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Conflict-free (vertex-)connection numbers of graphs with small diameter*

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Abstract

A path in an edge (vertex)-colored graph is called a conflict-free path if there exists at least one color that is used on only one of its edges (vertices). An edge (vertex)-colored graph is called conflict-free (vertex-) connected if for each pair of distinct vertices, there is a conflict-free path connecting them. For a connected graph G, the conflict-free (vertex-) connection number of G, denoted by $\mathrm{cfc}(G)$ (or $\mathrm{vcfc}(G)$), is defined as the smallest number of colors that are required to make G conflict-free (vertex-)connected. We use $C(G)$ to denote the subgraph induced by the set of all cut-edges of G. It is easy to see that $C(G)$ is a (possibly empty) forest. Let $h(G) = \max\{\mathrm{cfc}(T) : T \text{ isacomponentof } C(G)\}$. In this paper, we first give the exact value of $\mathrm{cfc}(T)$ for any tree T with diameter 2, 3 or 4. Based on this result, the conflict-free connection number is determined for any graph G with $\mathrm{diam}(G) \leq 4$ except for those graphs G with diameter 4 and $h(G) = 2$. In this case, we give some graphs with conflict-free connection numbers 2 and 3. For the conflict-free vertex-connection number, the exact value of $\mathrm{vcfc}(G)$ is determined for any graph G with $\mathrm{diam}(G) \leq 4$.

1 Introduction

In this paper, all graphs considered are simple, finite and undirected. We refer to Bondy and Murty's book [1] for notation and terminology in graph theory not defined here. Amongst all subjects of graph theory, chromatic theory is no doubt among the most interesting ones. In this paper, we mainly deal with the conflict-free (vertex-) connection coloring of graphs.

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In [8], Even, Lotker, Ron and Smorodinsky first introduced the hypergraph version of conflict-free (vertex-)coloring, motivated by the problem of assigning frequencies to different base stations in cellular networks. Since then, this coloring has received wide attention due to its practical application value.

Afterwards, Czap, Jendrol' and Valiska introduced the concept of conflict-free connection coloring in [5]. In an edge-colored graph, a path is called *conflict-free* if there is at least one color used on exactly one of its edges. This edge-colored graph is said to be *conflict-free connected* if every pair of distinct vertices of the graph is connected by a conflict-free path, and the coloring is called a *conflict-free connection coloring*. The *conflict-free connection number* of a connected graph G, denoted by $\mathrm{cfc}(G)$, is defined as the smallest number of colors required to make G conflict-free connected. There are many results on this topic; for more details, please refer to [2, 3, 4, 5, 6, 9]. It is easy to see that $1 \leq \mathrm{cfc}(G) \leq n-1$ for a connected graph G. The results on some derived concepts of conflict-free connection can be found in [10, 11].

Motivated by the above concept, Li, Zhang, Zhu, Mao, Zhao and Jendrol' [12] introduced the concept of *conflict-free vertex-connection*. A path in a vertex-colored graph is called *conflict-free* if there is at least one color that is used on exactly one of its vertices. This vertex-colored graph is said to be *conflict-free vertex-connected* if every two distinct vertices of the graph are connected by a conflict-free path, and the coloring is called a *conflict-free vertex-connection coloring*. The *conflict-free vertex-connection number* of a connected graph G, denoted by $\mathrm{vcfc}(G)$, is defined as the smallest number of colors required to make G conflict-free vertex-connected. In [7, 12, 13], several results about this notion were obtained. In particular, it was proved that the bounds $2 \leq \mathrm{vcfc}(G) \leq \lceil \log_2(n+1) \rceil$ hold for every connected graph G with order at least 2.

The path of order n is denoted by P_n. We use S_n to denote the *star graph* on n vertices and denote by $T(n_1, n_2)$ the *double star* in which the degrees of its two (adjacent) center vertices are n_1+1 and n_2+1, respectively. For a connected graph G, the *distance* between two vertices u and v is the minimum length of all paths between them, and we write it as $d_G(u,v)$. The *eccentricity* of a vertex v of G is defined by $\mathrm{ecc}_G(v) = \max_{u\in V(G)} d_G(u,v)$. The *diameter* of G is defined by $\mathrm{diam}(G) = \max_{v\in V(G)} \mathrm{ecc}_G(v)$ while the *radius* of G is defined by $rad(G) = \min_{v\in V(G)} \mathrm{ecc}_G(v)$. These parameters have much to do with graph structures and are very significant in the field of graph study. So it stimulates our interest to research on the conflict-free (vertex-)connections of graphs with small diameter.

In this paper, we first give the exact value of $\mathrm{cfc}(T)$ for any tree T with diameter $2, 3$ or 4. Based on this result, the conflict-free connection number is determined for any graph G with $\mathrm{diam}(G) \leq 4$ except for those graphs G with diameter 4 and $h(G) = 2$. In this case, we give some graphs with conflict-free connection numbers 2 and 3, respectively. For the conflict-free vertex-connection number, the exact value of $\mathrm{vcfc}(G)$ is determined for any graph G with $\mathrm{diam}(G) \leq 4$.

2 cfc-values for trees with diameter $2, 3$ or 4

For a connected graph G, let X denote the set of all cut-edges of G, and let $C(G)$ denote the subgraph induced by X. It is easy to see that $C(G)$ is a (possibly empty) forest. If $C(G)$ is not empty, then we let $h(G) = \max\{\mathrm{cfc}(T) : T \text{ is a component of } C(G)\}$. In [5], the authors obtained the following result.

Lemma 2.1. *[5] If G is a connected graph with cut-edges, then $h(G) \leq cfc(G) \leq h(G) + 1$. Moreover, the bounds are sharp.*

So, $h(G)$ is a crucial parameter to determine the conflict-free connection number of a connected graph G. Nevertheless, from the definition of $h(G)$, we can see that determining the value of $h(G)$ depends on determining the conflict-free connection numbers of trees. Therefore, in this section we first give the exact value of the conflict-free connection number of trees with diameter $2, 3$ or 4.

Theorem 2.2. *If a tree T has diameter 2 or 3, then $cfc(T) = \Delta(T)$.*

Proof. It is easy to see that T has diameter 2 if and only if it is a star S_n, and has diameter 3 if and only if it is a double star $T(n_1, n_2)$ $(n_1 \geq n_2)$. For the former case, every two edges of T must be colored differently in any conflict-free connection coloring, thus $\mathrm{cfc}(T) = \Delta(T)$. While in the latter case, we can obtain that $\mathrm{cfc}(T) = n_1 + 1 = \Delta(T)$ by a similar analysis. □

For a tree T of diameter 4, we denote by u the unique vertex with eccentricity two. The neighbors of u are pendent vertices $w_1, w_2, \ldots, w_\ell$ and $v_1, v_2, \ldots, v_r$ with degrees $p_1 \geq p_2 \geq \cdots \geq p_r \geq 2$. Certainly, $r + \ell = d(u)$. Observe that,

(i) in every conflict-free connection coloring c of T, the incident edges of every vertex must receive different colors.

Without loss of generality, set $c(uv_i) = i$ $(1 \leq i \leq r)$ and $c(uw_j) = r + j$ $(1 \leq j \leq \ell)$. Also observe that,

(ii) if one incident edge of v_i is assigned color j, then color i can not appear on any edge incident with v_j $(1 \leq i, j \leq r)$.

Actually, we are seeking for the minimum number of colors satisfying (i) and (ii).

Next we define a vector class S_r $(r \in \mathbb{N}^+)$. We say that an r-tuple $(s_1, s_2, s_3, \ldots, s_r)$ $(s_i\ (1 \leq i \leq r) \in \mathbb{N})$ belongs to S_r if and only if we can find a sequence of distinct pairs $(1, i_{1,1}), (1, i_{1,2}), \ldots, (1, i_{1,s_1}), (2, i_{2,1}), \ldots, (2, i_{2,s_2}), \ldots, (r, i_{r,s_r})$ the components of which are all from $[r]$ such that (1) the two components of every pair are different, (2) (i, j) and (j, i) $(1 \leq i, j \leq r)$ cannot both appear. Note that if $(s_1, s_2, s_3, \ldots, s_r) \in S_r$ then any permutation of its components also belongs to S_r. Thus we may suppose that $s_1 \geq s_2 \geq s_3 \geq \cdots \geq s_r$.

Lemma 2.3. *An r-tuple $(s_1, s_2, s_3, \ldots, s_r)$ $(s_i \in \mathbb{N}$ for $(1 \leq i \leq r))$ belongs to S_r if and only if $\sum_{i=1}^{j} s_i \leq (2r - 1 - j)j/2,\ (1 \leq j \leq r)$.*

Proof. First we show the necessity. If $(s_1, s_2, s_3, \dots, s_r) \in S_r$, then there is a sequence of pairs for them according to the definition. Suppose that none of $(i, j), (j, i)$ $(1 \leq i < j \leq r)$ appears. Then add the pair (i, j) to the sequence. Repeat this operation until nothing can be added. Finally there are $\frac{(r-1)r}{2}$ pairs and the corresponding r-tuple is $(s'_1, s'_2, s'_3, \dots, s'_r)$. Assume, to the contrary, that there exists a j such that $\sum_{i=1}^{j} s_i > \frac{(2r-1-j)j}{2}$. Then $\sum_{i=1}^{j} s'_i \geq \sum_{i=1}^{j} s_i > \frac{(2r-1-j)j}{2}$. Obviously, $j \neq r$. It is easy to observe that the number of distinct pairs (a, b) $(a, b \in \mathbb{N}, a \neq b, j+1 \leq a, b \leq r)$ is $(r-j)(r-j-1)$. Since exactly half of them appear in the sequence (one and only one of (a, b) and (b, a) is in the sequence), $\sum_{i=j+1}^{r} s'_i \geq \frac{(r-j)(r-j-1)}{2}$. However, this implies that $\frac{(r-1)r}{2} = \sum_{i=1}^{r} s'_i = \sum_{i=1}^{j} s'_i + \sum_{i=j+1}^{r} s'_i > \frac{(2r-1-j)j}{2} + \frac{(r-j)(r-j-1)}{2} = \frac{(r-1)r}{2}$, a contradiction. Thus the necessity holds.

For the sufficiency, we prove it by applying induction on r. When $r = 2$, the only pair satisfying the required condition is $(1, 0)$, and the corresponding sequence is then $(1, 2)$. Assume that the sufficiency holds for $r = p$. Consider the case $r = p+1$. For $(s_1, s_2, s_3, \dots, s_{p+1})$, suppose $s_1 = p - q$. We distinguish two cases to clarify.

Case 1. $s_{q+1} > s_{q+2}$. In this case, we prove that $(s_2 - 1, s_3 - 1, \dots, s_{q+1} - 1, s_{q+2}, \dots, s_{p+1}) \in S_p$. When $2 \leq j \leq q+1$, we have $\sum_{i=2}^{j}(s_i - 1) \leq (j-1)(p-q-1) < \frac{(2p-j)(j-1)}{2}$. When $q+2 \leq j \leq p+1$, $\sum_{i=2}^{q+1}(s_i - 1) + \sum_{i=q+2}^{j} s_i = \sum_{i=1}^{j} s_i - p \leq \frac{(2p-j+1)j}{2} - p = \frac{(2p-j)(j-1)}{2}$. Therefore, $(s_2 - 1, s_3 - 1, \dots, s_{q+1} - 1, s_{q+2}, \dots, s_{p+1}) \in S_p$, so there exists a sequence for it. By adding $(1, p+1), (2, p+1), \dots, (q, p+1), (p+1, q+1), (p+1, q+2), \dots, (p+1, p)$ to this sequence, we get a sequence satisfying (1), (2) for $(s_2, \dots, s_{p+1}, s_1)$, implying that $(s_1, s_2, \dots, s_{p+1})$ belongs to S_{p+1}.

Case 2. $s_{q+1} = s_{q+2}$. Obviously $q \neq 0$, since otherwise $\sum_{i=1}^{2} s_i = 2p > 2p - 1 = \frac{2(2(p+1)-1-2)}{2}$, a contradiction to the condition. Let b be the maximum subscript such that $s_{q+1} = s_b$. We distinguish two subcases to clarify.

Subcase 1. $s_2 > s_{q+1}$. Let a be the maximum subscript such that $s_a > s_{q+1}$. Again, we prove that $s' = (s'_1, s'_2, \dots, s'_p) = (s_2 - 1, s_3 - 1, \dots, s_a - 1, s_{a+1}, \dots, s_{b-q+a-1}, s_{b-q+a} - 1, \dots, s_b - 1, s_{b+1}, \dots, s_{p+1}) \in S_p$ (if $b = p+1$, then "$s_{b+1}, s_{b+2}, \dots$" does not exist). As in the discussion in **Case 1**, for $1 \leq j \leq a-1$ or $b-1 \leq j \leq p$, $\sum_{i=1}^{j} s'_i \leq \frac{(2p-1-j)j}{2}$. Thus if $s' \notin S_p$, the first j such that $\sum_{i=1}^{j-1} s'_i \leq \frac{(2p-j)(j-1)}{2}$ and $\sum_{i=1}^{j} s'_i > \frac{(2p-j-1)j}{2}$ must appear between a and $b-2$. Then we also deduce that $s'_j > p - j$, so $s'_i \geq p - j$ $(j+1 \leq i \leq b-1)$. However, this leads to $\frac{(2(p+1)-1-b)b}{2} \geq \sum_{i=1}^{b} s_i = \sum_{i=1}^{b-1} s'_i + p = \sum_{i=1}^{j} s'_i + \sum_{i=j+1}^{b-1} s'_i + p > \frac{(2p-j-1)j}{2} + (b-1-j)(p-j) + p > \frac{(2(p+1)-1-b)b}{2}$, a contradiction. Thus $s' \in S_p$. By an analysis similar to the one in **Case 1**, we can check that $(s_1, s_2, \dots, s_{p+1}) \in S_{p+1}$.

Subcase 2. $s_2 = s_{q+1}$. We prove that $s' = (s'_1, s'_2, \dots, s'_p) = (s_2, s_3, \dots, s_{b-q}, s_{b-q+1} - 1, \dots, s_{b-1} - 1, s_b - 1, s_{b+1}, \dots, s_{p+1}) \in S_p$ (if $b = p+1$, then "$s_{b+1}, s_{b+2}, \dots$" does not exist). Again, for $b-1 \leq j \leq p$, $\sum_{i=1}^{j} s'_i \leq \frac{(2p-1-j)j}{2}$. Thus if $s' \notin S_p$, the first j such that $\sum_{i=1}^{j-1} s'_i \leq \frac{(2p-j)(j-1)}{2}$ and $\sum_{i=1}^{j} s'_i > \frac{(2p-j-1)j}{2}$ must appear between 1 and $b-2$. By an analysis similar to the one in **Subcase 1**, we get a contradiction. Thus $s' \in S_p$, we can check that $(s_1, s_2, \dots, s_{p+1}) \in S_{p+1}$. The proof is complete. $\square$

Theorem 2.4. *Let T be a tree with diameter 4, and denote by u its unique vertex with eccentricity two. The neighbors of u are pendent vertices $w_1, w_2, \dots, w_\ell$ and $v_1, v_2, \dots, v_r$ with degrees $p_1 \geq p_2 \geq \cdots \geq p_r \geq 2$. Then $cfc(T) = \max\{r + b, d(u)\}$ where $b = \max\{\lceil \max\{\sum_{i=1}^{j} \frac{c_i}{j} : 1 \leq j \leq r\}\rceil, 0\}$ and $c_i = p_i - r + i - 1$ $(1 \leq i \leq r)$.*

Proof. Recall that we give color i to the edge uv_i $(1 \leq i \leq r)$ and color $r + j$ to the edge uw_j $(1 \leq j \leq \ell)$. We call the colors from $[r]$ the old colors. In any conflict-free connection coloring of T, we denote by h_i $(1 \leq i \leq r)$ the number of old colors used on the edges incident with v_i except uv_i. Obviously $(h_1, h_2, \dots, h_r) \in S_r$. In order to add new colors as few as possible, we are actually seeking for the number $a = \min\{\max\{p_i - 1 - h_i : 1 \leq i \leq r\} : (h_1, h_2, \dots, h_r) \in S_r\}$.

Let $c_i = p_i - r + i - 1$ $(1 \leq i \leq r)$, $b = \max\{\lceil \max\{\sum_{i=1}^{j} \frac{c_i}{j} : 1 \leq j \leq r\}\rceil, 0\}$. Suppose that $\max\{\sum_{i=1}^{j} \frac{c_i}{j} : 1 \leq j \leq r\}$ is obtained when $j = t$. Assume $a < b$. Then $a < \sum_{i=1}^{t} \frac{c_i}{t}$. Thus there exists an r-tuple $(h_1, h_2, \dots, h_r) \in S_r$ such that $h_i \geq p_i - 1 - a > p_i - 1 - \sum_{i=1}^{t} \frac{c_i}{t}$ for every i, $1 \leq i \leq r$. However, this implies that $\sum_{i=1}^{t} h_i > \sum_{i=1}^{t}(p_i - 1) - \sum_{i=1}^{t} c_i = \sum_{i=1}^{t}(r - i) = \frac{(2r-1-t)t}{2}$, a contradiction to $(h_1, h_2, \dots, h_r) \in S_r$ by Lemma 2.3. Thus $a \geq b$. Next, we only need to construct $(h_1, h_2, \dots, h_r) \in S_r$ with $b = \max\{p_i - 1 - h_i : 1 \leq i \leq r\}$. Let $h_i = \max\{p_i - 1 - b, 0\}$; we now verify that $(h_1, h_2, \dots, h_r)$ satisfies our demand. If $b = 0$, then for every j, $1 \leq j \leq r$, $\sum_{i=1}^{j} c_i \leq 0$, which means $\sum_{i=1}^{j}(p_i - 1) \leq \sum_{i=1}^{j}(r - i)$. So $\sum_{i=1}^{j} h_i = \sum_{i=1}^{j}(p_i - 1) \leq \sum_{i=1}^{j}(r - i) = \frac{(2r-1-j)j}{2}$ for every j, $1 \leq j \leq r$. Thus, $(h_1, h_2, \dots, h_r) \in S_r$ by Lemma 2.3. Besides, $b = 0 = \max\{p_i - 1 - h_i : 1 \leq i \leq r\}$. As a result, $a = b$. If $b = \lceil \max\{\sum_{i=1}^{j} \frac{c_i}{j} : 1 \leq j \leq r\}\rceil$, then $\sum_{i=1}^{j}(p_i - 1 - b) - \sum_{i=1}^{j}(r - i) = \sum_{i=1}^{j}(c_i - b) = j(\sum_{i=1}^{j} \frac{c_i}{j} - b) \leq 0$ for every j, $1 \leq j \leq r$. Note that $\sum_{i=1}^{j}(r - i) = \frac{(2r-1-j)j}{2}$. If $h_i = p_i - 1 - b$ for $1 \leq i \leq r$, then $(h_1, h_2, \dots, h_r) \in S_r$ by Lemma 2.3. Otherwise, let k be the minimum subscript such that $h_k = 0 \neq p_i - 1 - b$, then $\sum_{i=1}^{j} h_i \leq \frac{(2r-1-j)j}{2}$ for $1 \leq j \leq k - 1$ and $\sum_{i=1}^{j} h_i = \sum_{i=1}^{k-1} h_i \leq \frac{(2r-k)(k-1)}{2} \leq \frac{(2r-1-j)j}{2}$ for $k \leq j \leq r$. Again, $(h_1, h_2, \dots, h_r) \in S_r$ by Lemma 2.3. Easy to check that $b \geq \max\{p_i - 1 - h_i : 1 \leq i \leq r\}$, thus $a \leq b$. As a result, $a = b$ since $a \geq b$.

Combining Lemma 2.3 with the above analysis, we complete the proof. □

3 Results for graphs with diameter 2, 3 or 4

Based on the results in the above section for trees with diameter 2, 3 or 4, we are now ready to determine the values of cfc(G) and vcfc(G) for graphs G with diameter 2, 3 or 4. We first present some auxiliary lemmas that will be used in the sequel.

Lemma 3.1. *[5] Let u, v be distinct vertices and let $e = xy$ be an edge of a 2-connected graph G. Then there is a u-v path in G containing the edge e.*

Lemma 3.2. *[12] Let G be a 2-connected graph and w be a vertex of G. Then for any two distinct vertices u and v in G, there is a u-v path containing the vertex w.*

Lemma 3.3. *[12] If G is a connected graph with order at least 3, then $vcfc(G) = 2$ if and only if G is 2-connected or G has only one cut-vertex.*

Lemma 3.4. *[12] If G is a connected graph, then $vcfc(G) \leq rad(G) + 1$.*

For the conflict-free connection of graphs, the following results have already been obtained.

Lemma 3.5. *[5] If G is a noncomplete 2-connected graph, then $cfc(G) = 2$.*

Lemma 3.6. *[3] If G is a noncomplete 2-edge-connected graph, then $cfc(G) = 2$.*

Lemma 3.7. *[5] If G is a connected graph and $C(G)$ is a linear forest each component of which is of order 2, then $cfc(G) = 2$.*

Lemma 3.8. *[3] Let G be a connected graph with $h(G) \geq 2$. If there exists a unique component T of $C(G)$ such that $cfc(T) = h(G)$, and T has an optimal conflict-free connection coloring with a color used on exactly one edge of T, then $cfc(G) = h(G)$.*

We have calculated the exact value of $\mathrm{cfc}(T)$ for any tree T with $\mathrm{diam}(T) \leq 4$ in Section 2. If G is a connected graph with $\mathrm{diam}(G) \leq 4$, then any component of $C(G)$ must be a tree with diameter at most 4. Thus we can calculate $h(G)$ according to the theorems in Section 2.

For graphs with diameter 2, we have the following result.

Theorem 3.9. *If G is a connected graph with diameter 2, then $vcfc(G) = 2$ and $cfc(G) = \max\{2, h(G)\}$.*

Proof. Since G has diameter 2, G clearly has at most one cut-vertex. According to Lemma 3.3, $\mathrm{vcfc}(G) = 2$. If G is 2-edge-connected, then $\mathrm{cfc}(G) = 2$ by Lemma 3.6. Otherwise, $C(G)$ must be a star. Since every used color appears just once in an optimal conflict-free connection coloring of the star, $\mathrm{cfc}(G) = \max\{2, h(G)\}$ by Lemmas 3.7 and 3.8. □

For graphs with diameter 3, we have the following result. Recall that a vertex in a block of a graph G is called an *internal vertex* if it is not a cut-vertex of G.

Theorem 3.10. *If G is a connected graph with diameter 3, then $vcfc(G) \leq 3$ and $cfc(G) = \max\{2, h(G)\}$ except for the graph H depicted in Figure 1 which has conflict-free connection number $h(H) + 1 = 3$.*

Proof. If G contains at most one cut-vertex, then $\mathrm{vcfc}(G) = 2$ according to Lemma 3.3. We first claim that if we remove all internal vertices of the end blocks of G, then at most one block is left. Indeed, if there are two blocks B_1 and B_2 left, then we can always find two other blocks C_1, C_2 such that $V(B_i) \cap V(C_i) \neq \emptyset$ and $V(B_i) \cap V(C_{3-i}) = \emptyset$ $(i = 1, 2)$ and for every two internal vertices $u \in V(C_1), v \in V(C_2)$, every u-v path is a u-C_1-B_1-B_2-C_2-v path. However, this implies that the distance between u and v is at least 4, contradicting the fact that $\mathrm{diam}(G) = 3$. The

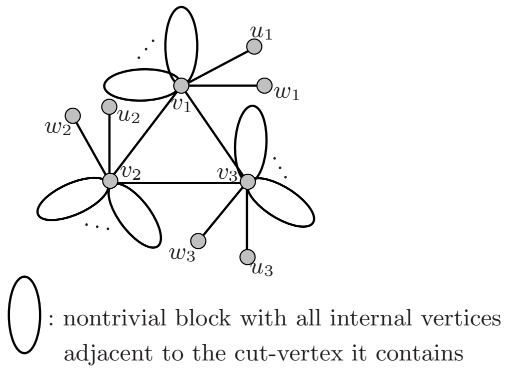

Figure 1: The graph H.

unique left block B_1 contains all cut-vertices of G. We then assign color 3 to one of them and color 2 to all remaining vertices of $V(B_1)$. Other unmentioned vertices of G share color 1. It is easy to check that G is conflict-free vertex-connected under this coloring. As a result, $\mathrm{vcfc}(G) \leq 3$.

The conflict-free connection number of G has been determined by Lemmas 3.6, 3.7 and 3.8 when $h(G) \leq 1$ or $h(G) \geq 2$ and there exists a unique component T such that $\mathrm{cfc}(T) = h(G)$. Since G has diameter 3, this T must have diameter at most 3 meaning it is a star or double star. Either of them has an optimal conflict-free connection coloring with a color used on exactly one edge (This color appears on the center edge of the double star). Thus we only need to consider the remaining cases. This implies that B_1 exists and it is nontrivial. Besides, every component of $C(G)$ is a star with its center belonging to B_1.

Let $h(G) = k$. If $k \geq 3$, since $\mathrm{cfc}(G) \geq k$, to prove $\mathrm{cfc}(G) = k$, we only need to provide a conflict-free connection k-coloring of G. To each component of $C(G)$, give a conflict-free connection coloring using $\{1, 2, \ldots, k\}$. Now, for each nontrivial block, give to two of its edges colors 2 and 3 respectively and all other edges color 1. It can be verified that G is conflict-free connected in this way.

When $k = 2$, we denote by n_1 the number of vertices of B_1 and ℓ the number of components of $C(G)$. If $\ell < n_1$, then there exists a vertex $v \in V(B_1)$ not belonging to any component of $C(G)$. Note that since $\mathrm{diam}(G) = 3$, the subgraph of B_1 induced by the vertices each of which belongs to some component of $C(G)$ is complete. We only need to give a conflict-free connection 2-coloring of G: The edges of each component of $C(G)$ receive different colors from $\{1, 2\}$. Randomly choose an edge e of B_1 incident with v and an edge for each of the other nontrivial blocks, then assign to them color 2. The remaining edges are given color 1. Since every nontrivial block B of G is 2-connected, there exists a u-v path in both $B \setminus w$ and $B \setminus e$ for any $u, v, w \in V(B), w \notin \{u, v\}, e \in E(B)$. Combining with Lemmas 3.1 and 3.2, we can choose freely if the u-v path uses or avoids any designated vertex or edge. Observing this, the checking process is very easy.

For the case $\ell = n_1$, certainly B_1 is complete with vertices $v_1, v_2, \ldots, v_{n_1}$. Since

$\mathrm{diam}(G) = 3$, for any end block of G, all its internal vertices are adjacent to the cut-vertex it contains. If $n_1 \geq 4$, we construct a conflict-free connection 2-coloring of G as follows: Assign different colors to the edges of each component of $C(G)$ from $\{1, 2\}$; give color 2 to all edges of the path $v_1v_2 \dots v_{n_1}$ and color 1 to the remaining edges of B_1. Observe that each edge of B_1 with color i ($i \in \{1, 2\}$) is contained in a triangle the other two edges of which receive distinct colors. Then pick one edge for each end block and give it color 2. Other edges are given color 1. The verification is similar.

Suppose $n_1 = 3$ with at least one component of $C(G)$ being P_2. Choose one such P_2 and give its edge color 1. Without loss of generality, assume that this edge is incident with $v_1 \in V(B_1)$, then pick one edge of B_1 incident with v_1 and give it color 2. Again, other edges of B_1 share color 1. We color the edges of other components of $C(G)$ and nontrivial blocks the way as we did in the previous paragraph. Obviously, this is a conflict-free connection 2-coloring for G.

If $n_1 = 3$ and every component of $C(G)$ is P_3, we show that two colors are not enough. Note that there are always two adjacent edges of B_1 sharing the same color if only two colors are used. Without loss of generality, suppose that the edges v_3v_1, v_3v_2 both have color 1. Let v_1u_1, v_2u_2 have color 1 and v_1w_1, v_2w_2 have color 2 where these edges are all cut-edges. It is easy to check that there is no conflict-free path either between u_1 and u_2 or between w_1 and w_2 no matter what color the edge v_1v_2 is assigned, a contradiction. Thus according to Lemma 2.1, $\mathrm{cfc}(G) = h(G) + 1 = 3$. □

Finally, we study the conflict-free (vertex-)connection number of graphs with diameter 4 in the next two results.

Theorem 3.11. *If G is a connected graph with diameter 4, then $vcfc(G) \leq 3$. Besides, $cfc(G) = 2$ if $h(G) \leq 1$ and $cfc(G) = h(G)$ if $h(G) \geq 3$.*

Proof. Since G has diameter 4, then after removing all internal vertices of the end blocks, the resulting graph has at most one cut-vertex. If there is none, we can give colors to vertices as we did in the proof of Theorem 3.10. Note that although the diameter of G is not the same, the graph structure is the same. For a connected graph G, it is its graph structure not diameter that determines the way of vertex-coloring to make it conflict-free vertex-connected. For example, if G is 2-connected, we can make G conflict-free vertex-connected by giving one of its vertices color 2 and all remaining ones color 1 no matter what the diameter of G is.

Otherwise, give color 3 to this cut-vertex v_1 and color 2 to all vertices of blocks incident with v_1 except for v_1. Finally, assign color 1 to all remaining vertices. Surely, G is conflict-free vertex-connected under this coloring.

Let $h(G) = k$. If $k \leq 1$, the result follows from Lemmas 3.6 and 3.7. If $k \geq 3$, we assign to $E(G)$ k colors as we did in the third paragraph of the proof of Theorem 3.10. For every pair of distinct vertices $u, v \in V(G)$, any path between them contains the same set of cut-edges. If they belong to the same component of $C(G)$, the conflict-free path is clear. Otherwise, since $\mathrm{diam}(G) = 4$, there are at most three cut-edges on the path. Thus at least one color of 2 and 3 (say 2) appears at most once. If it does not appear, then we can choose a u-v path using the 2-colored edge of a nontrivial block and avoiding all other such edges of the nontrivial blocks it goes through. Otherwise, the desired path is one avoiding all 2-colored edges of the nontrivial blocks it passes. Thus, k colors are enough in this case. $\square$

Corollary 3.12. *If G is a connected graph with $diam(G) \leq 4$, then $vcfc(G) = 3$ if and only if G has more than one cut-vertex.*

Proof. The result is an immediate corollary of Lemma 3.3, Theorems 3.9, 3.10 and 3.11. $\square$

If $h(G) = k = 2$, according to Lemma 2.1, we have $2 \leq \text{cfc}(G) \leq 3$. The situation in this case is complicated. Suppose there are exactly ℓ components of $C(G)$ with conflict-free connection number 2. Then for each $\ell \geq 2$, we give some graphs of diameter 4 with conflict-free connection numbers 2 and 3, respectively.

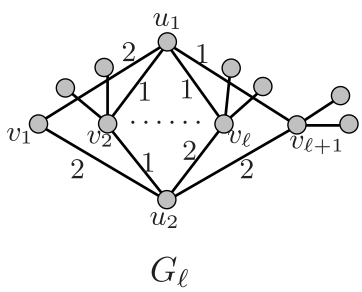

Figure 2: The graph G_ℓ with $\text{cfc}(G_\ell) = 2$ ($\ell \geq 2$).

See Figure 2 for the graph G_ℓ with $\text{cfc}(G_\ell) = 2$ ($\ell \geq 2$). Each v_i ($2 \leq i \leq \ell + 1$) of G_ℓ is the center of a P_3. We give each such P_3 colors 1 and 2 to its two edges, respectively. Besides, give color 1 to u_1v_i and 2 to u_2v_i ($3 \leq i \leq \ell+1$). The coloring for other edges are labelled in Figure 2. It is easy to check that this is a conflict-free connection 2-coloring for G_ℓ.

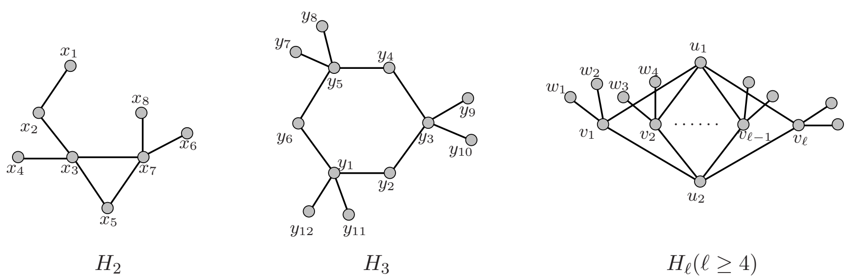

Figure 3: The graph H_ℓ with $\text{cfc}(H_\ell) = 3$ ($\ell \geq 2$).

The graph H_ℓ with $\text{cfc}(H_\ell) = 3$ ($\ell \geq 2$) is depicted in Figure 3. Suppose, to the contrary, that there exists a conflict-free connection 2-coloring c for H_ℓ. When

$\ell = 2$, without loss of generality, let $c(x_1x_2) = c(x_3x_4) = c(x_6x_7) = 1, c(x_2x_3) = c(x_7x_8) = 2$. Then if $c(x_3x_7) = 1$, to ensure a conflict-free path between x_4 and x_6, there must be $c(x_3x_5) \neq c(x_5x_7)$. However, there is no conflict-free path between x_1 and x_8, a contradiction. The case when $c(x_3x_7) = 2$ can be dealt with similarly. Thus $\mathrm{cfc}(H_2) = 3$. With the same method, we can deduce that $\mathrm{cfc}(H_3) = 3$.

For H_ℓ ($\ell \geq 4$), without loss of generality, set $c(v_1w_1) = c(v_2w_3) = 1, c(v_1w_2) = c(v_2w_4) = 2$. Suppose there exist two paths (say $u_1v_1u_2$ and $u_1v_2u_2$) with the same color between u_1 and u_2. Then there is no conflict-free path between w_1 and w_3 or w_2 and w_4, contradicting our assumption. If these two monochromatic paths receive different colors, then there is no w_1-w_4 conflict-free path, a contradiction. Assume that $c(u_1v_1) = c(u_1v_2) \neq c(u_2v_1) = c(u_2v_2)$. For the sake of the existence of conflict-free paths between w_1 and w_3, w_2 and w_4, there must be two monochromatic u_1-u_2 paths with different colors, a contradiction to our above analysis. Therefore, u_1 and u_2 are connected by at most three distinct paths, in contradiction with $\ell \geq 4$. As a result, $\mathrm{cfc}(H_\ell) = 3$ ($\ell \geq 4$).

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