptyset$. By (CF1), $xv_5 \notin E(G)$. Since G is $B(6,1)$-free, $xv_1 \in E(G)$. By (CF2), $N(x) \cap V(P_{10}) = \{v_1, v_2, v_3\}$. Similarly, $N(y) \cap V(P_{10}) = \{v_8, v_9, v_{10}\}$. Claim 3.1 holds.

Claim 3.2. Let $W_3 = (N(v_3) - \{v_2, v_4\}) - N(v_4)$ and $V_3 = (N(v_8) - \{v_7, v_9\}) - N(v_7)$. Then $W_3 = V_3 = \emptyset$.

Assume that $x \in W_3$. By Claim 3.1, $N(x) \cap V(P_{10}) = \{v_1, v_2, v_3\}$. Furthermore, if $x' \in N(v_1) \cap N(v_2) \cap N(v_3)$, then $x'v_4 \notin E(G)$ (otherwise, $G[\{x', v_3, v_4\} \cup \{v_5, v_6, v_7, v_8, v_9, v_{10}\} \cup \{v_1\}] = B(6, 1)$, a contradiction). So $W_3 = N(v_1) \cap N(v_2) \cap N(v_3)$. Let $W_2 = N(v_2) \cap N(v_1) - N(v_3)$ and $W_1 = (N(v_1) - \{v_2\}) - N(v_2)$, and let $w_i = |W_i|$ $(i = 1, 2, 3)$. Then $N(v_2) = W_2 \cup W_3 \cup \{v_1, v_3\}$, and $N(v_1) = W_1 \cup W_2 \cup W_3 \cup \{v_2\}$. Clearly, $G[W_1 \cup \{v_1\}]$, $G[W_2 \cup \{v_1, v_2\}]$, and $G[W_3]$ are complete graphs.

Let $y \in N(W_3) - \{v_1, v_2, v_3\}$. By (E4), $yv_4 \notin E(G)$. If $yv_3 \in E(G)$, then $y \in W_3$; if $yv_3 \notin E(G)$, then $yv_1 \in E(G)$, and so $y \in W_1 \cup W_2$. This imples that $N(W_3) \subseteq W_3 \cup W_1 \cup W_2 \cup \{v_1, v_2, v_3\}$, and $W_1 \cup W_2 \cup \{v_3\}$ is a cut in G. So we have $w_1 + w_2 \geq 3$. As $N(v_2) = W_2 \cup W_3 \cup \{v_1, v_3\}$, it follows that $w_2 + w_3 \geq 2$. If $w_2 = 0$, then $w_3 \geq 2$ and $w_1 \geq 3$. i As $N(W_3) - (W_3 \cup \{v_1, v_2, v_3\}) \subseteq W_1 \cup W_2 = W_1$, there is an edge joining W_1 and W_3. Thus $G[W_1 \cup W_3 \cup \{v_1, v_2, v_3\}]$ contains an 8-cycle, a contradiction. So $w_2 \geq 1$.

Consider $S = N(W_2) - (W_1 \cup W_2 \cup W_3 \cup \{v_1, v_2, v_3\}$. If $S = \emptyset$, then $W_1 \cup \{v_3\}$ is a cut in G. Thus $w_1 \geq 3$. It is clear that there is an edge joining W_1 and $W_2 \cup W_3$ (otherwise, $\{v_1, v_2, v_3\}$ is a cut in G, a contradiction). So $G[W_1 \cup W_2 \cup W_3 \cup \{v_1, v_2, v_3\}]$ contains an 8-cycle, a contradiction. So $S \neq \emptyset$. Let $y_1 \in W_2$. Also, let $z_1 \in S$. Then $y_1v_3, z_1v_1, z_1v_2 \notin E(G)$. By (E1), $N(y_1) \cap \{v_5, v_6, v_7, v_8\} = \emptyset$. By (CF1),

$y_1v_4 \notin E(G)$. Since G has no 8-cycles, $N(z_1) \cap \{v_5, v_6, v_7\} = \emptyset$. If $z_1v_3 \in E(G)$, since $z_1v_2 \notin E(G)$, we have $z_1v_4 \in E(G)$. By (E1), $N(z_1) \cap \{v_8, v_9, v_{10}\} = \emptyset$. Thus $G[\{z_1, v_3, v_4\} \cup \{v_5, v_6, v_7, v_8, v_9, v_{10}\} \cup \{v_2\}] = B(6,1)$, a contradiction. So $z_1v_3 \notin E(G)$. By (CF1), $z_1v_4 \notin E(G)$. Since $G[\{y_1, v_1, v_2\} \cup \{v_3, v_4, v_5, v_6, v_7, v_8\} \cup \{z_1\}] \neq B(6,1)$, $z_1v_8 \in E(G)$. By Claim 3.1, $N(z_1) \cap V(P_{10}) = \{v_8, v_9, v_{10}\}$.

Let $V_2 = N(v_9) \cap N(v_{10}) - N(v_8)$ and $V_1 = (N(v_{10}) - \{v_9\}) - N(v_9)$. As for the discussion on W_1, W_2 and W_3, there are $y_2 \in V_2$ and $z_2 \in N(V_2) - (V_1 \cup V_2 \cup V_3 \cup \{v_8, v_9, v_{10}\})$ such that $y_2z_2 \in E(G)$ and $N(z_2) \cap V(P_{10}) = \{v_1, v_2, v_3\}$. Now we have an 8-cycle $y_1z_1v_8v_9v_{10}y_2z_2v_2y_1$, a contradiction. So $W_3 = \emptyset$. Similarly, $V_3 = \emptyset$. Claim 3.2 holds.

By Claim 3.2, v_3 and v_4 have more than one common neighbor, and v_7 and v_8 have more than one common neighbor. Let $a_1, a_2 \in N(v_3) \cap N(v_4)$, and let $b_1, b_2 \in N(v_7) \cap N(v_8)$. By (E1), $N(a_i) \cap \{v_7, v_8, v_9, v_{10}\} = \emptyset$ $(i = 1, 2)$. If $v_1 \in N(a_1)$, then $N(a_1) \cap V(P_{10}) \subseteq \{v_1, v_2, v_3, v_4\}$ and then G has a $B(6,1) = G[\{a_1, v_3, v_4\} \cup \{v_1\} \cup \{v_5, v_6, v_7, v_8, v_9, v_{10}\}]$, a contradiction. So $v_1 \notin N(a_1)$. If $v_6 \in N(a_1)$, by (E2), $a_2v_5, a_2v_6, a_2v_2 \notin E(G)$. Thus $G[\{a_2, v_3, v_4\} \cup \{v_5, v_6, v_7, v_8, v_9, v_{10}\} \cup \{v_2\}\}]$ is a $B(6,1)$, a contradiction. So $a_1v_6 \notin E(G)$, and $N(a_1) \cap V(P_{10}) \subseteq \{v_2, v_3, v_4, v_5\}$. Similarly, $N(a_2) \cap V(P_{10}) \subseteq \{v_2, v_3, v_4, v_5\}$. Since G is $B(6,1)$-free, by (E2), we have either $N(a_1) \cap V(P_{10}) = N(a_2) \cap V(P_{10}) = \{v_2, v_3, v_4\}$ or $N(a_1) \cap V(P_{10}) = N(a_2) \cap V(P_{10}) = \{v_3, v_4, v_5\}$.

Suppose that $N(a_1) \cap V(P_{10}) = N(a_2) \cap V(P_{10}) = \{v_2, v_3, v_4\}$. By (E3), let $T_1 = (N(\{a_1, a_2, v_3\}) - \{a_1, a_2, v_3\}) - \{v_2, v_4\} = \{y_1, y_2\}$, $N(a_1) = \{v_2, v_3, v_4, y_1, y_2, a_2\}$, $N(a_2) = \{v_2, v_3, v_4, y_1, y_2, a_1\}$, and $N(v_3) = \{v_2, v_4, a_1, a_2, y_1, y_2\}$. Also, $|N(y_1) \cap \{v_2, v_4\}| = 1$. If $y_1v_4 \in E(G)$, then $G[\{y_1, v_3, v_4\} \cup \{v_5, v_6, v_7, v_8, v_9, v_{10}\} \cup \{v_2\}] = B(6,1)$; if $y_1v_2 \in E(G)$, then $G[\{y_1, v_2, v_3\} \cup \{v_4, v_5, v_6, v_7, v_8, v_9\} \cup \{v_1\}] = B(6,1)$, a contradiction. So $N(a_1) \cap V(P_{10}) = N(a_2) \cap V(P_{10}) = \{v_3, v_4, v_5\}$. Similarly, $N(b_1) \cap V(P_{10}) = N(b_2) \cap V(P_{10}) = \{v_6, v_7, v_8\}$.

By (E3) again, let $T_2 = (N(\{a_1, a_2, v_4\}) - \{a_1, a_2, v_4\}) - \{v_3, v_5\} = \{z_1, z_2\}$, $N(a_1) = \{v_3, v_4, v_5, z_1, z_2, a_2\}$, $N(a_2) = \{v_3, v_4, v_5, z_1, z_2, a_1\}$, and $N(v_4) = \{v_3, v_5, a_1, a_2, z_1, z_2\}$. Also, $|N(z_i) \cap \{v_3, v_5\}| = 1$ $(i = 1, 2)$. If $z_iv_3 \in E(G)$, then

$$G[\{z_i, v_3, v_4\} \cup \{v_5, v_6, v_7, v_8, v_9, v_{10}\} \cup \{v_2\}] = B(6,1),$$

a contradiction. So for $i = 1, 2$, $z_iv_5 \in E(G)$. Since G has no 8-cycles, $N(v_3) = \{a_1, a_2, v_2, v_4\}$. Similarly, by (E3), let $T_3 = (N(\{b_1, b_2, v_7\}) - \{b_1, b_2, v_7\}) - \{v_6, v_8\} = \{w_1, w_2\}$, $N(b_1) = \{v_6, v_7, v_8, w_1, w_2, b_2\}$, $N(b_2) = \{v_6, v_7, v_8, w_1, w_2, b_1\}$, and $N(v_7) = \{v_6, v_8, b_1, b_2, w_1, w_2\}$. Also, for $i = 1, 2$, $N(w_i) \cap \{v_6, v_8\} = \{v_6\}$. By (E1) and (E2), for $i = 1, 2$, $N(z_i) \cap V(P_{10}) = \{v_4, v_5\}$ and $N(w_i) \cap V(P_{10}) = \{v_6, v_7\}$. Also, we have $N(v_8) = \{v_7, v_9, b_1, b_2\}$. Since G is 4-connected, let $c_1 \in N(z_1) - \{v_4, v_5, a_1, a_2\}$ and $c_2 \in N(z_2) - \{v_4, v_5, a_1, a_2\}$. Then $N(c_i) \cap V(P_{10}) = \emptyset$ $(i = 1, 2)$.

Consider $N(v_{10})$. Let $x \in N(v_{10}) - \{v_9\}$. Then $N(x) \cap \{v_3, v_4, v_7, v_8\} = \emptyset$. Since G has no 8-cycles, $N(x) \cap \{v_5, v_6, z_1, z_2\} = \emptyset$, and $|N(x) \cap \{v_3, c_1, c_2\}| \leq 1$. Without loss of generality, we assume that $c_1x \notin E(G)$. Since $G[\{z_1, v_4, v_5\} \cup \{v_6, v_7, v_8, v_9, v_{10}, x\} \cup \{c_1\}] \neq B(6,1)$, $xv_9 \in E(G)$. Since $xv_8 \notin E(G)$, it follows that $G[N(v_{10})]$ is a complete graph. Since G is 4-connected, let $d \in N(N(v_{10})) - \{v_8, v_9, v_{10}\}$. Also, we assume that $dx \in E(G)$, where $x \in N(v_{10})$. Since G has no 8-cycles, $|N(d) \cap \{c_1, c_2, v_3\}| \leq 1$. Hence $|(N(d) \cup N(x)) \cap \{c_1, c_2, v_3\}| \leq 2$. There is a vertex $u \in \{c_1, c_2, v_3\}$ with $u \notin N(d) \cup N(x)$. Thus $G[\{z_1, v_4, v_5\} \cup \{v_6, v_7, v_8, v_9, x, d\} \cup \{u\}] = B(6, 1)$, a contradiction. □

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