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A conjecture on Gallai-Ramsey numbers of even cycles and paths

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Abstract

A *Gallai coloring* is a coloring of the edges of a complete graph without rainbow triangles, and a *Gallai k-coloring* is a Gallai coloring that uses at most k colors. Given an integer $k \geq 1$ and graphs $H_1, \dots, H_k$, the Gallai-Ramsey number $GR(H_1, \dots, H_k)$ is the least integer n such that every Gallai k-coloring of the complete graph K_n contains a monochromatic copy of H_i in color i for some $i \in \{1, 2, \dots, k\}$. When $H = H_1 = \cdots = H_k$, we simply write $GR_k(H)$. We study Gallai-Ramsey numbers of even cycles and paths. For all $n \geq 3$ and $k \geq 2$, let $G_i = P_{2i+3}$ be a path on $2i+3$ vertices for all $i \in \{0, 1, \dots, n-2\}$ and $G_{n-1} \in \{C_{2n}, P_{2n+1}\}$. Let $i_j \in \{0, 1, \dots, n-1\}$ for all $j \in \{1, 2, \dots, k\}$ with $i_1 \geq i_2 \geq \cdots \geq i_k$. The first author recently conjectured that $GR(G_{i_1}, G_{i_2}, \dots, G_{i_k}) = |G_{i_1}| + \sum_{j=2}^{k} i_j$. The truth of this conjecture implies that $GR_k(C_{2n}) = GR_k(P_{2n}) = (n-1)k + n + 1$ for all $n \geq 3$ and $k \geq 1$, and $GR_k(P_{2n+1}) = (n-1)k + n + 2$ for all $n \geq 1$ and $k \geq 1$. In this paper, we prove that the aforementioned conjecture holds for $n \in \{3, 4\}$ and all $k \geq 2$. Our proof relies only on Gallai's result and the classical Ramsey numbers $R(H_1, H_2)$, where $H_1, H_2 \in \{C_8, C_6, P_7, P_5, P_3\}$. We believe the recoloring method we develop here will be very useful for solving subsequent cases, and perhaps the conjecture.

1 Introduction

In this paper we consider graphs that are finite, simple and undirected. Given a graph G and a set $A \subseteq V(G)$, we use $|G|$ to denote the number of vertices of G, and $G[A]$ to denote the subgraph of G obtained from G by deleting all vertices in $V(G) \backslash A$. A graph H is an *induced subgraph* of G if $H = G[A]$ for some $A \subseteq V(G)$. We use P_n, C_n and K_n to denote the path, cycle and complete graph on n vertices, respectively. For any positive integer k, we write $[k]$ for the set $\{1, 2, \dots, k\}$. We use

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the convention "$A :=$" to mean that A is defined to be the right-hand side of the relation.

Given an integer $k \geq 1$ and graphs $H_1, \ldots, H_k$, the classical Ramsey number $R(H_1, \ldots, H_k)$ is the least integer n such that every k-coloring of the edges of K_n contains a monochromatic copy of H_i in color i for some $i \in [k]$. Ramsey numbers are notoriously difficult to compute in general. In this paper, we study Ramsey numbers of graphs in Gallai colorings, where a *Gallai coloring* is a coloring of the edges of a complete graph without rainbow triangles (that is, a triangle with all its edges colored differently). Gallai colorings naturally arise in several areas including: information theory [19]; the study of partially ordered sets, as in Gallai's original paper [12] (his result was restated in [17] in the terminology of graphs); and the study of perfect graphs [4]. There are now a variety of papers which consider Ramsey-type problems in Gallai colorings (see, e.g., [1, 2, 3, 5, 10, 15, 16, 18]). These works mainly focus on finding various monochromatic subgraphs in such colorings. More information on this topic can be found in [9, 11].

A *Gallai k-coloring* is a Gallai coloring that uses at most k colors. Given an integer $k \geq 1$ and graphs $H_1, \ldots, H_k$, the *Gallai-Ramsey number* $GR(H_1, \ldots, H_k)$ is the least integer n such that every Gallai k-coloring of K_n contains a monochromatic copy of H_i in color i for some $i \in [k]$. When $H = H_1 = \cdots = H_k$, we simply write $GR_k(H)$ and $R_k(H)$. Clearly, $GR_k(H) \leq R_k(H)$ for all $k \geq 1$ and $GR(H_1, H_2) = R(H_1, H_2)$. In 2010, Gyárfás, Sárközy, Sebő and Selkow [16] proved the general behavior of $GR_k(H)$.

Theorem 1.1 ([16]) *Let H be a fixed graph with no isolated vertices and let $k \geq 1$ be an integer. Then $GR_k(H)$ is exponential in k if H is not bipartite, linear in k if H is bipartite but not a star, and constant (does not depend on k) when H is a star.*

It turns out that for some graphs H (e.g., when $H = C_3$), $GR_k(H)$ behaves nicely, while the order of magnitude of $R_k(H)$ seems hopelessly difficult to determine. It is worth noting that finding exact values of $GR_k(H)$ is far from trivial, even when $|H|$ is small. We will utilize the following important structural result of Gallai [12] on Gallai colorings of complete graphs.

Theorem 1.2 ([12]) *For any Gallai coloring c of a complete graph G with $|G| \geq 2$, $V(G)$ can be partitioned into nonempty sets $V_1, \ldots, V_p$ with $p > 1$ so that at most two colors are used on the edges in $E(G) \backslash (E(G[V_1]) \cup \cdots \cup E(G[V_p]))$ and only one color is used on the edges between any fixed pair (V_i, V_j) under c.*

The partition given in Theorem 1.2 is a *Gallai partition* of the complete graph G under c. Given a Gallai partition $V_1, \ldots, V_p$ of the complete graph G under c, let $v_i \in V_i$ for all $i \in [p]$ and let $\mathcal{R} := G[\{v_1, \ldots, v_p\}]$. Then $\mathcal{R}$ is the *reduced graph* of G corresponding to the given Gallai partition under c. Clearly, $\mathcal{R}$ is isomorphic to K_p. It is worth noting that $\mathcal{R}$ does not depend on the choice of $v_1, \ldots, v_p$ because $\mathcal{R}$ can be obtained by first contracting each part V_i into a single vertex, say v_i, and then coloring every edge v_iv_j by the color used on the edges between V_i and V_j under c. By Theorem 1.2, all the edges in $\mathcal{R}$ are colored by at most two colors under c. One can see that any monochromatic copy of H in $\mathcal{R}$ under c will result in a monochromatic copy of H in G under c. It is not surprising that Gallai-Ramsey numbers $GR_k(H)$ are closely related to the classical Ramsey numbers $R_2(H)$. Recently, Fox, Grinshpun and Pach [9] posed the following conjecture on $GR_k(H)$ when H is a complete graph.

Conjecture 1.3 ([9]) *For all $t \geq 3$ and $k \geq 1$,*

$$GR_k(K_t) = \begin{cases} (R_2(K_t) - 1)^{k/2} + 1 & \text{if } k \text{ is even} \\ (t-1)(R_2(K_t) - 1)^{(k-1)/2} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

Recall that if $n < R_k(K_3)$, then there is a k-coloring c of the edges of K_n such that edges of every triangle in K_n are colored by at least two colors under c. A question of T. A. Brown (see [5]) asked: What is the largest number $f(k)$ of vertices of a complete graph can have such that it is possible to k-color its edges so that edges of every triangle are colored by exactly two colors? Chung and Graham [5] answered this question in 1983.

Theorem 1.4 ([5]) *For all $k \geq 1$, $f(k) = \begin{cases} 5^{k/2} & \text{if } k \text{ is even} \\ 2 \cdot 5^{(k-1)/2} & \text{if } k \text{ is odd.} \end{cases}$*

Clearly, $GR_k(K_3) = f(k) + 1$. By Theorem 1.4, Conjecture 1.3 holds for $t = 3$. The proof of Theorem 1.4 does not rely on Theorem 1.2. A simpler proof of this case using Theorem 1.2 can be found in [16]. The next open case, when $t = 4$, was recently settled in [21]. Gallai-Ramsey number of H, where $H \in \{C_4, P_5, C_6, P_6\}$, has also been studied, as well as general upper bounds for $GR_k(P_n)$ and $GR_k(C_n)$ that were first studied in [7, 10] and later improved in [18]. Gregory [14] proved in his thesis that $GR_k(C_8) = 3k+5$, but the proof was incomplete. We list some results in [7, 10, 18] below.

Theorem 1.5 ([7]) *For all $k \geq 1$,*

(a) $GR_k(P_n) = \lfloor \frac{n-2}{2} \rfloor k + \lceil \frac{n}{2} \rceil + 1$ *for* $n \in \{3, 4, 5, 6\}$.

(b) $GR_k(C_4) = k + 4$.

Theorem 1.6 ([10]) *For all $k \geq 1$, $GR_k(C_5) = 2^{k+1} + 1$ and $GR_k(C_6) = 2k + 4$.*

Theorem 1.7 ([18]) *For all $n \geq 3$ and $k \geq 1$,*

$$GR_k(C_{2n}) \leq (n-1)k + 3n \text{ and } GR_k(P_n) \leq \left\lfloor \frac{n-2}{2} \right\rfloor k + 3\left\lfloor \frac{n}{2} \right\rfloor.$$

More recently, Gallai-Ramsey numbers of odd cycles on at most 15 vertices have been completely settled by Bruce and Song [3] for C_7, Bosse and Song [1] for C_9 and C_{11}, and Bosse, Song and Zhang [2] for C_{13} and C_{15}. Very recently, the exact values of $GR_k(C_{2n+1})$ for $n \geq 8$ has been solved by Zhang, Song and Chen [23]. We summarize these results below.

Theorem 1.8 ([1, 2, 3]) *For $n \in \{3,4,5,6,7\}$ and all $k \geq 1$, $GR_k(C_{2n+1}) = n \cdot 2^k + 1$.*

In this paper, we study Gallai-Ramsey numbers of even cycles and paths. Note that $GR_k(H) = |H|$ for any graph H when $k = 1$. For all $n \geq 3$ and $k \geq 2$, let $G_{n-1} \in \{C_{2n}, P_{2n+1}\}$, $G_i := P_{2i+3}$ for all $i \in \{0, 1, \dots, n-2\}$. We want to determine the exact values of $GR(G_{i_1}, \dots, G_{i_k})$, where $i_j \in \{0, 1, \dots, n-1\}$ for all $j \in [k]$. By reordering colors if necessary, we assume that $i_1 \geq i_2 \geq \cdots \geq i_k$. The construction for establishing a lower bound for $GR(G_{i_1}, \dots, G_{i_k})$ for all $n \geq 3$ and $k \geq 2$ is similar to the construction given by Erdős, Faudree, Rousseau and Schelp in 1976 (see Section 2 in [6]) for classical Ramsey numbers of even cycles and paths. We recall their construction in the proof of Proposition 1.12. We list below the results on 2-colored Ramsey numbers of even cycles and paths that will be used in the proofs of Proposition 1.12 and Theorem 1.15.

Theorem 1.9 ([22]) *For all $n \geq 3$, $R_2(C_{2n}) = 3n - 1$.*

Theorem 1.10 ([8]) *For all integers n, m satisfying $2n \geq m \geq 3$, $R(P_m, C_{2n}) = 2n + \lfloor \frac{m}{2} \rfloor - 1$.*

Theorem 1.11 ([13]) *For all integers n, m satisfying $n \geq m \geq 2$, $R(P_m, P_n) = n + \lfloor \frac{m}{2} \rfloor - 1$.*

Proposition 1.12 *For all $n \geq 3$ and $k \geq 2$,*

$$GR(G_{i_1}, \dots, G_{i_k}) \geq |G_{i_1}| + \sum_{j=2}^{k} i_j,$$

where $n - 1 \geq i_1 \geq \cdots \geq i_k \geq 0$.

Proof. By Theorem 1.9, Theorem 1.10 and Theorem 1.11, the statement is true when $k = 2$. So we may assume that $k \geq 3$. To show that $GR(G_{i_1}, \dots, G_{i_k}) \geq |G_{i_1}| + \sum_{j=2}^{k} i_j$, we recall the construction given in [6]. Let G be a complete graph on $(|G_{i_1}| - 1) + \sum_{j=2}^{k} i_j$ vertices. Let $V_1, \dots, V_k$ be a partition of $V(G)$ such that $|V_1| = |G_{i_1}| - 1$ and $|V_j| = i_j$ for all $j \in \{2, 3, \dots, k\}$. Let c be a k-edge-coloring of G by first coloring all the edges of $G[V_j]$ by color j for all $j \in [k]$, and then coloring all the edges between V_{j+1} and $\bigcup_{\ell=1}^{j} V_\ell$ by color $j + 1$ for all $j \in [k-1]$. Then G contains neither a rainbow triangle nor a monochromatic copy of G_{i_j} in color j for all $j \in [k]$ under c. Hence, $GR(G_{i_1}, \dots, G_{i_k}) \geq |G| + 1 = |G_{i_1}| + \sum_{j=2}^{k} i_j$, as desired. ■

Motivated by the work developed in [14], the first author recently conjectured that the lower bound established in Proposition 1.12 is also the desired upper bound for $GR(G_{i_1}, \dots, G_{i_k})$ for all $n \geq 3$ and $k \geq 2$. We state it below (note that Conjecture 1.13 was first mentioned at the 49th Southeastern International Conference on Combinatorics, Graph Theory & Computing, Florida Atlantic University, Boca Raton, FL, March 5-9, 2018).

Conjecture 1.13 *For all $n \geq 3$ and $k \geq 2$,*

$$GR(G_{i_1}, \ldots, G_{i_k}) = |G_{i_1}| + \sum_{j=2}^{k} i_j,$$

where $n-1 \geq i_1 \geq \cdots \geq i_k \geq 0$.

Clearly, $GR_k(C_{2n}) \geq GR_k(P_{2n})$ and $GR_k(C_{2n}) \geq GR_k(M_n)$, where M_n denotes a matching of size n. It is worth noting that by letting $i_1 = \cdots = i_k = n-1$ and $G_{i_1} = C_{2n}$, the construction given in the proof of Proposition 1.12 yields that $(n-1)k + n + 1 \leq GR_k(P_{2n})$ and $(n-1)k + n + 1 \leq GR_k(M_n)$ for all $n \geq 3$ and $k \geq 1$ (the authors would like to thank Joseph Briggs, a Ph.D. student at the Carnegie-Mellon University, for pointing this out for M_n, at the 49th Southeastern International Conference on Combinatorics, Graph Theory & Computing, Florida Atlantic University, Boca Raton, FL, March 5-9, 2018). The truth of Conjecture 1.13 implies that $GR_k(C_{2n}) = GR_k(P_{2n}) = GR_k(M_n) = (n-1)k+n+1$ for all $n \geq 3$ and $k \geq 1$ and $GR_k(P_{2n+1}) = (n-1)k+n+2$ for all $n \geq 1$ and $k \geq 1$. As observed in [18], to completely solve Conjecture 1.13, one only needs to consider the case $G_{n-1} = C_{2n}$. We prove this in Proposition 1.14. The proof of Proposition 1.14 is similar to the proof of Theorem 7 given in [18]. We include a proof here for completeness.

Proposition 1.14 *For all $n \geq 3$ and $k \geq 2$, if Conjecture 1.13 holds for $G_{n-1} = C_{2n}$, then it also holds for $G_{n-1} = P_{2n+1}$.*

Proof. By the assumed truth of Conjecture 1.13 for $G_{n-1} = C_{2n}$, we may assume that $G_{i_1} = P_{2n+1}$. Then $i_1 = n-1$. We may further assume that $n-1 = i_1 = \cdots = i_t > i_{t+1} \geq \cdots \geq i_k$, where $t \in [k]$. By Proposition 1.12, $GR(G_{i_1}, \ldots, G_{i_k}) \geq (2n+1)+\sum_{j=2}^{k} i_j = 2+n+t(n-1)+\sum_{j=t+1}^{k} i_j$. We next show that $GR(G_{i_1}, \ldots, G_{i_k}) \leq 2+n+t(n-1)+\sum_{j=t+1}^{k} i_j$.

Let G be a complete graph on $2+n+t(n-1)+\sum_{j=t+1}^{k} i_j$ vertices and let $c: E(G) \to [k]$ be any Gallai coloring of G. Suppose G does not contain a monochromatic copy of G_{i_j} in color j for all $j \in [k]$. By the assumed truth of Conjecture 1.13 for $G_{n-1} = C_{2n}$, $GR(C_{2n}, \ldots, C_{2n}, G_{i_{t+1}}, \ldots, G_{i_k}) = 2n+(t-1)(n-1)+\sum_{j=t+1}^{k} i_j = 1+n+t(n-1)+\sum_{j=t+1}^{k} i_j$. Thus G must contain a monochromatic copy of $H := C_{2n}$ in some color $\ell \in [t]$ under c. We may assume that $\ell = 1$. Then for every vertex $u \in V(G)\backslash V(H)$, all the edges between u and $V(H)$ must be colored by exactly one color j for some $j \in \{2, \ldots, k\}$, because G contains neither a rainbow triangle nor a monochromatic copy of P_{2n+1} in color 1 under c. Thus, $V(G)\backslash V(H)$ can be partitioned into $V_2, V_3, \ldots, V_k$ such that all the edges between V_j and $V(H)$ are colored by color j for all $j \in \{2, \ldots, k\}$. It follows that for all $j \in \{2, \ldots, k\}$, $|V_j| \leq i_j$, because G does not contain a monochromatic copy of G_{i_j} in color j. But then $|G| = |H| + \sum_{j=2}^{k} |V_j| \leq 2n + \sum_{j=2}^{k} i_j = 1+n+t(n-1)+\sum_{j=t+1}^{k} i_j$, contrary to $|G| = 2+n+t(n-1)+\sum_{j=t+1}^{k} i_j$. ■

In this paper, we prove that Conjecture 1.13 is true for $n \in \{3, 4\}$ and all $k \geq 1$.

Theorem 1.15 *For $n \in \{3,4\}$ and all $k \geq 2$, let $G_i = P_{2i+3}$ for all $i \in \{0, 1, \ldots, n-2\}$, $G_{n-1} = C_{2n}$, and $i_j \in \{0, 1, \ldots, n-1\}$ for all $j \in [k]$ with $i_1 \geq i_2 \geq \cdots \geq i_k$. Then*

$$GR(G_{i_1}, \ldots, G_{i_k}) = |G_{i_1}| + \sum_{j=2}^{k} i_j.$$

Theorem 1.15 strengthens the results listed in Theorem 1.5, Theorem 1.6 and $GR_k(C_8) = 3k + 5$ given in [14]. Our proof relies only on Theorem 1.2 and Ramsey numbers $R(H_1, H_2)$, where $H_1, H_2 \in \{C_8, C_6, P_7, P_5, P_3\}$. Theorem 1.15, together with Proposition 1.14, implies that $GR_k(C_{2n}) = GR_k(P_{2n}) = GR_k(M_n) = (n-1)k + n + 1$ for $n \in \{3,4\}$ and all $k \geq 1$, and $GR_k(P_{2n+1}) = (n-1)k + n + 2$ for $n \in \{1,2,3,4\}$ and all $k \geq 1$. Hence, Theorem 1.15 yields a new and simpler proof of the known results on Gallai-Ramsey numbers of C_8, C_6 and P_n with $n \leq 7$. As mentioned earlier, the proof of $GR_k(C_8) = 3k + 5$ given in [14] was incomplete. We prove Theorem 1.15 in Section 2. In our completely new strategy, we developed an extremely useful recoloring method (in the proof of Claim 6 in Section 2) which we believe will assist in solving other cases, and possibly the conjecture. This method, together with new ideas, has been applied in [20] to prove that Conjecture 1.13 is true for $n \in \{5,6\}$ and all $k \geq 2$. Note that the method we developed here for even cycles and paths is very different from the method for odd cycles developed in [1, 2, 3].

2 Proof of Theorem 1.15

We are ready to prove Theorem 1.15. Let $n \in \{3,4\}$ and $k \geq 2$. By Proposition 1.12, it suffices to show that $GR(G_{i_1}, \ldots, G_{i_k}) \leq |G_{i_1}| + \sum_{j=2}^{k} i_j$.

By Theorem 1.9, Theorem 1.10 and Theorem 1.11, $GR(G_{i_1}, G_{i_2}) = R(G_{i_1}, G_{i_2}) = |G_{i_1}| + i_2$. We may assume that $k \geq 3$. Let $N := |G_{i_1}| + \sum_{j=2}^{k} i_j$. Since $GR_k(P_3) = 3$, we may assume that $i_1 \geq 1$ and so $N \geq 2i_1 + 3 \geq 5$. Let G be a complete graph on N vertices and let $c : E(G) \to [k]$ be any Gallai coloring of G such that all the edges of G are colored by at least three colors under c. We next show that G contains a monochromatic copy of G_{i_j} in color j for some $j \in [k]$. Suppose G contains no monochromatic copy of G_{i_j} in color j for any $j \in [k]$ under c. Such a Gallai k-coloring c is called a *bad coloring*. Among all complete graphs on N vertices with a bad coloring, we choose G with N minimum.

Consider a Gallai partition of G with parts $A_1, \ldots, A_p$, where $p \geq 2$. We may assume that $|A_1| \geq \cdots \geq |A_p| \geq 1$. Let $\mathcal{R}$ be the reduced graph of G with vertices $a_1, \ldots, a_p$, where $a_i \in A_i$ for all $i \in [p]$. By Theorem 1.2, we may assume that every edge of $\mathcal{R}$ is colored either red or blue. Since all the edges of G are colored by at least three colors under c, we see that $\mathcal{R} \neq G$ and so $|A_1| \geq 2$. By abusing the notation, we use i_b to denote i_j when the color j is blue. Similarly, we use i_r to denote i_j when the color j is red. Let

$$A_r := \{a_j \in \{a_2, \ldots, a_p\} \mid a_j a_1 \text{ is colored red in } \mathcal{R}\} \text{ and}$$
$$A_b := \{a_i \in \{a_2, \ldots, a_p\} \mid a_i a_1 \text{ is colored blue in } \mathcal{R}\}.$$

Let $R := \bigcup_{a_j \in A_r} A_j$ and $B := \bigcup_{a_i \in A_b} A_i$. Then $|A_1| + |R| + |B| = |G| = N$ and $\max\{|B|, |R|\} \neq 0$ because $p \geq 2$. Thus G contains a blue P_3 between B and A_1 or a red P_3 between R and A_1, and so $\max\{i_b, i_r\} \geq 1$. We next prove several claims.

Claim 1. Let $r \in [k]$ and let $s_1, \ldots, s_r$ be nonnegative integers with $s_1 + \cdots + s_r \geq 1$. If $i_{j_1} \geq s_1, \ldots, i_{j_r} \geq s_r$ for colors $j_1, j_2, \ldots, j_r \in [k]$, then for any $S \subseteq V(G)$ with $|S| \geq N - (s_1 + \cdots + s_r)$, $G[S]$ must contain a monochromatic copy of $G_{i^*_{j_q}}$ in color j_q for some $j_q \in \{j_1, \ldots, j_r\}$, where $i^*_{j_q} = i_{j_q} - s_q$.

Proof. Let $i^*_{j_1} := i_{j_1} - s_1, \ldots, i^*_{j_r} := i_{j_r} - s_r$, and $i^*_j := i_j$ for all $j \in [k] \backslash \{j_1, \ldots, j_r\}$. Let $i^*_\ell := \max\{i^*_j : j \in [k]\}$. Then $i^*_\ell \leq i_1$. Let $N^* := |G_{i^*_\ell}| + [(\sum_{j=1}^k i^*_j) - i^*_\ell]$. Then $N^* \geq 3$ and $N^* \leq N - (s_1 + \cdots + s_r) < N$ because $s_1 + \cdots + s_r \geq 1$. Since $|S| \geq N - (s_1 + \cdots + s_r) \geq N^*$ and $G[S]$ does not have a monochromatic copy of G_{i_j} in color j for all $j \in [k] \backslash \{j_1, \ldots, j_r\}$ under c, by minimality of N, $G[S]$ must contain a monochromatic copy of $G_{i^*_{j_q}}$ in color j_q for some $j_q \in \{j_1, \ldots, j_r\}$. ■

Claim 2. $|A_1| \leq n - 1$ and so G does not contain a monochromatic copy of a graph on $|A_1| + 1 \leq n$ vertices in any color $m \in [k]$ that is neither red nor blue.

Proof. Suppose $|A_1| \geq n$. We first claim that $i_b \geq |B|$ and $i_r \geq |R|$. Suppose $i_b \leq |B| - 1$ or $i_r \leq |R| - 1$. Then we obtain a blue G_{i_b} using the edges between B and A_1 or a red G_{i_r} using the edges between R and A_1, a contradiction. Thus $i_b \geq |B|$ and $i_r \geq |R|$, as claimed. Let $i^*_b := i_b - |B|$ and $i^*_r := i_r - |R|$. Since $|A_1| = N - |B| - |R|$, by Claim 1 applied to $i_b \geq |B|$, $i_r \geq |R|$ and A_1, $G[A_1]$ must have a blue $G_{i^*_b}$ or a red $G_{i^*_r}$, say the latter. Then $i_r > i^*_r$. Thus $|R| > 0$ and $G_{i^*_r}$ is a red path on $2i^*_r + 3$ vertices. Note that

$$\begin{aligned}
|A_1| &= |G_{i_1}| + \sum_{j=2}^{k} i_j - |B| - |R| \\
&\geq \begin{cases} |G_{i_r}| + i_b - |B| - |R| & \text{if } i_r \geq i_b \\ |G_{i_b}| + i_r - |B| - |R| & \text{if } i_r < i_b, \end{cases} \\
&\geq \begin{cases} |G_{i_r}| + i^*_b - |R| & \text{if } i_r \geq i_b \\ 2i_b + 2 + i_r - |B| - |R| \geq i^*_b + (2i_r + 3) - |R| & \text{if } i_r < i_b, \end{cases} \\
&\geq |G_{i_r}| - |R|.
\end{aligned}$$

Then

$$\begin{aligned}
|A_1| - |G_{i^*_r}| &\geq |G_{i_r}| - |G_{i^*_r}| - |R| \\
&= \begin{cases} (3 + 2i_r) - (3 + 2i^*_r) - |R| = |R| & \text{if } i_r \leq n - 2 \\ (2 + 2i_r) - (3 + 2i^*_r) - |R| = |R| - 1 & \text{if } i_r = n - 1. \end{cases}
\end{aligned}$$

But then $G[A_1 \cup R]$ contains a red G_{i_r} using the edges of the $G_{i_r^*}$ and the edges between $A_1 \backslash V(G_{i_r^*})$ and R, a contradiction. This proves that $|A_1| \le n-1$. Next, let $m \in [k]$ be any color that is neither red nor blue. Suppose G contains a monochromatic copy of a graph, say J, on $|A_1|+1$ vertices in color m. Then $V(J) \subseteq A_\ell$ for some $\ell \in [p]$. But then $|A_\ell| \ge |A_1|+1$, contrary to $|A_1| \ge |A_\ell|$. ■

For two disjoint sets $U, W \subseteq V(G)$, we say U is *blue-complete* (resp. *red-complete*) to W if all the edges between U and W are colored blue (resp. red) under c. For convenience, we say u is *blue-complete* (resp. *red-complete*) to W when $U = \{u\}$.

Claim 3. $\min\{|B|, |R|\} \ge 1$, $p \ge 3$ and B is neither red- nor blue-complete to R under c.

Proof. Suppose $B = \emptyset$ or $R = \emptyset$. By symmetry, we may assume that $R = \emptyset$. Then $B \neq \emptyset$ and so $i_b \ge 1$. By Claim 2, $|A_1| \le n-1 \le 3$ because $n \in \{3,4\}$. Then $|A_1| \le i_b + 2$. If $i_b \le |A_1|-1$, then $i_b \le n-2$ by Claim 2. Thus G_{i_b} is a blue path on $2i_b+3$ vertices and so

$$|B| = N - |A_1| \ge |G_{i_b}| - |A_1| = \begin{cases} i_b + 1 & \text{if } |A_1| = i_b + 2 \\ i_b + 2 & \text{if } |A_1| = i_b + 1. \end{cases}$$

But then we obtain a blue G_{i_b} using the edges between B and A_1. Thus $i_b \ge |A_1|$. Let $i_b^* := i_b - |A_1|$. By Claim 1 applied to $i_b \ge |A_1|$ and B, $G[B]$ must have a blue $G_{i_b^*}$. Since

$$\begin{aligned} |B| - |G_{i_b^*}| &\ge |G_{i_b}| - |G_{i_b^*}| - |A_1| \\ &= \begin{cases} (3+2i_b) - (3+2i_b^*) - |A_1| = |A_1| & \text{if } i_b \le n-2 \\ (2+2i_b) - (3+2i_b^*) - |A_1| = |A_1| - 1 & \text{if } i_b = n-1, \end{cases} \end{aligned}$$

we see that G contains a blue G_{i_b} using the edges of the $G_{i_b^*}$ and the edges between $B \backslash V(G_{i_b^*})$ and A_1, a contradiction. Hence $R \neq \emptyset$ and so $p \ge 3$ for any Gallai partition of G. It follows that B is neither red- nor blue-complete to R, otherwise $\{B \cup A_1, R\}$ or $\{B, R \cup A_1\}$ yields a Gallai partition of G with only two parts. ■

Claim 4. Let $m \in [k]$ be a color that is neither red nor blue. Then $i_m \le 1$. In particular, if $i_m = 1$, then $n = 4$ and G contains a monochromatic copy of P_3 in color m under c.

Proof. By Claim 2, G contains no monochromatic copy of P_n in color m under c. Suppose $i_m \ge 1$. Let $i_m^* := i_m - 1$. By Claim 1 applied to $i_m \ge 1$ and $V(G)$, G must have a monochromatic copy of $G_{i_m^*}$ in color m under c. Since $n \in \{3,4\}$ and G contains no monochromatic copy of P_n in color m, we see that $n = 4$ and $i_m^* = 0$. Thus $i_m = 1$ and G contains a monochromatic copy of P_3 in color m under c. ■

By Claim 3, $B \neq \emptyset$ and $R \neq \emptyset$. Since $|A_1| \ge 2$, we see that G has a blue P_3 using edges between B and A_1, and a red P_3 using edges between R and A_1. Thus

$i_b \geq 1$ and $i_r \geq 1$. Then $|G_{i_1}| \geq 5$ and so $N = |G_{i_1}| + \sum_{j=2}^{k} i_j \geq 6$. By Claim 2, $|A_1| \leq n-1$. If $|B| = |R| = 1$, then $N = |A_1|+|B|+|R| \leq n+1 \leq 5$, a contradiction. Thus $|B| \geq 2$ or $|R| \geq 2$. Since B is neither red- nor blue-complete to R, we see that G contains either a blue P_5 or a red P_5. Thus $i_1 \geq \max\{i_b, i_r\} \geq 2 \geq n-2$ because $n \in \{3, 4\}$. By Claim 4, we may assume that $\{i_b, i_r\} = \{i_1, i_2\}$. Then

$$|G_{i_1}| = \begin{cases} 2i_1 + 2 = 1 + n + i_1 & \text{if } i_1 = n-1 \\ 2i_1 + 3 = 1 + n + i_1 & \text{if } i_1 = n-2. \end{cases}$$

Therefore $N = |G_{i_1}| + \sum_{j=2}^{k} i_j = 1 + n + \sum_{j=1}^{k} i_j \geq 1 + n + i_b + i_r$.

Claim 5. $|B| \leq n-1$ or $|R| \leq n-1$.

Proof. Suppose $|B| \geq n$ and $|R| \geq n$. Let $H = (B, R)$ be the complete bipartite graph obtained from $G[B \cup R]$ by deleting all the edges with both ends in B or both ends in R. Then H has no blue P_{2n-3} with both ends in B, else, we obtain a blue C_{2n} because $|A_1| \geq 2$. Similarly, H has no red P_{2n-3} with both ends in R. For every vertex $v \in B \cup R$, let $d_b(v) := |\{u : uv \text{ is colored blue in } H\}|$ and $d_r(v) := |\{u : uv \text{ is colored red in } H\}|$. Let $x_1, \ldots, x_n \in B$, $y_1, \ldots, y_n \in R$ and $a_1, a_1^* \in A_1$ be all distinct. We next claim that $d_r(v) \leq n-2$ for all $v \in B$. Suppose, say, $d_r(x_1) \geq n-1$. Then $n = 4$ because H has no red P_{2n-3} with both ends in R. We may assume that x_1 is red-complete to $\{y_1, y_2, y_3\}$. Since H has no red P_5 with both ends in R, we see that for all $i \in \{2, 3, 4\}$ and every $W \subseteq \{y_1, y_2, y_3\}$ with $|W| = 2$, no x_i is red-complete to W. We may further assume that x_2y_1, x_2y_2, x_3y_1 are colored blue. Then x_4y_2 must be colored red, else, H has a blue P_5 with vertices x_3, y_1, x_2, y_2, x_4 in order. Thus x_4y_1, x_4y_3 are colored blue. But then H has a blue P_5 with vertices x_2, y_2, x_3, y_1, x_4 in order (when x_3y_2 is colored blue) or vertices x_2, y_1, x_3, y_3, x_4 in order (when x_3y_3 is colored blue), a contradiction. Thus $d_r(v) \leq n-2$ for all $v \in B$. Similarly, $d_b(u) \leq n-2$ for all $u \in R$. Then $|B||R| = |E(H)| = \sum_{v \in B} d_r(v) + \sum_{u \in R} d_b(u) \leq (n-2)|B| + (n-2)|R|$. Using inequality of arithmetic and geometric means, we obtain that $n = 4$, $|B| = |R| = 4$ and $d_r(v) = d_b(v) = 2$ for each $v \in B \cup R$. Thus the set of all the blue edges in H induces a 2-regular spanning subgraph of H. Since H has no blue C_8, we see that H must contain two vertex-disjoint copies of blue C_4. We may assume that y_1 is blue-complete to $\{x_1, x_2\}$ and y_2 is blue-complete to $\{x_3, x_4\}$. But then G contains a blue C_8 with vertices $a_1, x_1, y_1, x_2, a_1^*, x_3, y_2, x_4$ in order, a contradiction. ■

Claim 6. $|A_1| = 3$ and $n = 4$.

Proof. By Claim 2, $|A_1| \leq n-1 \leq 3$ because $n \in \{3, 4\}$. Note that $|A_1| = 3$ only when $n = 4$. Suppose $|A_1| = 2$. By Claim 2, G has no monochromatic copy of P_3 in color j for any $j \in \{3, \ldots, k\}$ under c. By Claim 4, $i_3 = \cdots = i_k = 0$ and so $N = 1 + n + \sum_{j=1}^{k} i_j = 1 + n + i_b + i_r$. We may assume that $A_1, \ldots, A_t$ are all the parts of order two in the Gallai partition $A_1, \ldots, A_p$ of G, where $t \in [p]$. Let $A_i := \{a_i, b_i\}$ for all $i \in [t]$. By reordering if necessary, each of $A_1, \ldots, A_t$ can be chosen as the largest part in the Gallai partition $A_1, \ldots, A_p$ of G. For all $i \in [t]$, let

$$A_b^i := \{a_j \in V(\mathcal{R}) \mid a_j a_i \text{ is colored blue in } \mathcal{R}\} \text{ and}$$
$$A_r^i := \{a_j \in V(\mathcal{R}) \mid a_j a_i \text{ is colored red in } \mathcal{R}\}.$$

Let $B^i := \bigcup_{a_j \in A_b^i} A_j$ and $R^i := \bigcup_{a_j \in A_r^i} A_j$. Then $|B^i| + |R^i| = N - |A_1| = n + i_b + i_r - 1 \geq n + 2$, because $\max\{i_b, i_r\} \geq 2$ and $\min\{i_b, i_r\} \geq 1$. Since each of $A_1, \ldots, A_t$ can be chosen as the largest part in the Gallai partition $A_1, \ldots, A_p$ of G, by Claim 5, either $|B^i| \leq n-1$ or $|R^i| \leq n-1$ for all $i \in [t]$. We claim that $|B^i| \neq |R^i|$ for all $i \in [t]$. Suppose $|B^i| = |R^i|$ for some $i \in [t]$. By Claim 5, $n + 2 \leq |B^i| + |R^i| \leq 2(n-1) \leq 6$. It follows that $|B^i| = |R^i| = 3$ and $n = 4$. Thus G has a blue P_5 between B^i and A_i and a red P_5 between R^i and A_i. It follows that $\min\{i_b, i_r\} \geq 2$. But then $|B^i| + |R^i| = n + i_b + i_r - 1 \geq 7$, a contradiction. This proves that $|B^i| \neq |R^i|$ for all $i \in [t]$. Let

$$E_B := \{a_i b_i \mid i \in [t] \text{ and } |R^i| < |B^i|\} \text{ and } E_R := \{a_i b_i \mid i \in [t] \text{ and } |R^i| > |B^i|\}.$$

We next apply the recoloring method. Let c^* be an edge-coloring of G obtained from c by recoloring all the edges in E_B blue and all the edges in E_R red. Then every edge of G is colored either red or blue under c^*. Since $|G| = 1 + n + i_b + i_r \geq R(G_{i_b}, G_{i_r})$ by Theorem 1.9, Theorem 1.10 and Theorem 1.11, we see that G must contain a blue G_{i_b} or a red G_{i_r} under c^*. By symmetry, we may assume that G has a blue $H := G_{i_b}$ under c^*. Then H contains no edges of E_R but must contain at least one edge of E_B, else, we obtain a blue G_{i_b} in G under c. We choose H so that $|E(H) \cap E_B|$ is minimal. We may further assume that $a_1 b_1 \in E(H)$. By the choice of c^*, $|R^1| \leq n-1$ and $|R^1| < |B^1|$. Then $|B^1| \geq 2$ and so G has a blue P_5 under c because B^1 is not red-complete to R^1. Thus $i_b \geq 2$. Let $W := V(G) \backslash V(H)$.

We next claim that $i_b = n - 1$. Suppose $2 \leq i_b \leq n-2$. Then $n = 4$, $i_b = 2$, $H = P_7$ and $|G| = 1 + n + i_b + i_r = 7 + i_r$. Thus $|W| = i_r$. Let $x_1, \ldots, x_7$ be the vertices of H in order. By symmetry, we may assume that $x_\ell x_{\ell+1} = a_1 b_1$ for some $\ell \in [3]$. Then $W \cup \{x_7\}$ must be red-complete to $\{a_1, b_1\}$ under c, else, say a vertex $u \in W \cup \{x_7\}$, is blue-complete to $\{a_1, b_1\}$ under c, then we obtain a blue $H' := P_7$ under c^* with vertices $x_1, \ldots, x_\ell, u, x_{\ell+1}, \ldots, x_6$ in order such that $|E(H') \cap E_B| < |E(H) \cap E_B|$, contrary to the choice of H. Thus $W \cup \{x_7\} \subseteq R^1$ and so $|R^1| \geq |W \cup \{x_7\}| = i_r + 1 \geq 2$. Note that G contains a red P_5 under c because $|R^1| \geq 2$ and R^1 is not blue-complete to B^1. Thus $i_r \geq 2$. Then $3 \leq i_r + 1 \leq |R^1| \leq 3$, which implies that $i_r = 2$ and $R^1 = W \cup \{x_7\}$. Thus $\{a_1, b_1\}$ is blue-complete to $V(H) \backslash \{x_\ell, x_{\ell+1}, x_7\}$. But then we obtain a blue $H' := P_7$ under c^* with vertices $x_1, \ldots, x_\ell, x_{\ell+2}, x_{\ell+1}, x_{\ell+3}, \ldots, x_7$ in order such that $|E(H') \cap E_B| < |E(H) \cap E_B|$, a contradiction. This proves that $i_b = n - 1$.

Since $i_b = n - 1$, we see that $H = C_{2n}$. Then $|G| = 1 + n + i_b + i_r = 2n + i_r$ and so $|W| = i_r$. Let $a_1, x_1, \ldots, x_{2n-2}, b_1$ be the vertices of H in order and let $W = V(G) \backslash V(H) := \{w_1, \ldots, w_{i_r}\}$. Then $x_1 b_1$ and $a_1 x_{2n-2}$ are colored blue under c because $\{a_1, b_1\} = A_1$. Suppose $\{x_j, x_{j+1}\}$ is blue-complete to $\{a_1, b_1\}$ under

c for some $j \in [2n-3]$. Then G has a blue $H' := C_{2n}$ under c^* with vertices $a_1, x_1, \ldots, x_j, b_1, x_{2n-2}, \ldots, x_{j+1}$ in order such that $|E(H') \cap E_B| < |E(H) \cap E_B|$, contrary to the choice of H. Thus, for all $j \in [2n-3]$, $\{x_j, x_{j+1}\}$ is not blue-complete to $\{a_1, b_1\}$. Since $\{x_1, x_{2n-2}\}$ is blue-complete to $\{a_1, b_1\}$ under c, we see that $x_2, x_{2n-3} \in R^1$ and then $|R^1 \cap \{x_2, \ldots, x_{2n-3}\}| = |R^1| = n-1$. Thus $R^1 = \{x_2, x_3\}$ when $n = 3$. By symmetry, we may assume that $R^1 = \{x_2, x_3, x_5\}$ when $n = 4$. Then $W \subseteq B^1$. Thus R^1 is red-complete to $\{a_1, b_1\}$ and W is blue-complete to $\{a_1, b_1\}$ under c. It follows that for any $w_j \in W$ and $x_m \in R^1$, $\{x_m, w_j\} \neq A_i$ for all $i \in [t]$. Then x_2 must be red-complete to W under c, else, say x_2w_1 is colored blue under c, then we obtain a blue $H' := C_{2n}$ under c^* with vertices $a_1, x_1, x_2, w_1, b_1, x_4$ (when $n = 3$) and vertices $a_1, x_1, x_2, w_1, b_1, x_4, x_5, x_6$ (when $n = 4$) in order such that $|E(H') \cap E_B| < |E(H) \cap E_B|$, a contradiction. Similarly, x_3 is red-complete to W under c, else, say x_3w_1 is colored blue under c, then we obtain a blue $H' := C_{2n}$ under c^* with vertices $b_1, x_4, x_3, w_1, a_1, x_1$ (when $n = 3$) and vertices $b_1, x_6, x_5, x_4, x_3, w_1, a_1, x_1$ (when $n = 4$) in order such that $|E(H') \cap E_B| < |E(H) \cap E_B|$, a contradiction. Thus $\{x_2, x_3\}$ is red-complete to W under c. Then for any $w_j \in W$, $\{x_1, w_j\} \neq A_i$ for all $i \in [t]$ since x_2x_1 is colored blue and x_2 is red-complete to W under c. If x_1w_j is colored blue under c for some $w_j \in W$, then we obtain a blue $H' := C_{2n}$ under c^* with vertices $a_1, w_j, x_1, \ldots, x_{2n-2}$ in order such that $|E(H') \cap E_B| < |E(H) \cap E_B|$, a contradiction. Thus $\{x_1, x_2, x_3\}$ is red-complete to W under c. Then $|W| = i_r \geq 2$ because G contains a red P_5 under c with vertices x_1, w_1, x_2, a_1, x_3 in order. But then we obtain a red C_{2n} under c with vertices $a_1, x_2, w_1, x_1, w_2, x_3$ in order (when $n = 3$) and $a_1, x_2, w_1, x_1, w_2, x_3, b_1, x_5$ in order (when $n = 4$), a contradiction. ■

By Claim 6, $|A_1| = 3$ and $n = 4$. Then $|B \cup R| = N - |A_1| \geq 2 + i_b + i_r \geq 5$ because $\max\{i_b, i_r\} \geq 2$ and $\min\{i_b, i_r\} \geq 1$. By symmetry, we may assume that $|B| \geq |R|$. Then $|B| \geq 3$ and so G has a blue P_7 because $|A_1| = 3$ and B is not red-complete to R. Thus $i_b = 3$. By Claim 5, $|R| \leq 3$. Then $i_r \geq |R|$, else, we obtain a red G_{i_r} because $|A_1| = 3$ and R is not blue-complete to B. Then $|B| \geq 2 + i_b + i_r - |R| \geq 5$. Thus $G[B \cup R]$ has no blue P_3 with both ends in B, else, we obtain a blue C_8 because $|A_1| = 3$ and $|B| \geq 5$. Let $i_b^* := 0$ and $i_r^* := i_r - |R| \leq 2$. By Claim 1 applied to $i_b = |A_1|$, $i_r \geq |R|$ and B, $G[B]$ must contain a red $P_{2i_r^*+3}$ with vertices, say $x_1, \ldots, x_{2i_r^*+3}$, in order. Let $R := \{y_1, \ldots, y_{|R|}\}$. Then no $y_j \in R$ is blue-complete to any $W \subseteq B$ with $|W| = 2$, in particular, when $W = \{x_1, x_{2i_r^*+3}\}$, because $G[B \cup R]$ has no blue P_3 with both ends in B. We may assume that x_1y_1 is colored red. Note that $G[R \cup A_1]$ has a red $P_{2|R|}$ with y_1 as an end. Then $G[\{x_1, \ldots, x_{2i_r^*+3}\} \cup R \cup A_1]$ has a red P_{2i_r+3}. It follows that $i_r = 3$. Let $a_1^* \in A_1 \backslash \{a_1\}$.

Suppose first that $x_{2i_r^*+3}$ is blue-complete to $R = \{y_1, \ldots, y_{|R|}\}$. Since $G[B \cup R]$ has no blue P_3 with both ends in B, we see that $\{x_{2i_r^*+3}\} = A_\ell$ for some $\ell \in [p]$, $B \backslash \{x_{2i_r^*+3}\}$ is red-complete to $\{y_1, \ldots, y_{|R|}\}$, and $x_{2i_r^*+3}$ is adjacent to at most one vertex, say $w \in B$, such that $wx_{2i_r^*+3}$ is colored blue. Thus $x_{2i_r^*+3}$ is red-complete to $B \backslash \{w, x_{2i_r^*+3}\}$. Let $w^* \in B \backslash \{x_1, x_2, x_3, w\}$. Since $B \backslash \{x_{2i_r^*+3}\}$ is red-complete to $\{y_1, \ldots, y_{|R|}\}$, we see that $\{x_1, \ldots, x_{2i_r^*+2}\}$ is red-complete to $\{y_1, \ldots, y_{|R|}\}$. If $w \notin \{x_2, \ldots, x_{2i_r^*+1}\}$, then we obtain a red C_8 with vertices $y_1, x_1, x_2, x_7, x_3, \ldots, x_6$ (when $i_r^* = 2$), vertices $a_1, y_1, x_1, x_2, x_5, x_3, x_4, y_2$ (when $i_r^* = 1$), and vertices

$a_1, y_1, x_2, x_3, w^*, y_2, a_1^*, y_3$ (when $i_r^* = 0$) in order, a contradiction. Thus $w \in \{x_2, \ldots, x_{2i_r^*+1}\}$. Then $i_r^* \geq 1$ and $x_1x_{2i_r^*+1}$ is colored red. But then we obtain a red C_8 with vertices $y_1, x_2, x_3, x_4, x_5, x_6, x_7, x_1$ (when $i_r^* = 2$) and vertices $a_1, y_1, x_2, x_3, x_4, x_5, x_1, y_2$ (when $i_r^* = 1$) in order, a contradiction. This proves that $x_{2i_r^*+3}$ is not blue-complete to R. Then $|R| \geq 2$, else, $|R| = 1$, $i_r^* = 2$ and x_7y_1 is colored red, which yields a red C_8 with vertices $y_1, x_1, \ldots, x_7$ in order, a contradiction. Thus $i_r^* \leq 1$. Next, suppose $x_{2i_r^*+3}$ is not blue-complete to $\{y_2, \ldots, y_{|R|}\}$, say $x_{2i_r^*+3}y_2$ is colored red. By assumption, x_1y_1 is red. We then obtain a red C_8 with vertices $a_1, y_1, x_1, \ldots, x_5, y_2$ (when $i_r^* = 1$) and vertices $a_1, y_1, x_1, x_2, x_3, y_2, a_1^*, y_3$ (when $i_r^* = 0$) in order, a contradiction. Thus $x_{2i_r^*+3}$ is blue-complete to $\{y_2, \ldots, y_{|R|}\}$ and so $x_{2i_r^*+3}y_1$ is colored red. By symmetry of x_1 and $x_{2i_r^*+3}$, x_1 must be blue-complete to $\{y_2, \ldots, y_{|R|}\}$. But then $G[B \cup R]$ has a blue P_3 with vertices $x_1, y_2, x_{2i_r^*+3}$ in order, a contradiction.

This completes the proof of Theorem 1.15. ■

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