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On an explicit correspondence between ***nbc***-basis, chambers and minimal complex for real supersolvable arrangements

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Abstract

In this paper we give a very natural description of the bijections between the set of cells in the minimal CW-complex homotopy equivalent to the complement of a complexified real supersolvable arrangement $\mathcal{A}$, the ***nbc***-basis of the Orlik-Solomon algebra associated to $\mathcal{A}$ and the set of chambers of $\mathcal{A}$. We use these bijections to describe a bijection between the symmetric group and the ***nbc***-basis of the braid arrangement.

1 Introduction

The theory of arrangements of hyperplanes is a subject intensively studied during the last 60 years. The main topic of this theory is the study of the complement of a set of hyperplanes in the space. It started in 1889 when Roberts gave a formula to count how many open disconnected regions there are when we cut the plane by removing a set of lines (see [12] for a detailed reference). A direct generalization of this problem, the removal of hyperplanes in higher dimensional spaces, stayed unsolved until 1975, when Zaslavsky gave a general counting formula in [22]. Those open regions are called *chambers*. In 1980 Orlik and Solomon introduced the well-known Orlik-Solomon algebra (see [11]) that is completely described by combinatorial methods, i.e. by the intersection lattice, and which computes the cohomology group with integer coefficients of the complement of a complex hyperplane arrangement.

The Orlik-Solomon algebra is a graded algebra with an additive basis ***nbc*** called the *non broken circuit* basis. It turns out that, when considering complexified real arrangements, i.e. the case in which the hyperplanes have real defining equations, the total number of elements in a non broken circuit basis equals the number of chambers of the underlining real arrangement. The correspondence between those

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two objects has been studied by many authors interested in the combinatorial aspects of the theory of arrangements of hyperplanes. For example, Barcelo and Goupil in [1] studied the case of arrangements coming from reflection groups and Gioan and Las Vergnas in [9], and Jewell and Orlik in [10], studied the general case.

More recently, Dimca and Papadima in [6], and Randell in [13] proved that the complement of a complex hyperplane arrangement is a minimal space, i.e. it has the homotopy type of a CW-complex with exactly as many k-cells as the k-th Betti number b_k or, in other words, as many k-cells as the cardinality of $\boldsymbol{nbc}_k$, i.e. of the elements of degree k in the non broken circuits basis. In 2007, Yoshinaga in [20], Salvetti and the first author in [15] gave a description of this minimal complex in the case of complexified real arrangements. Then the question arises on existence of "natural" bijections between the chambers of the real arrangement, the set of cells in the minimal CW-complex of the complexified one and the $\boldsymbol{nbc}$-basis. This question has been addressed by Delucchi in [4], where the author extended Jewell and Orlik bijection to matroids, and Yoshinaga in [21], where the author introduced a basis of the Orlik-Solomon algebra labeled by chambers.

It is worth noticing that all maps described so far in the literature are based on a similar construction which creates a correspondence between each element in the $\boldsymbol{nbc}$ and chambers contained in a cone defined by the hyperplanes of the fixed non broken circuit. This gives rise to a correspondence which assigns to each element of $\boldsymbol{nbc}$ not a single chamber, but a linear combination of chambers (see, for instance, [10] and [21, Example 3.3]). In this paper, we describe a map for supersolvable arrangements which assigns to each $\boldsymbol{nbc}$ exactly one and only one chamber. Differently from previous bijections, this construction is compatible with the supersolvable structure, meaning that the order on the hyperplanes is the one induced by the filtration of the supersolvable arrangement. This is particularly interesting in the case of the braid arrangement $\mathcal{A}(A_n)$, in which the supersolvable filtration is the natural inclusion of $\mathcal{A}(A_{n-1}) \subset \mathcal{A}(A_n)$ and hence hyperplanes in $\mathcal{A}(A_j)$ are smaller than hyperplanes in $\mathcal{A}(A_j) \setminus \mathcal{A}(A_i)$. This new bijection between the minimal complex and $\boldsymbol{nbc}$-basis (see Section 4, equation (5)) only involves elements of the intersection poset of the hyperplane arrangement endowed with an order $\lhd$ and the hyperplanes separating chambers from a previously fixed chamber C_0. In the case of supersolvable arrangements, the results in [5] allow for a purely combinatorial description of order $\lhd$. This, in turn, allows for an explicit bijection whose main example is the case of the braid arrangement described in Section 5. Here it is worth noticing that the set of hyperplanes separating the chambers from a fixed basis-chamber is in fact equivalent to the oriented-matroid data, up to reorientation (see [2]).

Moreover, if $\mathcal{A} = \{H_{ij} = \{x_i = x_j\}, 1 \leq i < j \leq n+1\}$ is the braid arrangement, then chambers are in one to one correspondence with elements of the symmetric group. Hence the map f defined in Section 4 also defines a new bijection between the symmetric group and the non broken circuit basis $\boldsymbol{nbc}$ associated to $\mathcal{A}$ (this is content of Section 5). In 1995 Barcelo and Goupil, joined with Garsia in [1] proved that if $\mathcal{A}(W)$ is the reflection arrangement associated to the Coxeter group (W, S), $\boldsymbol{nbc}(\mathcal{A}(W))$ is its non broken circuit basis and $H_r \in \mathcal{A}(W)$ is the hyperplane defined by the reflection $r \in W$, the map

$$g\colon \boldsymbol{nbc}(\mathcal{A}(W)) \longrightarrow W$$
$$(H_{r_1}, \ldots, H_{r_k}) \longmapsto w = r_1 \ldots r_k$$

is a bijection. However, this map is not compatible with Lefschetz argument. That is for any choice of a subspace V_k of dimension k in general position with respect to $\mathcal{A}(W)$, there is a chamber C corresponding to a reflection $w = r_1 \ldots r_k = g((H_{r_1}, \ldots, H_{r_k}))$ such that $C \cap V_k \neq \emptyset$. This can be already seen in $\mathcal{A}(A_2)$ (see Remark 3.1 in Section 3). In this paper, we provide a slightly different bijection such that for all $k = 0, \ldots, n$ each element in the group A_n, image of a k-uple of hyperplanes in $\boldsymbol{nbc}$, is in one to one correspondence with a chamber of the arrangement intersected by a k-dimensional subspace of a general flag of subspaces $\{V_k\}_{k=0,\ldots,n}$. Since the map g above, defined in [1], is for any reflection group it is a natural question whether the construction in this paper can be extended also to other reflection groups (even the non-supersolvable ones).

This paper is organized as follows. In Section 2, we recall the definitions of the minimal Salvetti's complex associated to complexified real arrangements and of the $\boldsymbol{nbc}$-basis for real supersolvable arrangements. In Section 3, we describe the example of $\mathcal{A}(A_3)$. In Section 4, we introduce a relation between the $\boldsymbol{nbc}$-basis and the minimal complex of a complexified real supersolvable arrangement $\mathcal{A}$ and prove that this relation is, in fact, a bijection. In Section 5, we give a description of this map in the special case of the braid arrangement providing a bijection between elements of the $\boldsymbol{nbc}$-basis and the permutations of the symmetric group.

2 Preliminaries

Let $\mathcal{A}$ be an essential affine hyperplane arrangement in $\mathbb{R}^d$, i.e. a finite set of affine real hyperplanes whose minimal nonempty intersections are points. Let us remark that, given an affine or central hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^d$ it is always possible to consider its essentialization $\mathrm{ess}(\mathcal{A})$ which is an essential arrangement having same homotopy type, and hence combinatorics, of $\mathcal{A}$ (see [17]). In the rest of the paper, we will always consider $\mathrm{ess}(\mathcal{A})$ even if not explicitly stated. Let $\mathcal{F} = \mathcal{F}(\mathcal{A})$ denote the set of closed strata of the induced stratification of $\mathbb{R}^d$. It is customary to endow $\mathcal{F}$ with a partial ordering $\prec$ given by reverse inclusion of topological closures (see[12]). The elements of $\mathcal{F}$ are called *faces* of the arrangement. The poset $\mathcal{F}$ is ranked by the *codimension* of the faces. The connected components of $\mathbb{R}^d \setminus \mathcal{A}$, corresponding to elements of $\mathcal{F}$ of maximal dimension, are called *chambers*. For any $F \in \mathcal{F}$, denote by $|F|$ the affine subspace spanned by F, called the *support* of F, and set

$$\mathcal{A}_F \ := \ \{H \in \mathcal{A} \mid F \subset H\}.$$

In [14], Salvetti constructed a regular CW-complex $\mathcal{S}(\mathcal{A})$ (denoted just by $\mathcal{S}$ if no confusion can arise) that is a deformation retract of the complement

$$\mathcal{M}(\mathcal{A}) := \ \mathbb{C}^d \setminus \bigcup_{H \in A} H_{\mathbb{C}},$$

of the complexification of $\mathcal{A}$, where $H_{\mathbb{C}} = H \otimes \mathbb{C}$.

The k-cells of $\mathcal{S}$ bijectively correspond to pairs $[C \prec F]$, where $F \in \mathcal{F}$, $\mathrm{codim}(F) = k$ and C is a chamber. A cell $[D \prec G]$ is in the boundary of $[C \prec F]$ if $G \prec F$ and the chambers D, C are contained in the same chamber of $\mathcal{A}_G$.

2.1 Salvetti-Settepanella minimal complex

In [15], Salvetti and the first author constructed a minimal complex homotopy equivalent to the complement $\mathcal{M}(\mathcal{A})$ of a complexified real arrangement $\mathcal{A}$. The main ingredients of this construction are Forman's Discrete Morse Theory and Salvetti's complex. They explicitly constructed a combinatorial gradient vector field over $\mathcal{S}$ whose critical cells correspond to the cells of the minimal complex. This vector field is related to a given system of polar coordinates in $\mathbb{R}^d$ which is *generic* with respect to the arrangement $\mathcal{A}$. This generic system of coordinates allow the authors of [15] to give a total order $\lhd$ on the faces $\mathcal{F}$ that is the key to describe both, gradient vector field and critical cells. In this paper we are mainly interested in the latter.

In more detail, let $\{V_k\}_{k=0,\dots,d}$ be a flag of affine subspaces in general position in $\mathbb{R}^d$, such that $\dim(V_k) = k$ for every $i = 0, \dots, d$ and such that the polar coordinates $(\rho, \theta_1, \dots, \theta_{d-1})$ of every point in a bounded face of $\mathcal{A}$ satisfy $\rho > 0$ and $0 < \theta_i < \pi/2$, for every $i = 1, \dots, d-1$ (see [15, Section 4.2] for the precise description). Every face F is labeled by the coordinates of the point in its closure that has, lexicographically, least polar coordinates. The *polar ordering* associated to such a flag is the total order $\lhd$ on $\mathcal{F}$ obtained by ordering the faces lexicographically according to their labels. This extends the order in which V_{d-1} intersects the faces in V_d while rotating around V_{d-2}. If two faces share the same label, thus the same minimal point r, the ordering is determined by the flag induced on the copy of V_{d-1} that is rotated "just past F" and the ordering it generates by induction on the dimension (see [15, Definition 4.7]). The k-cells of the minimal complex will be the critical k-cells (see [15, Theorem 6])

$$\mathrm{Crit}_k(\mathcal{S}) = \left\{ [C \prec F] \;\middle|\; \begin{array}{l} \mathrm{codim}(F) = k,\ F \cap V_k \neq \emptyset, \\ G \lhd F \text{ for all } G \text{ with } C \prec G \precneqq F \end{array} \right\} \tag{1}$$

(equivalently, $F \cap V_k$ is the maximum in polar ordering among all facets of $C \cap V_k$).

Notation 2.1. We denote by $\mathsf{ch}(\mathcal{A})$ the set of chambers of $\mathcal{A}$ and by $\mathrm{Crit}(\mathcal{S}) = \cup_{k=0}^{d} \mathrm{Crit}_k(\mathcal{S})$ the union of sets $\mathrm{Crit}_k(\mathcal{S})$ of critical k-cells.

2.2 Salvetti-Settepanella minimal complex for supersolvable arrangements

The class of "strictly linearly fibered" arrangements was introduced by Falk and Randell [8] in order to generalize the techniques of Fadell and Neuwirth's proof [7] of asphericity of the braid arrangement (involving a chain of fibrations). Later on, Terao [18] recognized that strictly linearly fibered arrangements are exactly those which intersection lattice is supersolvable [16]. Since then these arrangements are known as *supersolvable arrangements*, and attracted considerable consideration. See [12] and [17] for more details.

Definition 2.2. *A central arrangement $\mathcal{A}$ of hyperplanes in $\mathbb{R}^d$ is called* supersolvable *if there is a filtration*

$\mathcal{A} = \mathcal{A}_d \supset \mathcal{A}_{d-1} \supset \cdots \supset \mathcal{A}_2 \supset \mathcal{A}_1$ *such that*

(1) $\operatorname{rank}(\mathcal{A}_i) := \operatorname{codim}(\bigcap_{H\in\mathcal{A}_i} H) = i$ *for all* $i = 1, \ldots, d$ *;*

(2) for every two $H, H' \in \mathcal{A}_i$ *there exists some* $H'' \in \mathcal{A}_{i-1}$ *such that* $H \cap H' \subset H''$.

Let us remark that the above definition of supersolvable arrangement can be given in the more general case of complex arrangements.

In [5], Delucchi and the first author gave a more general combinatorial description of the minimal complex constructed in [15] and briefly described in Subsection 2.1. This allowed them to give a very handy order $\lhd$ in the case of supersolvable arrangements.

Given an affine real arrangement of hyperplanes $\mathcal{A}$ in $\mathbb{R}^d$, following [5], we call a flag $\{V_k\}_{k=0,\ldots,d}$ of affine subspaces *general flag* if every one of its subspaces is in general position with respect to $\mathcal{A}$ and if, for every $k = 0, \ldots d-1$, V_k does not intersect any bounded chamber of the arrangement $\mathcal{A} \cap V_{k+1}$. Note that this is a less restrictive hypothesis than the one required for being a *generic* flag in [15].

Fixed a real arrangement $\mathcal{A}$ in $\mathbb{R}^d$ and a general flag $\{V_k\}_{k=0,\ldots,d}$, denote by

$$\mathcal{P}^k(\mathcal{A}) := \{p \in \mathcal{F}^k \mid p \cap V_k \neq \emptyset\}$$

the set of faces that intersect V_k. In the rest of the paper, we will call those faces *critical faces* and denote them by the letter p to make it easier for the reader to distinguish them from the faces $F \in \mathcal{F}^k$ such that $F \cap V_k = \emptyset$. In [5], the authors showed that, in case $\mathcal{A} = \mathcal{A}_d \supset \mathcal{A}_{d-1} \supset \cdots \supset \mathcal{A}_2 \supset \mathcal{A}_1$ is a supersolvable arrangement, the general flag $\{V_k\}_{k=0,\ldots,d}$ can be chosen in such a way that there exists $R \in \mathbb{R}$ such that for all $k = 1, \ldots, d-1$, every element of $\mathcal{P}^k(\mathcal{A}_{d-1})$ is contained in a ball of radius R centered in V_0, that contains no elements of $\mathcal{P}^k(\mathcal{A}_d) \setminus \mathcal{P}^k(\mathcal{A}_{d-1})$. With this choice of a general flag, they proved (see [5, Theorem II.3.4]) that a real supersolvable arrangement $\mathcal{A}$ always admits a total order $\lhd$, that they called *recursive special order* (see [5, Definition II.1.1]), which satisfies the following properties. Given two critical faces $p, r \in \mathcal{P}(\mathcal{A}) = \cup_{k=0}^{d} \mathcal{P}^k(\mathcal{A})$, then $p \lhd r$ if one of the following occurs:

(i) $p \in \mathcal{P}^h(\mathcal{A})$, $r \in \mathcal{P}^k(\mathcal{A})$ for $h < k$;

(ii) there is k such that $p, r \in \mathcal{P}^k(\mathcal{A})$ and we can write $p_0 := \min_{\lhd}\{p' \in \mathcal{P}^{k-1}(\mathcal{A}) \mid p \subset |p'|\}$, $r_0 := \min_{\lhd}\{p' \in \mathcal{P}^{k-1}(\mathcal{A}) \mid r \subset |p'|\}$,

   (a) either $p_0 \lhd r_0$,

(b) or $p_0 = r_0$ and there exists a sequence of faces

$$p_0 \prec p_1 \succ r_1 \prec p_2 \succ r_2 \cdots \prec p,$$

such that $\mathrm{codim}(p_i) = \mathrm{codim}(r_i)+1 = \mathrm{codim}(p)$, and every r_i, p_i intersect $|p_0| \cap V_k$, and $p_i \neq r$ for all i.

(iii) If $p \in \mathcal{P}^k(\mathcal{A}_{i-1})$ and $r \in \mathcal{P}^k(\mathcal{A}_i) \setminus \mathcal{P}^k(\mathcal{A}_{i-1})$ lie in the support of the same $(k+1)$-codimensional face.

Any order $\lhd$ on the faces $\mathcal{F}$ induced by a general flag $\{V_k\}_{k=0,\ldots,d}$ induces an order $\lhd_{\mathcal{A}}$ on hyperplanes of $\mathcal{A}$ as follows

$$H \lhd_{\mathcal{A}} H' \text{ if and only if } p_H \lhd p_{H'},$$

where $p_H, p_{H'} \in \mathcal{P}^1(\mathcal{A})$ are the only two faces such that $|p_H| = H, |p_{H'}| = H'$.

By [(iii)], the order $\lhd$ can be chosen in such a way that the following property holds

$$\text{if } H \in \mathcal{A}_i \setminus \mathcal{A}_{i-1}, H' \in \mathcal{A}_j \setminus \mathcal{A}_{j-1} \text{ with } i < j, \text{ then } H \lhd_{\mathcal{A}} H'. \tag{2}$$

As no confusion can arise, we will denote the order $\lhd_{\mathcal{A}}$ simply by $\lhd$.

2.3 *nbc*-basis for supersolvable arrangements.

Let us briefly recall some basic facts on the Orlik-Solomon algebra and its ***nbc***-basis.

Fix an arbitrary order $\lhd$ on a central arrangement $\mathcal{A}$ in $\mathbb{R}^d$. Then an ordered k-tuple $(H_1, \ldots, H_{k+1})$, with $H_1 \lhd \ldots \lhd H_{k+1}$, is *independent* if $\mathrm{rank}(\cap_{i=1}^{k+1} H_i) = k+1$, and it is *dependent* otherwise. It is called a *circuit* if it is minimally dependent, that is $(H_1, \ldots, H_{k+1})$ is dependent, while $(H_1, \ldots, \hat{H}_p, \ldots, H_{k+1})$ is independent for any $1 \leq p \leq k+1$.

An ordered independent k-tuple $(H_1, \ldots, H_k)$ is a *broken circuit* if there exists an hyperplane $H \lhd H_1$ such that $(H, H_1, \ldots, H_k)$ is a circuit. It is well-known that a basis for the Orlik-Solomon algebra of the arrangement $\mathcal{A}$ is given by all ordered k-tuples $(H_1, \ldots, H_k)$, $1 \leq k \leq d$, that do not contain any broken circuit. Such a basis is called a *non broken circuit basis*, or simply ***nbc**-basis*.

Björner and Ziegler (see [3, Corollary 2.9]) proved that in a supersolvable arrangement a k-tuple $(H_1, \ldots, H_k)$ does not contain a broken circuit if and only if it does not contain a 2-broken circuit. From this we get the following proposition.

Proposition 2.3. *Let $\mathcal{A}$ be a supersolvable arrangement in $\mathbb{R}^d$ together with an order $\lhd$ that satisfies property* (2)*. If, for any $H_i \in \mathcal{A}$, h_i denote the index such that $H_i \in \mathcal{A}_{h_i}$ and $H_i \notin \mathcal{A}_{h_i-1}$, then the set*

$$\boldsymbol{nbc}_k(\mathcal{A}) = \{(H_1, \ldots, H_k) \in \mathcal{A}^k \mid H_i \in \mathcal{A}_{h_i}, h_i < h_{i+1}\}$$

*is a **nbc**-basis of the k-stratum of the Orlik-Solomon algebra associated to $\mathcal{A}$.*

Proof.[1] By Björner and Ziegler's result, it is enough to check that k-tuples $(H_1, \ldots, H_k) \in \boldsymbol{nbc}_k$ do not contain couples (H_i, H_j) that are broken circuits. Since $\mathcal{A}$ is a supersolvable arrangement, if H_i, H_j are hyperplanes that belong to the same subarrangement $\mathcal{A}_{h_i+1} \setminus \mathcal{A}_{h_i}$ then there exists $H \in \mathcal{A}_{h_i}$ such that $H_i \cap H_j \subset H$, that is (H, H_i, H_j) is a broken circuit.

On the other hand, if $H_i \in \mathcal{A}_{h_i+1} \setminus \mathcal{A}_{h_i}$ and $H_j \in \mathcal{A}_{h_j+1} \setminus \mathcal{A}_{h_j}$ belong to different subarrangements with $h_i < h_j$, then for any $H \lhd H_i$ we get that $\text{rank}(H \cap H_i \cap H_j) = 3$. Indeed if $H \in \mathcal{A}_h$, $h < h_i$ this is obvious while, if $H \in \mathcal{A}_{h_i}$ then there exists $H' \in \mathcal{A}_{h_i-1}$ such that $H \cap H_i = H' \cap H_i$ and $\text{rank}(H \cap H_i \cap H_j) = \text{rank}(H' \cap H_i \cap H_j) = 3$. $\square$

Following the previous proposition we denote

$$\boldsymbol{nbc}(\mathcal{A}) := \cup_k \boldsymbol{nbc}_k(\mathcal{A}).$$

For the sake of simplicity, when no confusion arises, we will omit $\mathcal{A}$ in the rest of the paper and we will simply denote $\boldsymbol{nbc}_k(\mathcal{A})$ by $\boldsymbol{nbc}_k$ and $\boldsymbol{nbc}(\mathcal{A})$ by $\boldsymbol{nbc}$. Similarly, we will simply denote $\mathcal{P}^k(\mathcal{A})$ by $\mathcal{P}^k$ and $\mathcal{P}(\mathcal{A})$ by $\mathcal{P}$.

3 The braid arrangement $\mathcal{A}(A_3)$.

In this section, we illustrate how the construction described in Section 2 works in the case of the braid arrangement in $\mathbb{R}^4$,

$$\mathcal{A}(A_3) = \{H_{i,j} = \{x_i = x_j\}, 1 \leq i < j \leq 4\}\ ^2.$$

In Figure 1 is depicted a general flag $\{V_k\}_{k=0,\ldots,3}$ which induces a recursive special order on the faces of $\mathcal{A}(A_3)$, as described in Subsection 2.2. In particular, V_0 is chosen inside the chamber C_0. For the sake of simplicity we assume that C_0 corresponds to the identity element in the symmetric group. The walls of C_0 in the section V_2 are the hyperplanes $H_{1,2}$, corresponding to permutation $(1,2) \simeq s_1 \in A_3$, $H_{2,3}$, corresponding to permutation $(2,3) \simeq s_2 \in A_3$, and $H_{3,4}$, corresponding to permutation $(3,4) \simeq s_3 \in A_3$. The order induced by V_1 on the hyperplanes corresponds exactly to the one described in Subsection 2.3. Indeed $H_{1,2} \in \mathcal{A}(A_1)$ is smaller than $H_{1,3}, H_{2,3} \in \mathcal{A}(A_2) \setminus \mathcal{A}(A_1)$ smaller than $H_{1,4}, H_{2,4}, H_{3,4} \in \mathcal{A}(A_3) \setminus \mathcal{A}(A_2)$. The order inside each $\mathcal{A}(A_i) \setminus \mathcal{A}(A_{i-1})$ is a lexicographic order on indices of the hyperplanes. Notice that this is the usual order on $\boldsymbol{nbc}$ for the braid arrangement case.

While Figure 1 describes how 1-codimensional critical faces, i.e. 1-codimensional faces intersected by V_1, are ordered, in Figure 2 the rotation of V_1 inside V_2 "just passing" the 2-codimensional critical faces p_i^2 is described. The construction in Subsection 2.2 essentially tells us that the flag $\{V_k\}_{k=0,\ldots,3}$ can be chosen in such a way that first all the 2-codimensional faces belonging to hyperplane $H_{1,2}$ are ordered following $H_{1,2}$ starting from its 1-critical face, and hence faces belonging to $H_{1,3}$ and $H_{2,3}$

[1] This proposition is a well-known fact but we could not find a detailed proof of it anywhere, so we provide it here.

[2] Note that $\text{ess}(\mathcal{A})$ is an arrangement in $\mathbb{R}^3$

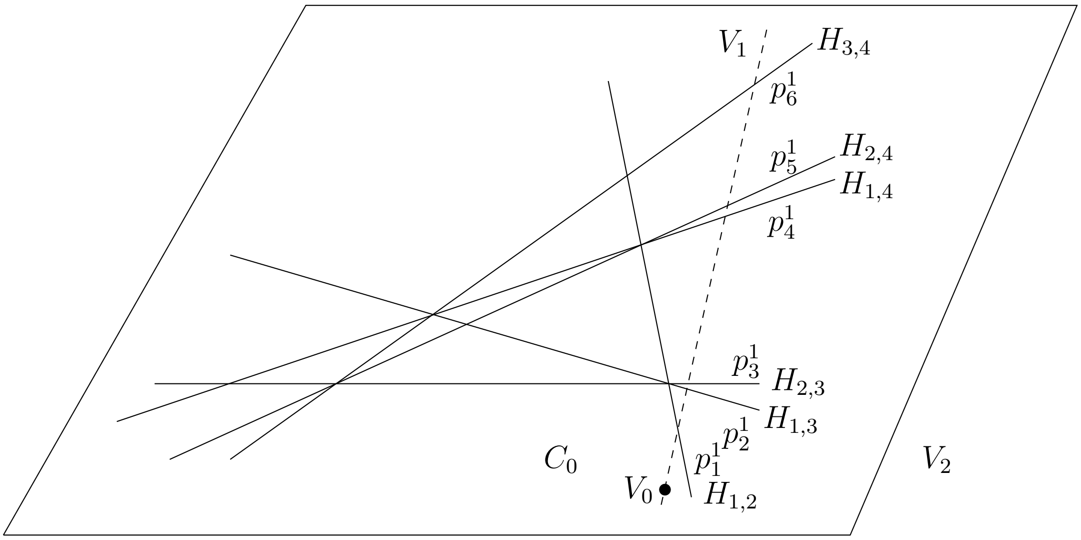

Figure 1: Section $V_2 \supset V_1 \supset V_0 \in C_0$ with 0-critical face C_0 and 1-critical faces p_i^1, for $i = 1, \ldots, 6$.

follow respecting the order of hyperplanes $H_{1,2} \lhd H_{1,3} \lhd H_{2,3}$ (see Figure 2). Note that this also induces an order between non critical faces. So, for example, when $V_1 = V_1^-(p_1^2)$ is rotated just passing p_1^2, we obtain $V_1^+(p_1^2)$ which induces an order on 1-codimensional faces it crosses exactly as it happens along V_1, i.e. $F_1^1 \lhd F_2^1 \lhd F_3^1$ with F_1^1 the face having p_1^2, p_6^2 in its closure, F_2^1 the face having p_1^2, p_4^2 in its closure and F_3^1 the face having p_1^2, p_2^2 in its closure (see Figure 2). Order on the chambers is analogous (see Figure 3).

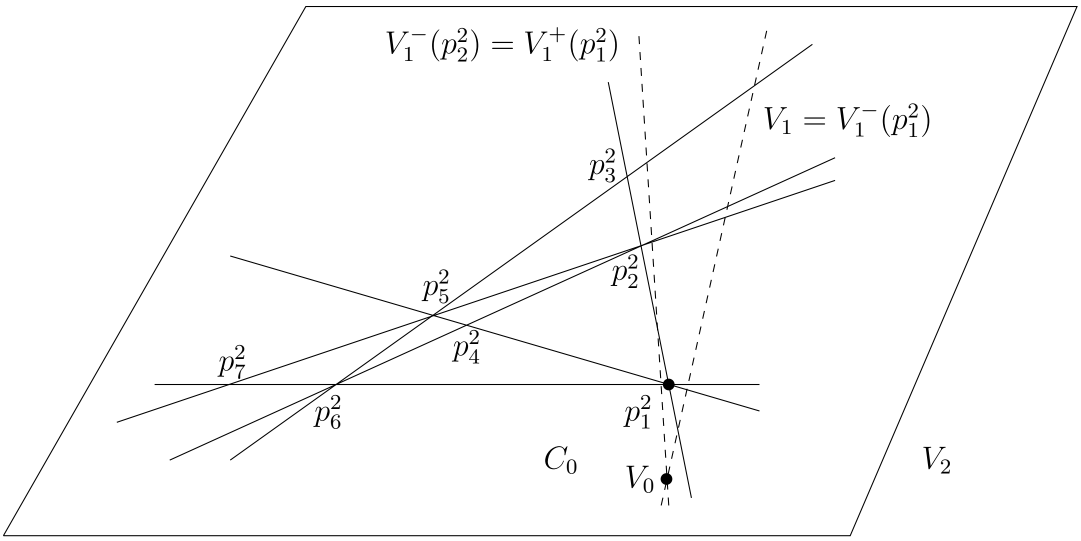

Figure 2: Rotation of V_1 inside V_2 around V_0 to order the 2-critical faces p_i^2, with $p_i^2 \lhd p_j^2$ if and only if $i < j$.

In the case of $\mathcal{A}(A_3)$ we get the correspondence described in Table 1. Note that in this table the correspondence with chambers that will be described in Section 4 is

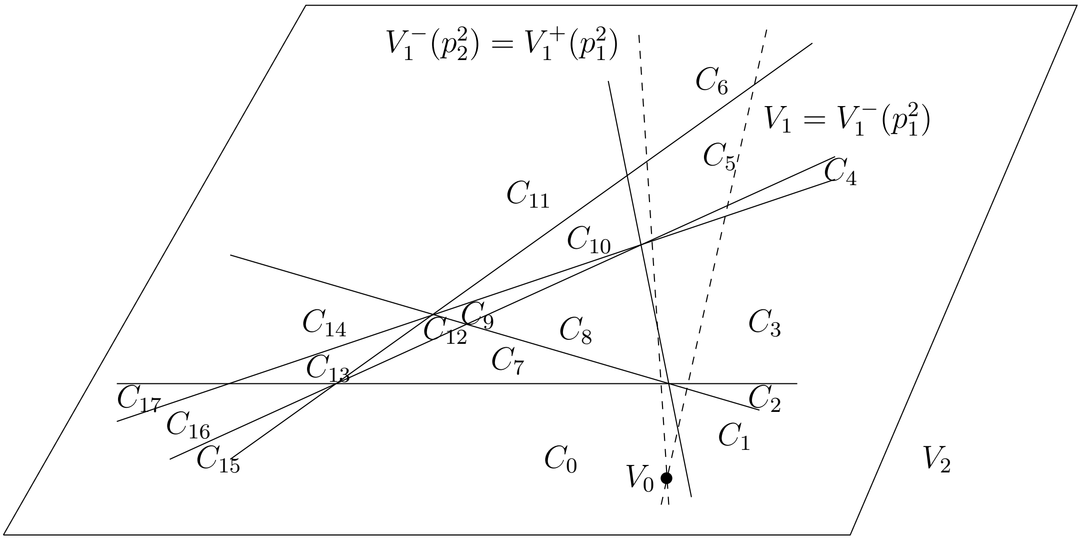

Figure 3: Chambers in the section V_2 ordered following $V_1^{\pm}(p_i^2)$.

k	k-critical cell	$\boldsymbol{nbc}_k$	Chamber
0	$[C_0 \prec C_0]$	$\emptyset$	C_0
1	$[C_0 \prec p_1^1]$	$H_{1,2}$	$C_1 = \mathrm{op}_{p_1^1}(C_0)$
1	$[C_1 \prec p_2^1]$	$H_{1,3}$	$C_2 = \mathrm{op}_{p_2^1}(C_1)$
1	$[C_2 \prec p_3^1]$	$H_{2,3}$	$C_3 = \mathrm{op}_{p_3^1}(C_2)$
1	$[C_3 \prec p_4^1]$	$H_{1,4}$	$C_4 = \mathrm{op}_{p_4^1}(C_3)$
1	$[C_4 \prec p_5^1]$	$H_{2,4}$	$C_5 = \mathrm{op}_{p_5^1}(C_4)$
1	$[C_5 \prec p_6^1]$	$H_{3,4}$	$C_6 = \mathrm{op}_{p_6^1}(C_5)$
2	$[C_1 \prec p_1^2]$	$(H_{1,2}, H_{1,3})$	$C_8 = \mathrm{op}_{p_1^2}(C_1)$
2	$[C_2 \prec p_1^2]$	$(H_{1,2}, H_{2,3})$	$C_7 = \mathrm{op}_{p_1^2}(C_2)$
2	$[C_3 \prec p_2^2]$	$(H_{1,2}, H_{1,4})$	$C_{10} = \mathrm{op}_{p_2^2}(C_3)$
2	$[C_4 \prec p_2^2]$	$(H_{1,2}, H_{2,4})$	$C_9 = \mathrm{op}_{p_2^2}(C_4)$
2	$[C_5 \prec p_3^2]$	$(H_{1,2}, H_{3,4})$	$C_{11} = \mathrm{op}_{p_3^2}(C_5)$
2	$[C_8 \prec p_4^2]$	$(H_{1,3}, H_{2,4})$	$C_{12} = \mathrm{op}_{p_4^2}(C_8)$
2	$[C_9 \prec p_5^2]$	$(H_{1,3}, H_{1,4})$	$C_{14} = \mathrm{op}_{p_5^2}(C_9)$
2	$[C_{10} \prec p_5^2]$	$(H_{1,3}, H_{3,4})$	$C_{13} = \mathrm{op}_{p_5^2}(C_{10})$
2	$[C_7 \prec p_6^2]$	$(H_{2,3}, H_{2,4})$	$C_{16} = \mathrm{op}_{p_6^2}(C_7)$
2	$[C_{12} \prec p_6^2]$	$(H_{2,3}, H_{3,4})$	$C_{15} = \mathrm{op}_{p_6^2}(C_{12})$
2	$[C_{13} \prec p_7^2]$	$(H_{2,3}, H_{1,4})$	$C_{17} = \mathrm{op}_{p_7^2}(C_{13})$

Table 1: Correspondence between critical k-cells, $\boldsymbol{nbc}_k$ and chambers in $\mathcal{A}(A_3)$ up to $k = 2$.

also described. Precise definition of $\mathrm{op}_p(C)$, the opposite chamber of C with respect to a given face p used in Table 1, is provided in equation (6) in Section 4.

Remark 3.1. Finally, note that in the bijection described in [1] by Barcelo, Goupil and Garsia, the hyperplanes $H_{1,2}, H_{1,3}$ and $H_{2,3}$, as 1 non broken circuits, correspond, respectively, to reflections $s_1, s_1s_2s_1, s_2$ which, in turn, correspond to chambers C_1, C_3 and C_7 (as labeled in Figure 3). Already from this basic example, it is possible to see how there is no way for those three chambers to be all intersected by the same 1-dimensional space in general position with respect to the arrangement $\mathcal{A}(A_3)$.

4 Orlik-Solomon algebra and minimal complex

In this section, $\mathcal{A}$ is a supersolvable arrangement in $\mathbb{R}^d$ endowed with a recursive special order $\lhd$ induced by a generic flag $\{V_k\}_{k=0,\dots,d}$ of affine subspaces as described in Subsection 2.2.

4.1 A natural relation

If $p \in \mathcal{P}^k$ is a k-critical face and $|p| = \cap_{j=1}^m H'_j$ is its support, then p is the only k-codimensional face that contains the intersection $|p| \cap V_k$. That is, there is a natural bijection between elements of the intersection poset $\mathcal{L}(\mathcal{A})$ and critical faces. Moreover, by the properties of supersolvable arrangements described in the Definition 2.2, to get a k-codimensional intersection $|p|$ in the poset $\mathcal{L}(\mathcal{A})$ of a supersolvable arrangement it is enough to consider a k-tuple $(H_1, \dots, H_k) \in \boldsymbol{nbc}_k$ such that $\cap_{i=1}^k H_i = \cap_{j=1}^m H'_j = |p|$. Notice that this is not a bijection. With the previous notations, if $H_j \in \mathcal{A}_{i_j} \setminus \mathcal{A}_{i_j-1}$ and $H \neq H_j$ is another hyperplane in $\mathcal{A}_{i_j} \setminus \mathcal{A}_{i_j-1}$ that contains p, then $(H_1, \dots, H_j, \dots, H_k)$ and $(H_1, \dots, H, \dots, H_k)$, with H in the j-th position, are both k-tuples in $\boldsymbol{nbc}_k$ with intersection equal the support $|p|$ of p.

Let $H \in \mathcal{A}_{i_j} \setminus \mathcal{A}_{i_j-1}$ be a hyperplane that contains the critical face p, we define the set

$$[H]_p := \{H' \in \mathcal{A}_{i_j} \setminus \mathcal{A}_{i_j-1} \mid p \subset H'\}.$$

Then to any critical k-codimensional face p is attached one and only one k-tuple of classes of hyperplanes

$$[p] := ([H_1]_p, \dots, [H_k]_p). \tag{3}$$

Notice that since a $(k-1)$-face F such that $F \prec G$ verifies, by definition of $\prec$, that G is contained in its closure, it is a straightforward remark that, if p' is a critical $(k-1)$-face such that $p' \prec p$, then there exists an index $1 \leq j \leq k$ such that $[p'] = ([H'_1]_{p'}, \dots, \widehat{[H'_j]_{p'}}, \dots, [H'_k]_{p'})$, $H'_i \in [H_i]_p$. Notice that the inclusion $[H'_1]_{p'} \subseteq [H'_1]_p$ holds.

Definition 4.1. *Given two chambers $C, C' \in \mathsf{ch}(\mathcal{A})$ and an hyperplane H in $\mathcal{A}$, we define*

$$(C \mid C')_H := \begin{cases} -1 & \text{if } H \text{ separates } C \text{ and } C', \\ 1 & \text{otherwise .} \end{cases}$$

Let C_0 be the chamber containing V_0, with the previous notations, define

$$f_k \colon \mathrm{Crit}_k(\mathcal{S}) \longrightarrow \boldsymbol{nbc}_k \tag{4}$$

as $f_k([C \prec p]) = (H_1, \dots, H_k)$ if and only if

(i) $\cap_{i=1}^k H_i = |p|$;

(ii) if $(-1)^{k-j} = -1$, then

$$H_j = \min_{\lhd}\{H \in [H_j]_p \mid (C \mid C_0)_H = -1\};$$

(iii) if $(-1)^{k-j} = 1$, then

$$H_j = \max_{\lhd}\{H \in [H_j]_p \mid (C \mid C_0)_H = 1\};$$

(iv) H_k is a wall of C and $(C \mid C_0)_{H_k} = 1$.

The above map naturally defines a function

$$f \colon \mathrm{Crit}(\mathcal{S}) \longrightarrow \boldsymbol{nbc} \tag{5}$$

between the critical cells of Salvetti's complex and the $\boldsymbol{nbc}$-basis of the Orlik-Solomon algebra of the supersolvable arrangement $\mathcal{A}$.

4.2 Preliminary Notations and Lemmas

Before proving that the maps f_k, $k = 0, \dots, d$, defined in equation (4), are well defined bijective maps, we need some notation and lemmas.

Notation 4.2. For a given critical cell $p \in \mathcal{P}^k(\mathcal{A})$ denote by $\overline{p}$ the point $p \cap V_k$ in V_k and by $V_{k-1}^+(p)$ the copy of V_{k-1} that is rotated in V_k around V_{k-2} "just past $\overline{p}$" and $V_{k-1}^-(p)$ the copy of V_{k-1} that is rotated in V_k around V_{k-2} "just before $\overline{p}$" , as in Figure 4. Let us remark that "just past $\overline{p}$" means that, when V_{k-1} rotates moving between $V_{k-1}^-(p)$ and $V_{k-1}^+(p)$, the rotation is as small as possible in such a way that p is the only k-face crossed. Following notation in [15], given a chamber C and a facet p, we will denote by $C.p$ the unique chamber of $\mathcal{A}$ containing p in its closure and lying in the same chamber as C in $\mathcal{A}_p$.

Let us remark that, by the construction of the polar ordering, all faces $F \prec p$ such that $F \cap V_{k-1}^+(p) \neq \emptyset$ and $F \cap V_{k-1}^-(p) = \emptyset$ satisfy $F \rhd p$. Moreover, given a chamber $C \in \mathsf{ch}(\mathcal{A})$ and a critical face $p \in \mathcal{P}(\mathcal{A}), C \prec p$, the k-cell $[C \prec p]$ is critical if and only if $C \cap V_{k-1}^-(p)$ is a bounded chamber in $V_{k-1}^-(p)$.

In the rest of the paper we will often deal with chambers C such that $C \cap V_{k-1}^-(p)$ (respectively $C \cap V_{k-1}^+(p)$) is bounded in $V_{k-1}^-(p)$ (respectively $V_{k-1}^+(p)$). In this case, for the sake of simplicity, we will say that C is bounded in $V_{k-1}^-(p)$ (respectively $V_{k-1}^+(p)$).

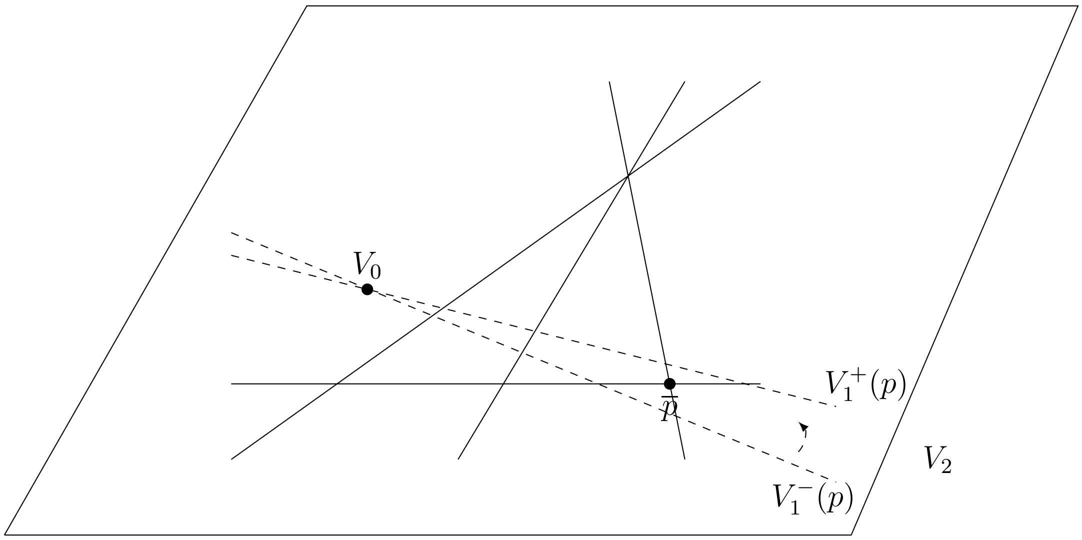

Figure 4: Rotation of V_1 in V_2 around V_0 "just past $\overline{p}$".

If $F \in \mathcal{F}$ is a face of a chamber $C \in \mathsf{ch}(\mathcal{A})$, then define $\mathrm{op}_F(C) \in \mathsf{ch}(\mathcal{A})$ as the unique chamber such that the set of hyperplanes of $\mathcal{A}$ that separates C and $\mathrm{op}_F(C)$ equals $\mathcal{A}_F$. The map

$$\begin{aligned} \eta\colon \mathrm{Crit}(\mathcal{S}) &\longrightarrow \mathsf{ch}(A) \\ [C \prec p] &\mapsto \mathrm{op}_p(C) \end{aligned} \tag{6}$$

is a bijection (see [15] and Figure 2 and Table 1 for an example).

We remark that if $[C \prec p]$ is critical then C is bounded in $V_{k-1}^-(p)$ by hyperplanes in $\mathcal{A}_p$. Indeed, by construction, $V_{k-1}^-(p)$ and $V_{k-1}^+(p)$ are chosen "just before p" and "just past p", i.e. we can assume that they intersect a small ball in V_k centered at p that intersects only hyperplanes in $\mathcal{A}_p$. Since C is bounded in $V_{k-1}^-(p)$ by hyperplanes in $\mathcal{A}_p$ then $\mathrm{op}_p(C)$ is a bounded chamber in $V_{k-1}^+(p)$ and the following lemma holds.

Lemma 4.3. *If $[C \prec p]$ is a critical k-cell and $\breve{p} = \min_{\triangleleft}\{p' \in \mathcal{P}^{k-1}(\mathcal{A}) \mid p \in |p'|\}$, then $[\mathrm{op}_{\breve{p}}(C.\breve{p}) \prec \breve{p}]$ is a critical $(k-1)$-cell.*

Proof. If $[C \prec p]$ is a critical k-cell, then C is bounded in $V_{k-1}^-(p)$ by hyperplanes in $\mathcal{A}_p$. Hence there exists at least one face $F \in \mathcal{F}^{k-1}$ of codimension $k-1$ with $p \subset \overline{F}$, i.e. F is the intersection of hyperplanes in $\mathcal{A}_p$, $F \subset \overline{C}$ and F is smaller than C in the local order inside $V_{k-1}^-(p)$. Let $\breve{F} \in \mathcal{F}^{k-1}$ denote the smallest of the faces F with the above property. Let $C(\mathcal{A}_p)$ denote the chamber of $\mathcal{A}_p$ containing C. Then any chamber C' contained in $C(\mathcal{A}_p)$ with $C' \cap V_{k-1} \neq \emptyset$ is a bounded chamber in V_{k-1}, as it is contained in $C(\mathcal{A}_p) \cap V_{k-1}$. In particular, if $\breve{p} \in \mathcal{P}^{k-1}(\mathcal{A})$ is the critical $k-1$ face such that $|\ \breve{p}\ |=|\ \breve{F}\ |$, then $C.\breve{p}$ is contained in $C(\mathcal{A}_p) \cap V_{k-1}$ and it is bounded too. Finally the flag in V_{k-1} and the flag in $V_{k-1}^-(p)$ induce the same order on faces of $\mathcal{A}_p$. That is $C.\breve{p} \triangleright \breve{p}$, since C is bigger than $\breve{F}$ in the local order inside $V_{k-1}^-(p)$, and $\breve{p} = \min_{\triangleleft}\{p' \in \mathcal{P}^{k-1}(\mathcal{A}) \mid p \in |p'|\}$, since $\breve{F}$ is the smallest face too.

By $C.\breve{p}$ bounded in V_{k-1} and $C.\breve{p} \rhd \breve{p}$, it follows that $C.\breve{p} \cap V^+_{k-2}(\breve{p}) \neq \emptyset$ is a bounded chamber in $V^+_{k-2}(\breve{p})$ and $C.\breve{p} \cap V^-_{k-2}(\breve{p}) = \emptyset$ (indeed if $C.\breve{p} \cap V^-_{k-2}(\breve{p}) \neq \emptyset$ then, by construction, $C.\breve{p} \lhd \breve{p}$). That is $C.\breve{p}$ is bounded in $V^+_{k-2}(\breve{p})$ by hyperplanes in $\mathcal{A}_{\breve{p}}$ and hence $\mathrm{op}_{\breve{p}}(C.\breve{p})$ is a bounded chamber in $V^-_{k-2}(\breve{p})$ and $[\mathrm{op}_{\breve{p}}(C.\breve{p}) \prec \breve{p}]$ is a critical $(k-1)$-cell. □

4.3 Bijection between *nbc* and critical cells.

In this subsection, we prove that the maps f_k, defined in equation (4), are well defined bijective maps, for all $k = 0, \ldots, d$.

Lemma 4.4. *If $p \in \mathcal{P}^k$ is a critical k-face with $[p] = ([H_1]_p, \ldots, [H_k]_p)$, then there exists one and only one critical $(k-1)$-face $\breve{p}$ such that $\breve{p} \prec p$ and $[\breve{p}] = ([H_1]_p, \ldots, [H_{k-1}]_p)$.*

Proof. Let $p \in \mathcal{P}^k$ be a critical k-face with $[p] = ([H_1]_p, \ldots, [H_k]_p)$, $H_i \in \mathcal{A}_{h_i} \setminus \mathcal{A}_{h_{i-1}}$ and let $p_1, p_2 \prec p$ be two critical $(k-1)$-faces with $[p_1] = ([H'_1]_{p_1}, \ldots, [H'_{k-1}]_{p_1})$ and $[p_2] = ([H''_1]_{p_2}, \ldots, [H''_{k-1}]_{p_2})$, where $H'_i, H''_i \in [H_i]_p$, $i = 1, \ldots, k-1$. If $p_1 \neq p_2$, then $\mid p_1 \mid \cap \mid p_2 \mid$ would be a space of codimension $\geq k$ that contains p and this is not possible as $\mid p_1 \mid \cap \mid p_2 \mid$ is an element in the intersection lattice of the arrangement $\mathcal{A}_{h_{k-1}}$ while $\mid p \mid \subset H_k$ and $H_k \in \mathcal{A}_{h_k} \setminus \mathcal{A}_{h_{k-1}}$. Then there is a unique critical $(k-1)$-face $\breve{p} \prec p$, $[\breve{p}] = ([\breve{H}_1]_{\breve{p}}, \ldots, [\breve{H}_{k-1}]_{\breve{p}})$ and it follows that $[\breve{H}_i]_{\breve{p}} = [H_i]_p$ for any $i = 1, \ldots, k-1$. □

From now on, given a critical k-cell $[C \prec p]$ we will always denote by $\breve{p}$ the unique critical $(k-1)$-face satisfying the condition in Lemma 4.4.

Lemma 4.5. *If $[C \prec p] \in \mathrm{Crit}_k(\mathcal{S})$ is a critical k-cell then $[\mathrm{op}_{\breve{p}}(C.\breve{p}) \prec \breve{p}]$ is a critical $(k-1)$-cell.*

Proof. It follows from Lemma 4.3 and the fact that $\breve{p} = \min_{\lhd}\{p' \in \mathcal{P}^{k-1} \mid p \in\mid p' \mid\}$ by property [(iii)] of the recursive order $\lhd$. □

Lemma 4.6. *Let $p \in \mathcal{P}^k$ be a critical k-face, $[p] = ([H_1]_p, \ldots, [H_k]_p)$. If $[C \prec p] \in \mathrm{Crit}_k(\mathcal{S})$ is a critical k-cell, then C has exactly one wall $H \in [H_k]_p$ that satisfies $(C \mid C_0)_H = 1$.*

Proof. Let $C.\breve{p}(\mathcal{A}_{\breve{p}})$ be the chamber of the arrangement $\mathcal{A}_{\breve{p}}$ that contains the chamber $C.\breve{p}$ and hence C. Let $[C' \prec p]$ be another critical k-cell with $C' \neq C$ and $C' \subset C.\breve{p}(\mathcal{A}_{\breve{p}})$. Then C and C' have to be separated by at least one hyperplane and, as they belong to the same chamber of $\mathcal{A}_{\breve{p}}$, they are separated by a hyperplane $H \in [H_k]_p$. It can be easily verified that $(C \mid C_0)_H = 1$ if and only if $(C' \mid C_0)_H = -1$.

Vice versa, any hyperplane $H \in [H_k]_p$ intersects the chamber $C.\breve{p}(\mathcal{A}_{\breve{p}})$ and hence it is the separating hyperplane of two different chambers contained in $C.\breve{p}(\mathcal{A}_{\breve{p}})$. That is for each hyperplane $H \in [H_k]_p$ there is a chamber $C' \subset C.\breve{p}(\mathcal{A}_{\breve{p}})$ such that H is a wall of C' and $(C' \mid C_0)_H = 1$.

We are now ready to prove uniqueness. If $H, H' \in [H_k]_p$ are two hyperplanes walls of C such that $(C \mid C_0)_H = 1$ and $(C \mid C_0)_{H'} = 1$ then, by supersolvability, there exists $\overline{H} \in [H_{k-1}]_p$ such that $H \cap H' \subset \overline{H}$. It follows that $\operatorname{rank}(H \cap H' \cap \overline{H}) = 2$ and hence H, H' and $\overline{H}$ cannot be walls of the same chamber C, that is either H or H' separates chamber C from a chamber $\overline{C}$ having $\overline{H}$ as wall. By the order chosen on the supersolvable arrangement it follows that an ordered minimal path from C_0 will cross $\overline{C}$ first and then C, i.e. H' separates C from C_0 which contradicts $(C \mid C_0)_{H'} = 1$. □

Theorem 4.7. *The four conditions stated after* (4) *i determine a well-defined map* f_k*, and the map is a bijection between* $\operatorname{Crit}_k(\mathcal{S})$ *and* $\boldsymbol{nbc}_k$.

Proof. We will prove the theorem by induction on the dimension k of the critical cells in $\operatorname{Crit}(\mathcal{S})$. The theorem holds trivially for the critical 0-cell that corresponds to the empty set.

Let $[C \prec p]$ be a critical k-cell. Then, by Lemma 4.5, the $(k-1)$-cell $[\operatorname{op}_{\check{p}}(C.\check{p}) \prec \check{p}]$ is critical and, by inductive hypothesis, there exists one and only one $(k-1)$-tuple of hyperplanes $(H_1, \ldots, H_{k-1}) \in \boldsymbol{nbc}_{k-1}$ such that $(H_1, \ldots, H_{k-1}) = f_{k-1}([\operatorname{op}_{\check{p}}(C.\check{p}) \prec \check{p}])$. Moreover, since when crossing a face p the order of hyperplanes in $\mathcal{A}_p$ is reversed and since C and $C.\check{p}$ belong to the same chamber $C.\check{p}(\mathcal{A}_{\check{p}})$ of the arrangement $\mathcal{A}_{\check{p}}$ and $C.\check{p}$ and $\operatorname{op}_{\check{p}}(C.\check{p})$ are opposite chambers with respect to $\check{p}$, it follows that

$$\begin{aligned} &\min_{\lhd}\{H \in [H_j]_p \mid (C \mid C_0)_H = -1\} \\ &\qquad = \min_{\lhd}\{H \in [H_j]_p \mid (C.\check{p} \mid C_0)_H = -1\} \\ &\qquad = \max_{\lhd}\{H \in [H_j]_p \mid (\operatorname{op}_{\check{p}}(C.\check{p}) \mid C_0)_H = 1\}, \end{aligned}$$

for $j = 1, \ldots, k-1$. Analogously,

$$\begin{aligned} &\max_{\lhd}\{H \in [H_j]_p \mid (C \mid C_0)_H = 1\} \\ &\qquad = \max_{\lhd}\{H \in [H_j]_p \mid (C.\check{p} \mid C_0)_H = 1\} \\ &\qquad = \min_{\lhd}\{H \in [H_j]_p \mid (\operatorname{op}_{\check{p}}(C.\check{p}) \mid C_0)_H = -1\}, \end{aligned}$$

for $j = 1, \ldots, k-1$.

Then, if $H \in [H_k]_p$ is the only hyperplane satisfying Lemma 4.6, the k-tuple $(H_1, \ldots, H_{k-1}, H)$ satisfies all conditions of (4) and it is clearly the only element in $f_k([C \prec p])$, that is f_k is a map.

We need to verify that f_k is bijective. Since $\boldsymbol{nbc}_k$ and $\operatorname{Crit}_k(\mathcal{S})$ are finite sets of the same cardinality, it is enough to show that f_k is injective. Let $[C' \prec p]$ be a critical k-cell with $f_k([C' \prec p]) = f_k([C \prec p])$. Then, by inductive hypothesis, $\operatorname{op}_{\check{p}}(C.\check{p}) = \operatorname{op}_{\check{p}}(C'.\check{p})$, that is $C.\check{p} = C'.\check{p}$ and hence $C.\check{p}(\mathcal{A}_{\check{p}}) = C'.\check{p}(\mathcal{A}_{\check{p}})$. By Lemma 4.6 there is only one hyperplane $H' \in [H_k]_p$ satisfying condition (iv) in the definition of the map f_k, that is $C = C'$. Hence f_k is injective. □

As immediate corollaries, we get the following results.

Corollary 4.8. *The map f defined in* (5) *is a bijection.*

Corollary 4.9. *The map $\eta^{-1}f$ is a bijection between* $\mathsf{ch}(\mathcal{A})$ *and* ***nbc***.

Remark 4.10. In the construction of the map f_k, we used properties of supersolvable arrangements, but the main argument behind its description and construction is merely a geometrical one. In fact, the map could be extended in general for any complexified arrangement in a similar way to what has been done for different maps by other authors, such as Yoshinaga in [21], Delucchi in [4], Jewell and Orlik in [10] and Gioan and Las Vergnas in [9]. The interest of this map is its handy and natural description that allows applications such as the one in the subsequent section. A natural question is to which extent and how this map can be generalised without losing its simple description. A partial answer about the non triviality of this question is given by the following example.

4.4 Nice arrangements

A natural generalization of the notion of supersolvable arrangements is the one of nice arrangements introduced by Terao in [19].

Fix an arrangement $\mathcal{A}$ in $\mathbb{R}^d$. A partition $\pi = (\pi_1, \ldots, \pi_s)$ of $\mathcal{A}$ is called *independent* if for any $H_i \in \pi_i \subset \mathcal{A}$, the hyperplanes $H_1, \ldots, H_s$ are independent, i.e. $\text{rank}(H_1 \cap \cdots \cap H_s) = s$.

Consider now $X \in L(\mathcal{A})$ and $\pi = (\pi_1, \ldots, \pi_s)$ a partition of $\mathcal{A}$. Then the *induced partition* π_X is a partition of the arrangement $\mathcal{A}_X$ whose blocks are the subsets $\pi_i \cap \mathcal{A}_X$, for $i = 1, \ldots, s$, which are not empty.

Definition 4.11. *A partition $\pi = (\pi_1, \ldots, \pi_s)$ of $\mathcal{A}$ is called* nice *if*

1. *π is independent;*
2. *for any $X \in L(\mathcal{A})$, the induced partition π_X contains a block which is a singleton unless $\mathcal{A}_X = \emptyset$.*

We will call $\mathcal{A}$ a nice arrangement *if it admits a nice partition.*

A supersolvable arrangement $\mathcal{A}$ is a nice arrangement with $s = d - 1$ and $\pi_i = \mathcal{A}_{i+1} \setminus \mathcal{A}_i$.

Nice arrangements have been introduced by Terao since they answered the question of which arrangements have their Orlik-Solomon algebra factorizable. In particular, $\pi = (\pi_1, \ldots, \pi_s)$ is a nice partition of $\mathcal{A}$ if and only if the Orlik-Solomon algebra of $\mathcal{A}$, viewed as $\mathbb{Z}$-module factorizes as

$$A(\mathcal{A}) = (\mathbb{Z} \oplus B(\pi_1)) \otimes \cdots \otimes (\mathbb{Z} \oplus B(\pi_s)),$$

where $B(\pi_i)$ denotes the submodule of $A^1(\mathcal{A})$ spanned by the hyperplanes in π_i.

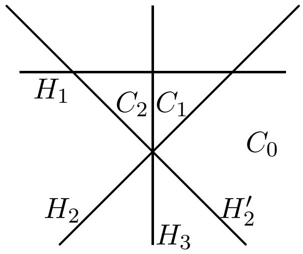

Figure 5: Nice arrangement.

Example 4.12. Let us consider the arrangement $\mathcal{A}$ described in Figure 5. The cone $c\mathcal{A}$ over $\mathcal{A}$ is supersolvable and hence nice. Consider the partition defined by $\pi_1 := \{H_1\}, \pi_2 := \{H_2, H_2'\}$ and $\pi_3 := \{H_3\}$. This partition is nice, but $\pi_1 \subset \pi_1 \cup \pi_2$ is not supersolvable arrangement, that is the partition $\pi = (\pi_1, \pi_2, \pi_3)$ is not compatible with supersolvable structure of $c\mathcal{A}$. If we replace the ***nbc***-basis obtained by supersolvable filtration $\mathcal{A} = \{H_1, H_2, H_2', H_3\} \supset \mathcal{A}_2 = \{H_1, H_2\} \supset \{H_1\}$ with the one obtained using partition π, the map f_2 defined in 4 should associate to the critical 2-cell $[C_2 \prec p]$ a 3-tuple of hyperplanes with H_3 as last entry since $\pi_3 := \{H_3\}$. But this clearly does not satisfy condition (iv) in the definition of the map f_k.

Let us remark that in the above example no recursive order is compatible with the nice partition π.

5 Braid arrangement

In this section, we describe the isomorphism between the symmetric group and the ***nbc***-basis of the Orlik-Solomon algebra for the braid arrangement in $\mathbb{R}^{n+1}$

$$\mathcal{A}(A_n) = \{H_{i,j} = \{x_i = x_j\}, 1 \leq i < j \leq n+1\}.$$

In order to simplify the notation, we indicate by A_n both the Coxeter group and the symmetric group on $n+1$ elements[3], acting by permutations of the coordinates, $\mathcal{A} = \mathcal{A}(A_n)$ is the braid arrangement and $\mathcal{S}(A_n)$ is the associated CW-complex.

Notice that $\mathcal{A}$ is a supersolvable arrangement with filtration given by $\mathcal{A}_1 = \{H_{1,2}\}$ and $\mathcal{A}_j \setminus \mathcal{A}_{j-1} = \{H_{1,j+1}, \ldots, H_{j,j+1}\}$, for $j = 2, \ldots, n$.

In [15], the authors gave a tableaux description of $\mathcal{S}(A_n)$ and constructed singular tableaux, that is tableaux corresponding to critical faces.

5.1 Tableaux description of $\mathcal{S}(A_n)$ and singular tableux

Given a system of coordinates in $\mathbb{R}^{n+1}$ it is possible to describe $\mathcal{S}(A_n)$ through certain tableaux as follow.

[3]With the obvious correspondence $s_i = (i, i+1)$.

Every k-cell $[C \prec F]$ is represented by a tableau with $n+1$ boxes and $n+1-k$ rows (aligned on the left), filled with all the integers in $\{1, \dots, n+1\}$. There is no monotony condition on the length of the rows. One has:

- $(x_1, \dots, x_{n+1})$ is a point in F if and only if:

1. i and j belong to the same row if and only if $x_i = x_j$,

2. i belongs to a row less than the one containing j if and only if $x_i < x_j$;

- the chamber C belongs to the half-space $x_i < x_j$ if and only if:

1. either the row which contains i is less than the one containing j or

2. i and j belong to the same row and the column which contains i is less than the one containing j.

Notice that the geometrical action of A_n on the stratification induces a natural action on the complex $\mathcal{S}(A_n)$, which, in terms of tableaux, is given by a left action of A_n: σ. T is the tableau with the same shape as T, and with entries permuted through σ.

Denote by $\mathbf{T}(A_n)$ the set of "row-standard" tableaux, i.e. with entries increasing along each row. Each face in the stratification $\mathcal{F}(A_n)$ corresponds to an equivalence class of tableaux, where the equivalence is up to row preserving permutations. Let $\mathbf{T^k}(A_n)$ be the set of tableaux of dimension k (briefly, k-tableaux), i.e. tableaux with exactly $n+1-k$ rows. Moreover, write $T \prec T'$ if and only if $F \prec F'$, where the tableaux T and T' correspond respectively to F and F'.

Define the following operations between tableaux:

1. $T * T'$ is the new tableau obtained by attaching vertically T' below T;

2. $T *_i h$ is the tableau obtained by attaching the one-box tableau with entry h to the i-th row of T;

3. T^{op} is the tableau obtained from T by reversing the row order. Notice that $(T * T')^{op} = T'^{op} * T^{op}$.

Fix k integers $1 < j_1 < \cdots < j_k \leq n+1$ and, for any $1 \leq h \leq k+1$, let T_h be the 0-tableau (that is, the one-column tableau) with entries $J_h = \{j_{h-1}+1, \dots, j_h - 1\}$ in the natural order (set $j_0 = 0$, $j_{k+1} = k+2$). Then, for any suitable choice of integers $i_1, \dots, i_k$ define a k-tableau

$$T^k = ((\cdots((((T_1^{op} *_{i_1} j_1) * T_2)^{op} *_{i_2} j_2) * T_3)^{op} \cdots)^{op} *_{i_k} j_k) * T_{k+1}. \tag{7}$$

In [15], the authors proved that there exists a system of polar coordinates, generic with respect to $\mathcal{A}(A_n)$, such that a k-facet p is critical if and only if the tableau T_p which represents p is of the form in (7). Moreover the induced order $\lhd$ between critical k-facets p equals the order between *critical* k-tableaux induced by the order between sequences of pairs $((j_1, i_1), \dots, (j_k, i_k))$, $(j_t, i_t) < (j'_t, i'_t)$ if and only if either $j_t < j'_t$ or $j_t = j'_t$ and $i_t > i'_t$. While a critical k-tableau is smaller than an critical

h-one, with $k \neq h$, if and only if $k < h$. This ordering is special recursive ordering. As no confusion can arise, we will denote this order among critical tableaux by $\lhd$.

Example 5.1. In the case $\mathcal{A}(A_3)$ described in Section 3, following the notations in Figures 1 and 2, the polar order on critical faces is

$$C_0 \lhd p_1^1 \lhd \ldots \lhd p_6^1 \lhd p_1^2 \lhd \ldots \lhd p_7^2 \lhd p^3$$

which exactly correspond to the ordered sequence of corresponding tableaux:

$$\begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \end{ytableau} \lhd \begin{ytableau} 1 & 2 \\ 3 \\ 4 \end{ytableau} \lhd \begin{ytableau} 2 \\ 1 & 3 \\ 4 \end{ytableau} \lhd \begin{ytableau} 2 & 3 \\ 1 \\ 4 \end{ytableau} \lhd \begin{ytableau} 3 \\ 2 \\ 1 & 4 \end{ytableau} \lhd \begin{ytableau} 3 \\ 2 & 4 \\ 1 \end{ytableau} \lhd \begin{ytableau} 3 & 4 \\ 2 \\ 1 \end{ytableau} \lhd \begin{ytableau} 1 & 2 & 3 \\ 4 \end{ytableau}$$

$$\lhd \begin{ytableau} 3 \\ 1 & 2 & 4 \end{ytableau} \lhd \begin{ytableau} 3 & 4 \\ 1 & 2 \end{ytableau} \lhd \begin{ytableau} 1 & 3 \\ 2 & 4 \end{ytableau} \lhd \begin{ytableau} 1 & 3 & 4 \\ 2 \end{ytableau} \lhd \begin{ytableau} 1 \\ 2 & 3 & 4 \end{ytableau} \lhd \begin{ytableau} 1 & 4 \\ 2 & 3 \end{ytableau} \lhd \begin{ytableau} 1 & 2 & 3 & 4 \end{ytableau}.$$

Notice, for example, that the critical 1-face p_1^1 with support $\mid p_1^1 \mid= H_{1,2}$ corresponds to the tableau $\begin{ytableau} 1 & 2 \\ 3 \\ 4 \end{ytableau}$ and, analogously, the critical 2-face p_7^2, $\mid p_7^2 \mid= H_{1,4} \cap H_{2,3}$ corresponding to couple of hyperplanes $(H_{1,4}, H_{2,3})$ corresponds to tableau $\begin{ytableau} 1 & 4 \\ 2 & 3 \end{ytableau}$.

In a similar way, following the algorithm described above, it is possible to order all faces, including chambers, and give a direct correspondence between chambers and elements of the group. So, for example, the chamber C_1 in Figure 3 corresponding to the reflection $s_1 = (1, 2)$ corresponds to the tableau $\begin{ytableau} 2 \\ 1 \\ 3 \\ 4 \end{ytableau}$, while the tableau $\begin{ytableau} 3 \\ 2 \\ 1 \\ 4 \end{ytableau}$ (which is exactly opposite permutation of numbers $\{1, 2, 3\}$ of the identity corresponding to C_0) corresponds to the chamber $C_3 = \mathrm{op}_{p_1^1}(C_0)$ corresponding to the reflection $s_1 s_2 s_1$ and so on.

We will now describe how to attach to each critical cell a 0-tableau to construct a bijection between the ***nbc***-basis of the Orlik-Solomon algebra of $\mathcal{A}(A_n)$ and the symmetric group A_n.

5.2 Non broken circuits of the symmetric group

Fix p a critical k-face, $[p] = ([H_{i_1}]_p, \ldots, [H_{i_k}]_p)$. Consider then a critical k-cell $[C \prec p]$ with $f_k([C \prec p]) = (H'_1, \ldots, H'_k)$, where $H'_j \in [H_{i_j}]_p$. Because we are considering the braid arrangement, for any $H \in \mathcal{A}$ we can write $H = H_{(s,t)}$ for some $1 \leq s < t \leq n+1$. Hence we can write

$$([H_{i_1}]_p, \ldots, [H_{i_k}]_p) = ([H_{(s_1,t_1)}]_p, \ldots, [H_{(s_k,t_k)}]_p)$$

and

$$(H'_1, \ldots, H'_k) = (H_{(s'_1,t_1)}, \ldots, H_{(s'_k,t_k)}).$$

Let T_p be the tableau attached to the critical k-face p. Then T_p is of the form described in (7). Suppose that T_p has rows $(T_p)_1, \ldots, (T_p)_{n+1-k}$.

We will describe how to attach to $[C \prec p]$ one and only one 0-tableaux $T_{[C\prec p]}$ starting from the tableau T_p.

Definition 5.2. *Given a tableau T, define the map*

$$\mathfrak{r}_T \colon \{1, \ldots, n+1\} \longrightarrow \{1, \ldots, n+1\}$$

such that $\mathfrak{r}_T(j)$ is the row of T where j is. If $T = T_F$ for a face F, we will simply write $\mathfrak{r}_F$ instead of $\mathfrak{r}_{T_F}$.

Remark 5.3. By construction, if $H_{s,t}$ is a hyperplane such that $H_{s,t} \supset p$ then s and t lie in the same row of the tableau T_p, that is $\mathfrak{r}_p(s) = \mathfrak{r}_p(t)$, while if $H_{s,t} \not\supset p$ then $\mathfrak{r}_p(s) \neq \mathfrak{r}_p(t)$. Moreover, if C is a chamber, that is a 0-codimensional facet, then it is represented by a column tableau T_C. Let C_0 be the base chamber corresponding to the tableau with entry i in the i-th row. By construction, if the hyperplane $H_{s,t}$, $s < t$, separates the chambers C and C_0 then $\mathfrak{r}_C(s) > \mathfrak{r}_C(t)$, while $\mathfrak{r}_C(s) < \mathfrak{r}_C(t)$ otherwise. It is easy to see that if $H_{s,t}$ is a wall of C, then $|\mathfrak{r}_C(s) - \mathfrak{r}_C(t)| = 1$, that is s and t belong to consecutive rows.

Definition 5.4. *Consider $f_k([C \prec p]) = (H_{(s'_1,t_1)}, \ldots, H_{(s'_k,t_k)})$. Then we define an order $<_{[C\prec p]}$ on the set of integers $\{s'_1, \ldots, s'_k, t_1, \ldots, t_k\}$ as follows*

i) if $(-1)^{k-j} = 1$, then $t_j <_{[C\prec p]} s'_j$;

ii) if $(-1)^{k-j} = -1$, then $s'_j <_{[C\prec p]} t_j$;

iii) $t_k <_{[C\prec p]} s'_k$ are consecutive numbers in the order $<_{[C\prec p]}$.

Proposition 5.5. *The relation $<_{[C\prec p]}$ is a total order on the entries of $(T_p)_i$, for all $i = 1, \ldots, n+1-k$.*

Proof. We will prove the statement using induction on k. Suppose $k = 1$. Then the statement is obvious as T_p is a tableau with $n-1$ rows of length one and one row of length two with entries $\{s_1, t_1\}$ and, by definition, $t_1 <_{[C\prec p]} s_1$.

Suppose now $k > 1$ and let $(H_{(s'_1,t_1)}, \ldots, H_{(s'_{k-1},t_{k-1})})$ be the $k-1$-tuple obtained removing the last entry of $f_k([C \prec p]) = (H_{(s'_1,t_1)}, \ldots, H_{(s'_k,t_k)})$. By Lemma 4.4 and Corollary 4.8, there exists a unique critical $(k-1)$-cell $[C' \prec \breve{p}]$ attached to it. Hence we can consider the tableau $T_{\breve{p}}$ and we know by induction that $<_{[C'\prec\breve{p}]}$ is a total order on the rows of $T_{\breve{p}}$. $T_p \prec T_{\breve{p}}$ differ only by the rows that contains s'_k and t_k as $\mathfrak{r}_{\breve{p}}(s'_k) \neq \mathfrak{r}_{\breve{p}}(t_k)$ while $\mathfrak{r}_p(s'_k) = \mathfrak{r}_p(t_k)$. As a consequence we just need to prove that $<_{[C\prec p]}$ is a total order on the row $\mathfrak{r}_p(s'_k) = \mathfrak{r}_p(t_k)$. In all the other rows of T_p the order is given by

$$b <_{[C\prec p]} a \quad \text{if and only if} \quad a <_{[C'\prec\breve{p}]} b.$$

Notice that, by tableaux definition, $\sharp(T_{\check{p}})_{\mathfrak{r}_{\check{p}}(t_k)} = 1$ and hence $(T_p)_{\mathfrak{r}_p(s'_k)} = (T_{\check{p}})_{\mathfrak{r}_{\check{p}}(s'_k)} \cup \{t_k\}$. By condition iii) in Definition 5.4 $t_k <_{[C \prec p]} s'_k$ are consecutive numbers in the order $<_{[C \prec p]}$ then the rest of the row is naturally ordered by $a \in (T_{\check{p}})_{\mathfrak{r}_{\check{p}}(s'_k)} \setminus \{s'_k\}$, $a <_{[C' \prec \check{p}]} s'_k$ implies $a <_{[C \prec p]} t_k$ and, similarly, $s'_k <_{[C' \prec \check{p}]} a$ implies $t_k <_{[C \prec p]} a$, i.e. $<_{[C \prec p]}$ is a total order. □

For all $i = 1, \dots, n+1-k$, denote by $T^i_{[C \prec p]}$ the column tableau obtained transposing the i-th row $(T_p)_i$ of the tableau T_p with entries ordered from upper to bottom by $<_{[C \prec p]}$. Define

$$T_{[C \prec p]} := T^1_{[C \prec p]} * \cdots * T^{n+1-k}_{[C \prec p]}.$$

The following theorem holds.

Theorem 5.6. *The map* $\mathfrak{T}\colon \mathrm{Crit}(\mathcal{S}(A_n)) \longrightarrow \mathbf{T^0}(A_n)$ *defined by* $\mathfrak{T}([C \prec p]) = T_{[C \prec p]}$ *is a bijection.*

By Definition 5.4, we get that the bijection $\mathfrak{T}$ factorizes through the bijection f. That is, there exists a bijection

$$g\colon \boldsymbol{nbc}(A_n) \longrightarrow \mathbf{T^0}(A_n)$$

between non broken circuit basis of the Orlik-Solomon algebra $A(A_n)$ and 0-tableaux such that $\mathfrak{T} = g \circ f$. Moreover, as tableaux in $\mathbf{T^0}(A_n)$ naturally correspond to permutations in the symmetric group A_n on one hand and to chambers $C \simeq [C \prec C]$ of the braid arrangement $\mathcal{A}(A_n)$ on the other hand, if we consider the map η described in (6), we get the following diagram of maps

$$\begin{array}{ccc} \mathrm{Crit}(\mathcal{S}(A_n)) & \xrightarrow{f} & \boldsymbol{nbc}(A_n) \\ \eta\downarrow & & g\downarrow \\ \mathsf{ch}(A_n) & \xrightarrow{\varphi} & \mathbf{T^0}(A_n) \simeq A_n \end{array} \tag{8}$$

where φ is the bijection described in Section 5.1. The following Theorem holds.

Theorem 5.7. *The diagram in* (8) *is a commutative diagram of bijective maps.*

Example 5.8. In the case of $\mathcal{A}(A_3)$, described in Section 3, Table 1 can be completed adding the corresponding tableaux. We obtain Table 2. Notice that the tableaux in the last column of Table 2 are the transpose of the corresponding tableaux, i.e. one row tableaux instead of one column ones, simply for graphic reasons.

We remark that in this construction, the inverse map g^{-1} of the map g can be obtained by direct computation as follows. Let $\mathcal{T}(A_n)$ be the set of all critical tableaux computed as in equation (7). Given a permutation $w \in A_n$ and the associated tableau T_w, then $T_w = T_{C'}$ for a chamber $C' = \mathrm{op}_p(C) = \eta([C \prec p])$ where p is the smallest critical face in the order $\lhd$ such that $C' \prec p$ and hence

$$T_p = \min_{\lhd}\{T \in \mathcal{T}(A_n) \mid T_{C'} \prec T\}.$$

k	k-**critical cell**	$\boldsymbol{nbc}_k$	**Chamber**	**Tableau**
0	$[C_0 \prec C_0]$	$\emptyset$	C_0	1 2 3 4
1	$[C_0 \prec p_1^1]$	$H_{1,2}$	$C_1 = \mathrm{op}_{p_1^1}(C_0)$	2 1 3 4
1	$[C_1 \prec p_2^1]$	$H_{1,3}$	$C_2 = \mathrm{op}_{p_2^1}(C_1)$	2 3 1 4
1	$[C_2 \prec p_3^1]$	$H_{2,3}$	$C_3 = \mathrm{op}_{p_3^1}(C_2)$	3 2 1 4
1	$[C_3 \prec p_4^1]$	$H_{1,4}$	$C_4 = \mathrm{op}_{p_4^1}(C_3)$	3 2 4 1
1	$[C_4 \prec p_5^1]$	$H_{2,4}$	$C_5 = \mathrm{op}_{p_5^1}(C_4)$	3 4 2 1
1	$[C_5 \prec p_6^1]$	$H_{3,4}$	$C_6 = \mathrm{op}_{p_6^1}(C_5)$	4 3 2 1
2	$[C_1 \prec p_1^2]$	$(H_{1,2}, H_{1,3)}$	$C_8 = \mathrm{op}_{p_1^2}(C_1)$	3 1 2 4
2	$[C_2 \prec p_1^2]$	$(H_{1,2}, H_{2,3})$	$C_7 = \mathrm{op}_{p_1^2}(C_2)$	1 3 2 4
2	$[C_3 \prec p_2^2]$	$(H_{1,2}, H_{1,4})$	$C_{10} = \mathrm{op}_{p_2^2}(C_3)$	3 4 1 2
2	$[C_4 \prec p_2^2]$	$(H_{1,2}, H_{2,4})$	$C_9 = \mathrm{op}_{p_2^2}(C_4)$	3 1 4 2
2	$[C_5 \prec p_3^2]$	$(H_{1,2}, H_{3,4})$	$C_{11} = \mathrm{op}_{p_3^2}(C_5)$	4 3 1 2
2	$[C_8 \prec p_4^2]$	$(H_{1,3}, H_{2,4})$	$C_{12} = \mathrm{op}_{p_4^2}(C_8)$	1 3 4 2
2	$[C_9 \prec p_5^2]$	$(H_{1,3}, H_{1,4})$	$C_{14} = \mathrm{op}_{p_5^2}(C_9)$	4 1 3 2
2	$[C_{10} \prec p_5^2]$	$(H_{1,3}, H_{3,4})$	$C_{13} = \mathrm{op}_{p_5^2}(C_{10})$	1 4 3 2
2	$[C_7 \prec p_6^2]$	$(H_{2,3}, H_{2,4})$	$C_{16} = \mathrm{op}_{p_6^2}(C_7)$	1 4 2 3
2	$[C_{12} \prec p_6^2]$	$(H_{2,3}, H_{3,4})$	$C_{15} = \mathrm{op}_{p_6^2}(C_{12})$	1 2 4 3
2	$[C_{13} \prec p_7^2]$	$(H_{2,3}, H_{1,4})$	$C_{17} = \mathrm{op}_{p_7^2}(C_{13})$	4 1 2 3

Table 2: Correspondence in diagram (8) in case $\mathcal{A}(A_3)$ up to $k = 2$.

Then, given $T_{C'}$ and T_p it is possible to retrieve the *nbc*-tuple of hyperplanes associated to w via Definition 5.4.

Notice that the map g is slightly different from the map constructed by Barcelo and Goupil in [1]. If, from one side, g is a less natural map, from the group point of view, than Barcelo and Goupil's one[4], on the other side it verifies the property that all the chambers corresponding to reflections $w \simeq (H_{(s_1,t_1)}, \ldots, H_{(s_k,t_k)})$ intersect a k-dimensional subspace. So far this property, i.e. Lefschetz type argument, seems to be quite important in order to study (co)homology related problems for $\mathcal{A}$. Moreover, the map g behaves properly with respect to the natural inclusion $A_i \subset A_{i+1}$.

An interesting question is whether this construction can be extended to other reflection groups.

[4]and from Jewell-Orlik one from geometric point of view

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