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Some results on the f-chromatic index of graphs whose f-core has maximum degree 2

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Abstract

Let G be a graph and $f : V(G) \to \mathbb{N}$ be a function. An f-coloring of a graph G is an edge coloring such that each color appears at each vertex $v \in V(G)$ at most $f(v)$ times. The minimum number of colors needed to f-color G is called the f-chromatic index of G and is denoted by $\chi'_f(G)$. It was shown that for every graph G, $\Delta_f(G) \leq \chi'_f(G) \leq \Delta_f(G) + 1$, where $\Delta_f(G) = \max_{v\in V(G)}\lceil \frac{d_G(v)}{f(v)} \rceil$. A graph G is said to be f-Class 1

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if $\chi'_f(G) = \Delta_f(G)$, and f-Class 2, otherwise. Also, G_{Δ_f} is the induced subgraph of G on $\{v \in V(G) : \frac{d_G(v)}{f(v)} = \Delta_f(G)\}$. In this paper, we show that if G is a connected graph with $\Delta(G_{\Delta_f}) \leq 2$ and G has an edge cut of size at most $\Delta_f(G) - 2$ which is a star, then G is f-Class 1. Also, we prove that if G is a connected graph and every connected component of G_{Δ_f} is a unicyclic graph or a tree and G_{Δ_f} is not 2-regular, then G is f-Class 1. Moreover, we show that except one graph, every connected claw-free graph G whose f-core is 2-regular with a vertex v such that $f(v) \neq 1$ is f-Class 1.

1 Introduction

All graphs considered in this paper are simple and finite. Let G be a graph. The number of vertices of G is called the order of G and is denoted by $|G|$. Also, $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. The degree of a vertex v in G is denoted by $d_G(v)$ and $N_G(v)$ denotes the set of all vertices adjacent to v. Moreover, for $S \subseteq V(G)$, we denote the neighbor set of S in G by $N_G(S)$. Also, let $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of G, respectively. A *star graph* is a graph containing a vertex adjacent to all other vertices and with no extra edges. A *matching* in a graph is a set of pairwise non-adjacent edges. An *edge cut* is a set of edges whose removal produces a subgraph with more connected components than the original graph. If the edge cut is the edge set of a star, then we call it *star cut*. Moreover, a graph is *k-edge connected* if the minimum number of edges whose removal would disconnect the graph is at least k. We mean $G \setminus H$, the induced subgraph on $V(G) \setminus V(H)$. For two subsets S and T of $V(G)$, where $S \cap T = \emptyset$, $e_G(S, T)$ denotes the number of edges with one end in S and other end in T. For a subset $X \subseteq V(G)$, we denote the induced subgraph of G on X by $\langle X \rangle$. An induced $K_{1,3}$ is called a *claw*. A graph is called *claw-free* if it contains no claw. Moreover, a graph G is called a *unicyclic* graph if it is connected and contains exactly one cycle.

A *k-edge coloring* of a graph G is a function $f : E(G) \longrightarrow L$, where $|L| = k$ and $f(e_1) \neq f(e_2)$, for every two adjacent edges e_1, e_2 of G. The minimum number of colors needed to color the edges of G is called the *chromatic index* of G and is denoted by $\chi'(G)$. Vizing [6] proved that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$, for any graph G. A graph G is said to be *Class* 1 if $\chi'(G) = \Delta(G)$ and *Class* 2 if $\chi'(G) = \Delta(G)+1$. A graph G is called *critical* if G is connected, Class 2 and $\chi'(G \setminus e) < \chi'(G)$, for every edge $e \in E(G)$. Also, G_Δ is the induced subgraph on all vertices of degree $\Delta(G)$.

For a function f which assigns a positive integer $f(v)$ to each vertex $v \in V(G)$, an *f-coloring* of G is an edge coloring of G such that each vertex v has at most $f(v)$ edges colored with the same color. The minimum number of colors needed to f-color G is called the *f-chromatic index* of G, and denoted by $\chi'_f(G)$. For a graph G, if $f(v) = 1$ for all $v \in V(G)$, then the f-coloring of G is reduced to the proper edge coloring of G. Let $\Delta_f(G) = \max_{v \in V(G)} \lceil \frac{d_G(v)}{f(v)} \rceil$. A graph G is said to be *f-Class* 1 if

$\chi'_f(G) = \Delta_f(G)$ and f-*Class* 2, otherwise. Also, we say that G has a $\Delta_f(G)$-coloring if G is f-Class 1. A vertex v is called an f-*maximum vertex* if $d_G(v) = f(v)\Delta_f(G)$. A graph G is called f-*critical* if G is connected, f-Class 2 and $\chi'_f(G \setminus e) < \chi'_f(G)$, for every edge $e \in E(G)$. The f-*core* of a graph G is the induced subgraph of G on the f-maximum vertices and denoted by G_{Δ_f}. The following example presents an f-Class 1 graph.

Example 1.1 Let G be a graph represented in Figure 1 with $f(v_1) = f(v_2) = 2$ and $f(v_i) = 1$, for $i = 3, \ldots, 7$. It is easy to see that $\Delta_f(G) = 2$ and $G_{\Delta_f} = K_3$. Now, by assigning color α to the edges $\{v_1v_6, v_1v_5, v_2v_3, v_2v_4\}$ and color β to the edges $\{v_1v_2, v_1v_7, v_2v_5\}$, one can see that G is f-Class 1.

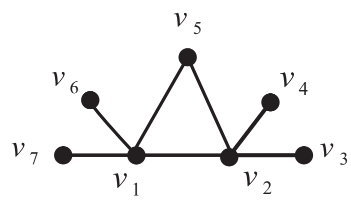

Figure 1: An f-Class 1 graph

In [3], Hakimi and Kariv obtained the following three results.

Theorem 1.1 [3] *Let G be a graph. Then*

$$\Delta_f(G) \leq \chi'_f(G) \leq \max_{v \in V(G)} \lceil \frac{d_G(v) + 1}{f(v)} \rceil \leq \Delta_f(G) + 1.$$

Theorem 1.2 [3] *Let G be a bipartite graph. Then G is f-Class* 1.

Theorem 1.3 [3] *Let G be a graph and $f(v)$ be even, for all $v \in V(G)$. Then G is f-Class* 1.

The following result is due to Zhang and Liu, who gave a series of sufficient conditions for a graph G to be f-Class 1 based on the f-core of G.

Theorem 1.4 [8] *Let G be a graph. If G_{Δ_f} is a forest, then G is f-Class* 1.

In [5], some properties of f-critical graphs are given. In the following, we review one of them.

Theorem 1.5 *For every vertex v of an f-critical graph G, v is adjacent to at least $2f(v)$ f-maximum vertices and G contains at least three f-maximum vertices.*

There are some results on proper edge colorings of graphs as follows:

Theorem 1.6 [4] *Let G be a connected Class 2 graph with $\Delta(G_\Delta) \leq 2$. Then:*

1. *G is critical;*
2. *$\delta(G_\Delta) = 2$;*
3. *$\delta(G) = \Delta(G) - 1$, unless G is an odd cycle.*

Theorem 1.7 [1] *Let G be a connected graph such that $\Delta(G_\Delta) \leq 2$. Suppose that G has an edge cut of size at most $\Delta(G) - 2$ which is a matching or a star. Then G is Class 1.*

Theorem 1.8 [1] *Let G be a connected graph. If every connected component of G_Δ is a unicyclic graph or a tree and G_Δ is not 2-regular, then G is Class 1.*

Theorem 1.9 [9] *If G is a connected f-Class 2 graph with $\Delta(G_{\Delta_f}) \leq 2$, then*

(i) *G is f-critical;*

(ii) *$\delta(G_{\Delta_f}) = 2$;*

(iii) *$V(G) = N_G(V(G_{\Delta_f}))$;*

(iv) *$f(v) = 1$ for all $v \in V(G_{\Delta_f})$;*

(v) *$d_G(v) = f(v)\Delta_f(G) - 1$, for each $v \in V(G) \setminus V(G_{\Delta_f})$.*

Theorem 1.10 [2] *Let G be a connected graph such that $\Delta(G_{\Delta_f}) \leq 2$. Suppose that G has an edge cut of size at most $\Delta_f(G) - 2$ which is a matching. Then G is f-Class 1 and G has a $\Delta_f(G)$-coloring in which the edges of the edge cut have different colors.*

In this paper, we generalize Theorems 1.7 and 1.8 to f-coloring of graphs. Moreover, we show that, with the exception of one graph, every connected claw-free graph G whose f-core is 2-regular and a vertex v, such that $f(v) \neq 1$ for some, is f-Class 1.

2 Results

In this section, we generalize Theorems 1.7 and 1.8 and we obtain some results in f-coloring of claw-free graphs whose f-core is 2-regular. First we want to prove that if a connected graph G with $\Delta(G_{\Delta_f}) \leq 2$ has an edge cut of size at most $\Delta_f(G) - 2$ which is a star, then G is f-Class 1. To do this, we need the following two lemmas.

Lemma 2.1 [2] *Let G be a connected graph with $\Delta(G_{\Delta_f}) \leq 2$. Suppose that $F = \{uv_1, \ldots, uv_k\}$, $k \leq \Delta_f(G) - 2$, is an edge cut of G and $f(u) = 1$. Then G is f-Class 1.*

Lemma 2.2 *Let G be a graph. If $G_{\Delta_f} = \emptyset$, then G is f-Class 1.*

Proof. Let $v \in V(G)$ be a vertex such that $\Delta_f(G) = \lceil \frac{d_G(v)}{f(v)} \rceil$. Let H be the graph obtained from G by adding $(\lceil \frac{d_G(v)}{f(v)} \rceil - \frac{d_G(v)}{f(v)}) f(v)$ new vertices, all adjacent to v. Let $f' : V(H) \longrightarrow \mathbb{N}$ be a function defined by

$$f'(z) = \begin{cases} f(z) & z \in V(G), \\ 1 & z \in V(H) \setminus V(G). \end{cases}$$

Clearly, $|V(H_{\Delta_f})| = 1$ and $\Delta_f(H) = \Delta_f(G)$. Now, by Theorem 1.4, H is f-Class 1 and so is G. □

The following theorem together with Theorem 1.10 generalizes Theorem 1.7.

Theorem 2.1 *Let G be a connected graph, $\Delta_f(G) \geq 3$ and $\Delta(G_{\Delta_f}) \leq 2$. Suppose that G has a star cut of size at most $\Delta_f(G) - 2$. Then G is f-Class 1.*

Proof. Let $F = \{uv_1, \dots, uv_k\}$ be a minimal star cut of G. If $k = 1$, we are done by Theorem 1.10. Next, we assume that $2 \leq k \leq \Delta_f(G) - 2$. Also, let X be the vertex set of the connected component of $G \setminus F$ containing u and let Y be $V(G) \setminus X$. Let G_1 and G_2 be the induced subgraphs on X and Y, respectively. Then $u \in V(G_1)$ and $v_i \in V(G_2)$, for $i = 1, \dots, k$. By Lemma 2.1 we can assume that $f(u) \geq 2$. For a contradiction assume that G is f-Class 2. Since $\Delta(G_{\Delta_f}) \leq 2$, by Theorem 1.9, we get that G is f-critical, and because $f(u) \geq 2$, by Theorem 1.5, we conclude that $u \notin V(G_{\Delta_f})$. Thus by Theorem 1.9, $d_G(u) = f(u)\Delta_f(G) - 1 \geq 2\Delta_f(G) - 1$. Let $N_{G_1}(u) = \{w_1, \dots, w_t\}$. This means that $t \geq \Delta_f(G) + 1$. By the minimality of F, we can assume that for every component S of $G_1 \setminus \{u\}$, we have $|N_{G_1}(u) \cap V(S)| \geq k \geq 2$. Let D be one of the components of $G_1 \setminus \{u\}$ such that $w_1, w_t \in V(D)$. Add two new vertices x and y to $G_1 \setminus \{u\}$ and join x and y to $\{w_1, \dots, w_{\Delta_f(G)-k}\}$ and $\{w_{\Delta_f(G)-k+1}, \dots, w_t\}$, respectively. Then call the resultant graph by H. Clearly, $d_H(x) = \Delta_f(G) - k$ and $d_H(y) = t - (\Delta_f(G) - k) = (d_G(u) - k) - (\Delta_f(G) - k) = d_G(u) - \Delta_f(G)$. Also, add a new vertex z to G_2 and join it to $\{v_1, \dots, v_k\}$ and call it by K. Let $f' : V(H \cup K) \longrightarrow \mathbb{N}$ be a function defined by

$$f'(v) = \begin{cases} f(v) & v \in V(G), \\ 1 & v \in \{x, z\}, \\ f(u) - 1 & v = y. \end{cases}$$

Note that H and K are connected. Moreover, $\max\{\Delta_{f'}(H), \Delta_{f'}(K)\} \leq \Delta_f(G)$, because

$$\frac{d_G(v)}{f'(v)} = \begin{cases} \frac{d_G(v)}{f(v)} \leq \Delta_f(G) & v \in V(G), \\ \Delta_f(G) - k < \Delta_f(G) & v = x, \\ k \leq \Delta_f(G) - 2 < \Delta_f(G) & v = z, \\ \frac{d_G(u)-\Delta_f(G)}{f(u)-1} = \frac{f(u)\Delta_f(G)-1-\Delta_f(G)}{f(u)-1} < \Delta_f(G) & v = y. \end{cases}$$

and since $|V(G_i) \cap V(G_{\Delta_f})| \geq 2$, for $i = 1, 2$, $\Delta_{f'}(H) = \Delta_{f'}(K) = \Delta_f(G)$. Moreover, note that by adding the new vertices x, y and z, $d_H(v) = d_G(v)$ and $d_K(v) = d_G(v)$ for every $v \in V(G) \setminus \{u\}$. This implies that $\Delta(H_{\Delta_{f'}}) = \Delta(K_{\Delta_{f'}}) = \Delta(G_{\Delta_f})$.

We claim that both H and K are f'-Class 1. Note that if H is f'-Class 2, then by Theorem 1.9, $d_H(x) = f'(x)\Delta_{f'}(H) - 1 = \Delta_f(G) - 1$, but $d_H(x) = \Delta_f(G) - k \leq \Delta_f(G) - 2$, a contradiction. So, there exists an f'-coloring ϕ of H by colors $\{1, \ldots, \Delta_{f'}(H)\}$. Similarly, there is an f'-coloring θ of K by colors $\{1, \ldots, \Delta_{f'}(K)\}$ and the claim is proved.

By a suitable permutation of colors, one may assume that

$$\{\phi(xw_1), \ldots, \phi(xw_{\Delta_f(G)-k}), \theta(zv_1), \ldots, \theta(zv_k)\}$$

are distinct. Now, define an f-coloring $c : E(G) \longrightarrow \{1, \ldots, \Delta_f(G)\}$ as follows:

$$\begin{cases} c(e) = \phi(e) & \text{for every } \ e \in E(G_1 \setminus \{u\}), \\ c(e') = \theta(e') & \text{for every } \ e' \in E(G_2), \\ c(uv_i) = \theta(zv_i) & \text{for } \ i = 1, \ldots, k, \\ c(uw_i) = \phi(xw_i) & \text{for } \ i = 1, \ldots, \Delta_f(G) - k, \\ c(uw_i) = \phi(yw_i) & \text{for } \ i = \Delta_f(G) - k + 1, \ldots, t. \end{cases}$$

Since $f'(y) = f(u) - 1$, we conclude that G is f-Class 1 which is a contradiction and the proof is complete. □

Now, we want to prove another result on f-coloring of graphs which classifies some families of f-Class 1 graphs. We need the following lemma subsequently.

Lemma 2.3 [7] *Let C denote the set of colors available to color the edges of a simple graph G. Suppose that $e = uv$ is an uncolored edge in G, and graph $G \setminus \{e\}$ is f-colored with the colors in C. If for every neighbor x of either u or v, there exists a color α_x which appears at most $f(x) - 1$ times at vertex x, then there exists an f-coloring of G using colors of C.*

In fact the following result can be derived either from the main result in [7] and Theorem 1.9, or from the main result of [9]. We give a proof here, which is distinct from the above.

Theorem 2.2 *Let G be a connected graph. If every connected component of G_{Δ_f} is a unicyclic graph or a tree and G_{Δ_f} is not 2-regular, then G is f-Class 1.*

Proof. First suppose that $\Delta(G_{\Delta_f}) \leq 2$. For a contradiction, assume that G is f-Class 2. By Theorem 1.9, G_{Δ_f} is 2-regular, which is a contradiction. So one may suppose that $\Delta(G_{\Delta_f}) \geq 3$. Now the proof is by induction on $m = |E(G_{\Delta_f})|$. Since $\Delta(G_{\Delta_f}) \geq 3$, we have $m \geq 3$. First assume that $m = 3$. Since G_{Δ_f} is not 2-regular and $\Delta(G_{\Delta_f}) \geq 3$, we have $G_{\Delta_f} = K_{1,3}$. Now, by Theorem 1.4, G is f-Class 1 and we are done.

Now let G be a graph and let $t = |E(G_{\Delta_f})|$. Assume that the assertion holds for all graphs with fewer than m edges, where $m < t$. Note that since $\Delta(G_{\Delta_f}) \geq 3$ and

G_{Δ_f} is not 2-regular, there exists an edge $e = uv \in E(G_{\Delta_f})$ such that $d_{G_{\Delta_f}}(v) = 1$ and $d_{G_{\Delta_f}}(u) \geq 2$. Let $H = G \setminus \{e\}$ with the function $f : V(G) \to \mathbb{N}$. We would like to show that H is f-Class 1. Two cases may occur.

First assume that H is connected. If $\Delta(H_{\Delta_f}) \geq 3$, then by the induction hypothesis we are done. If $\Delta(H_{\Delta_f}) \leq 2$ and H_{Δ_f} is not 2-regular, then by Theorem 1.9, H is f-Class 1. Thus assume that H_{Δ_f} is 2-regular. Note that by deleting the edge $e = uv$, it is not hard to see that since $\Delta(G_{\Delta_f}) \geq 3$, G_{Δ_f} is a disjoint union of some cycles and the graph shown in the following figure:

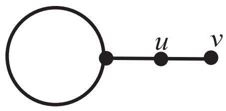

Figure 2: A part of G_{Δ_f}

Now, by Theorem 1.9, H is f-critical and so by Theorem 1.5, u should have at least two neighbors in H_{Δ_f}, a contradiction.

Next assume that H is not connected. Let P and Q be two connected components of H such that $u \in V(P)$ and $v \in V(Q)$. Since $d_{G_{\Delta_f}}(u) \geq 2$, we have $\Delta_f(P) = \Delta_f(G)$. Now, if $\Delta(P_{\Delta_f}) \geq 3$, then by the induction hypothesis, P is f-Class 1. If $\Delta(P_{\Delta_f}) \leq 2$ and P_{Δ_f} is not 2-regular, then by Theorem 1.9, P is f-Class 1. Thus assume that P_{Δ_f} is 2-regular. Then it is not hard to see that G_{Δ_f} is the disjoint union of some unicycles, trees and the graph shown in the Figure 2. Now, by Theorem 1.9, P is f-critical and so by Theorem 1.5, u should have at least two neighbors in P_{Δ_f}, a contradiction and P is f-Class 1. Now, we want to show that Q is f-Class 1, too. First note that if $\Delta_f(Q) < \Delta_f(G)$, then by Theorem 1.1, Q has an f-coloring with colors $\{1, \ldots, \Delta_f(G)\}$. So, assume that $\Delta_f(Q) = \Delta_f(G)$. Now, if $Q_{\Delta_f} = \emptyset$, then by Theorem 2.2, Q is f-Class 1. If $\Delta(Q_{\Delta_f}) \geq 3$, then Q is f-Class 1 by the induction hypothesis. Thus, we can assume that $\Delta(Q_{\Delta_f}) \leq 2$. Now, if Q_{Δ_f} is not 2-regular, then by Theorem 1.9, Q is f-Class 1. Thus assume that Q_{Δ_f} is 2-regular. Then it is not hard to see that G_{Δ_f} is the disjoint union of some unicycles, trees and the graph shown in the Figure 2. Now, by Theorem 1.9, Q is f-critical and so by Theorem 1.5, v should have at least two neighbors in Q_{Δ_f}, a contradiction and Q is f-Class 1. Now, since for every $x \in N_G(v) \setminus \{u\}$, we have $x \notin V(G_{\Delta_f})$, there exists a color α_x which appears at most $f(x) - 1$ times in x and so by Lemma 2.3, G is f-Class 1 and we are done. □

Theorem 2.3 *Let G be a connected claw-free graph with $\Delta(G_{\Delta_f}) \leq 2$. If there exists a vertex $v \in V(G)$ such that $f(v) \neq 1$ and $G \neq W$, where W is the graph shown in Figure 3, then G is f-Class 1.*

Proof. For a contradiction assume that G is f-Class 2. Then by Theorem 1.9, G is f-critical and G_{Δ_f} is 2-regular. Now, by Theorem 1.5, $f(u) = 1$, for every

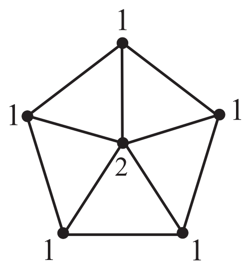

Figure 3: The graph W (the value of each vertex z denotes $f(z)$)

$u \in V(G_{\Delta_f})$ and so by the definition we have

$$d_G(u) = \Delta_f(G), \text{ for every } u \in V(G_{\Delta_f}). \tag{1}$$

Note that, if $\Delta_f(G) = 2$, then since G is connected and G_{Δ_f} is 2-regular, $G = G_{\Delta_f}$ and there is no vertex v with $f(v) \neq 1$. Thus we can assume that

$$\Delta_f(G) \geq 3. \tag{2}$$

Let $H = G \setminus G_{\Delta_f}$. Now, if $|N_{G_{\Delta_f}}(x)| \geq 7$, for some $x \in V(H)$, then clearly there exists an independent set of size 3 in $N_{G_{\Delta_f}}(x)$ which implies that G has a claw, a contradiction. Thus we have

$$|N_{G_{\Delta_f}}(x)| \leq 6, \text{ for every } x \in V(H). \tag{3}$$

Now, to prove the theorem, first we need the following claim:

Claim 1. $f(z) \leq 2$, for every $z \in V(G)$.

Proof of Claim 1. To see this for a contradiction, assume that there exists a vertex $z \in V(G)$ such that $f(z) \geq 3$. By Theorem 1.5, for every $u \in V(G_{\Delta_f})$, $f(u) = 1$. Thus, $z \in V(H)$. Now, by (3) and Theorem 1.5, we conclude that $|N_{G_{\Delta_f}}(z)| = 6$ and $f(z) = 3$. Then by Theorem 1.9, $d_G(z) = 3\Delta_f(G) - 1$ and so $d_H(z) = 3\Delta_f(G) - 7$. Now, we want to show that for every $w \in N_H(z)$,

$$|N_{G_{\Delta_f}}(w) \cap N_{G_{\Delta_f}}(z)| \geq 3.$$

Suppose otherwise and note that there are at least 4 vertices, say $u_1, u_2, u_3, u_4 \in N_{G_{\Delta_f}}(z)$, such that $wu_i \notin E(G)$, for $i = 1, \ldots, 4$. Since G_{Δ_f} is 2-regular, with no loss of generality, we can assume that $u_1u_2 \notin E(G)$. Then $\langle\{u_1, u_2, w, z\}\rangle$ is a claw, a contradiction. Thus, we conclude that for every $w \in N_H(z)$, $|N_{G_{\Delta_f}}(w) \cap N_{G_{\Delta_f}}(z)| \geq 3$ and so $e_G(N_{G_{\Delta_f}}(z), N_H(z)) \geq 3(3\Delta_f(G) - 7)$. Moreover, since for every $u \in V(G_{\Delta_f})$, $d_G(u) = \Delta_f(G)$, we conclude that $e_G(N_{G_{\Delta_f}}(z), N_H(z)) \leq 6(\Delta_f(G) - 3)$. Thus, $3(3\Delta_f(G) - 7) \leq e_G(N_{G_{\Delta_f}}(z), N_H(z)) \leq 6(\Delta_f(G) - 3)$, which yields that $\Delta_f(G) \leq 1$, a contradiction and the claim is proved.

Now, by the assumption of the theorem and Claim 1, we can assume that there exists a vertex $v \in V(H)$ such that $f(v) = 2$. Then, by Theorem 1.9, $d_G(v) =$

$2\Delta_f(G)-1$. By Theorem 1.5 and using (3), we have $4 \le |N_{G_{\Delta_f}}(v)| \le 6$. Thus, three cases may occur:

Case 1. $|N_{G_{\Delta_f}}(v)| = 4$.
Let $N_{G_{\Delta_f}}(v) = \{u_1, \dots, u_4\}$ and $N_H(v) = \{w_1, \dots, w_{2\Delta_f(G)-5}\}$. Since G_{Δ_f} is 2-regular, with no loss of generality, there are two non-adjacent vertices $u_1, u_2 \in N_{G_{\Delta_f}}(v)$. Since G is claw-free, $u_1w_i \in E(G)$ or $u_2w_i \in E(G)$, for $i = 1, \dots, 2\Delta_f(G)-5$. Thus, $2\Delta_f(G) - 5 \le e_G(N_H(v), \{u_1, u_2\}) \le 2(\Delta_f(G) - 3)$, a contradiction.

Case 2. $|N_{G_{\Delta_f}}(v)| = 5$.
Let $N_{G_{\Delta_f}}(v) = \{u_1, \dots, u_5\}$ and $N_H(v) = \{w_1, \dots, w_{2\Delta_f(G)-6}\}$. First note that since G is claw-free, $N_{G_{\Delta_f}}(v)$ does not contain an independent set of size 3 and so it can be easily checked that $\langle N_{G_{\Delta_f}}(v)\rangle$ is one of two following graphs:

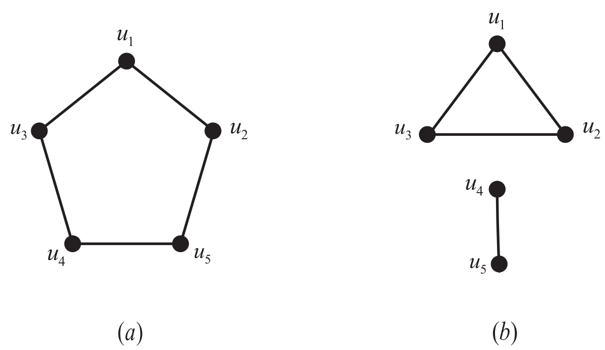

Figure 4: $\langle N_{G_{\Delta_f}}(v)\rangle$ when $|\langle N_{G_{\Delta_f}}(v)\rangle| = 5$

Three subcases may occur:

(i) $\Delta_f(G) = 3$.
We have $d_G(v) = 2\Delta_f(G) - 1 = 5$. Now, if $\langle N_{G_{\Delta_f}}(v)\rangle = C_5$, then G is the graph W shown in 3, a contradiction. Thus, assume that $\langle N_{G_{\Delta_f}}(v)\rangle$ is the graph shown in Figure 4(b). By Theorem 1.9, since G_{Δ_f} is 2-regular, there exists $u_6 \in N_{G_{\Delta_f}}(u_5)\backslash\{u_4\}$ and $u_7 \in N_{G_{\Delta_f}}(u_4)\backslash\{u_5\}$. Now, we divide the proof of this subcase into two parts:

• $u_6 \ne u_7$. Let $L = G\backslash\{v, u_1, \dots, u_5\}$. Now, add a new vertex x to L and join x to u_6 and u_7. Call the resultant graph L'. Let $f' : V(L') \longrightarrow \mathbb{N}$ be a function defined by

$$f'(z) = \begin{cases} f(z) & z \in V(L), \\ 1 & z = x. \end{cases}$$

Note that since $\Delta_f(G) = 3$, we have $d_G(u) = 3$, for every $u \in V(G_{\Delta_f})$. Now, since $d_G(v) = 5$, we conclude that L' is connected and $\Delta_{f'}(L') = \Delta_f(G) = 3$. Note that since $d_{L'}(x) = 2$, we have $x \notin V(L'_{\Delta_{f'}})$ and so $\delta(L'_{\Delta_{f'}}) = 1$. Now, since $\Delta(L'_{\Delta_{f'}}) \le 2$ and $L'_{\Delta_{f'}}$ is not 2-regular, by Theorem 1.9, L' has an f'-coloring call θ, with colors $\{1, 2, 3\}$. Without loss of generality, assume that $\theta(xu_7) = 1$ and $\theta(xu_6) = 2$. Now,

define an f-coloring $c : E(G) \rightarrow \{1, 2, 3\}$ as follows.
Define $c(e) = \theta(e)$, for every $e \in E(L)$ and

$$\begin{cases} c(u_4u_7) = c(vu_1) = c(vu_5) = c(u_2u_3) = 1 \\ c(u_5u_6) = c(vu_4) = c(vu_3) = c(u_1u_2) = 2 \\ c(u_4u_5) = c(vu_2) = c(u_1u_3) = 3. \end{cases}$$

• $u_6 = u_7$. Since $\Delta_f(G) = 3$, we have $d_G(u_6) = 3$ and so u_6 has a neighbor t, where $t \notin \{v, u_1, \dots, u_5\}$. Noting that $d_G(v) = 5$ and $d_G(u_i) = 3$, for $i = 1, \dots, 6$, we conclude that tu_6 is a cut edge for G and by Theorem 2.1, G is f-class 1, a contradiction.

(ii) $\Delta_f(G) = 4$.
Clearly, $d_G(v) = 2\Delta_f(G) - 1 = 7$ and $N_H(v) = \{w_1, w_2\}$. Now, we divide the proof of this subcase into two parts:

• $\langle N_{G_{\Delta_f}}(v)\rangle$ is the graph shown in Figure 4(a).
Since G is claw-free, noting that $u_1u_4 \notin E(G)$, we have $u_1w_1 \in E(G)$ or $u_4w_1 \in E(G)$. Without loss of generality assume that $u_1w_1 \in E(G)$. Moreover, since $\langle\{v, u_1, u_4, w_2\}\rangle$ is not a claw and $N_G(u_1) = \{v, u_2, u_3, w_1\}$, we have $u_4w_2 \in E(G)$. Similarly, since $\langle\{v, u_1, u_5, w_2\}\rangle$ is not a claw and $N_G(u_1) = \{v, u_2, u_3, w_1\}$, we conclude that $u_5w_2 \in E(G)$. Also, since $\langle\{v, u_2, u_4, w_1\}\rangle$ is not a claw and $N_G(u_4) = \{v, u_3, u_5, w_2\}$, we obtain that $u_2w_1 \in E(G)$. Moreover, since $\langle\{v, u_3, u_5, w_1\}\rangle$ is not a claw and $N_G(u_5) = \{v, u_2, u_4, w_2\}$, $u_3w_1 \in E(G)$. Clearly, $\langle\{v, u_2, u_3, w_2\}\rangle$ is a claw which is a contradiction.

• $\langle N_{G_{\Delta_f}}(v)\rangle$ is the graph shown in Figure 4(b). Similar to the previous argument, one can assume that $\{u_1w_1, u_2w_1, u_3w_1, u_4w_2, u_5w_2\} \subseteq E(G)$. Now, since $d_G(w_1) \geq 4$, $f(w_1) \geq 2$. By Claim 1 we conclude that $f(w_1) = 2$ and so by Theorem 1.9, $d_G(w_1) = 7$. Assume that $N_G(w_1) = \{v, v_1, v_2, v_3, u_1, u_2, u_3\}$. Note that since $d_G(u_4) = d_G(u_5) = 4$ and $\{v, w_2\} \subseteq N_G\{u_4, u_5\}$, we conclude that $\{v_1, v_2, v_3\} \cap \{u_4, u_5\} = \varnothing$. Now, by Theorem 1.5, we have $|N_{G_{\Delta_f}}(w_1)| \geq 4$. If $|N_{G_{\Delta_f}}(w_1)| = 4$, then by Case 1, we are done. So, we can assume that $|N_{G_{\Delta_f}}(w_1)| \geq 5$. Without loss of generality, assume that

$$v_1, v_2 \in V(G_{\Delta_f}). \tag{4}$$

Also, since $\langle\{u_1, v_i, v_j, w_1\}\rangle$ is not a claw, for $i, j = 1, 2, 3$, $i \neq j$ and $N_G(u_1) = \{v, u_2, u_3, w_1\}$, we obtain that

$$\langle\{v_1, v_2, v_3\}\rangle = K_3, \tag{5}$$

Now, we claim that $v_3 \neq w_2$. For a contradiction assume that $v_3 = w_2$. Then $d_G(w_2) \geq 6$ and since $w_2 \notin V(G_{\Delta_f})$, we have $f(w_2) = 2$. Let $N_G(w_2) = \{v, v_1, v_2, u_4, u_5, w_1, y\}$, where $y \notin \{u_1, u_2, u_3\}$. Since $\langle\{v_1, u_4, w_2, y\}\rangle$ and $\langle\{v_1, u_5, w_2, y\}\rangle$ are not claws, we conclude that $\langle\{u_4, u_5, y\}\rangle$ is a K_3 in G_{Δ_f} and so $yv_1 \notin E(G)$. Then $\langle\{v, v_1, w_2, y\}\rangle$ is a claw, a contradiction and the claim holds. Consider $L = G \setminus$

$\{v, u_1, u_2, u_3, w_1\}$. Add a new vertex x to L and join x to u_5, w_2, v_2, v_3. Call the resultant graph L'.

Let $f' : V(L') \longrightarrow \mathbb{N}$ be a function defined by

$$f'(z) = \begin{cases} f(z) & z \in V(L), \\ 1 & z = x. \end{cases}$$

Clearly, by (5), L' is connected and $v_1, u_4 \notin V(L'_{\Delta_{f'}})$. If $v_3 \notin V(G_{\Delta_f})$, then clearly $\delta(L'_{\Delta_{f'}}) = 1$ and $\Delta(L'_{\Delta_{f'}}) \leq 2$ and by Theorem 1.9, L' is f'-Class 1. So assume that $v_3 \in V(G_{\Delta_f})$. Clearly, $L'_{\Delta_{f'}}$ is not 2-regular and each of the components is a unicyclic graph or a tree. By Theorem 2.2, L' has an f'-coloring, say θ, with colors $\{1, 2, 3, 4\}$. Without loss of generality, assume that $\theta(xu_5) = 1$, $\theta(xw_2) = 2$, $\theta(xv_3) = 3$ and $\theta(xv_2) = 4$. Define an f-coloring $c : E(G) \to \{1, 2, 3, 4\}$ as follows.

Let $c(e) = \theta(e)$, for every $e \in E(L)$, $c(vu_5) = 1$, $c(vw_2) = 2$, $c(v_3w_1) = 3$, $c(v_2w_1) = 4$ and $c(vu_4) = a$, $c(v_1w_1) = b$, where a and b are the colors missed in coloring θ in u_4 and v_1, respectively.

By a suitable f-coloring of $\langle\{v, u_1, u_2, u_3, w_1\}\rangle$, we extend the f'-coloring θ of L' to an f-coloring c of G, using four colorings given in Figure 5. For $(a, b) = (2, 4)$, the Figure 5(i) works. If $(a, b) = (1, 3)$, then interchange two colors 1 and 2, and two colors 3 and 4 in igure 5(i). For $(a, b) = (1, 4)$ or $(a, b) = (2, 3)$, interchange two colors 1 and 2, and two colors 3 and 4 in Figure 5(i), respectively. For $(a, b) \in \{(4, 2), (4, 1), (3, 2)\}$, we can use the same method given in Figure 5(ii). If $a, b \in \{3, 4\}$, then 5(iii) works. For $a, b \in \{1, 2\}$, Figure 5(iv) works, and for $(a, b) = (3, 1)$, Figure 5(v) works.

(iii) $\Delta_f(G) \geq 5$.
Consider $G \setminus \{v\}$. Now, add two new vertices v_1 and v_2 to $G \setminus \{v\}$, join v_1 to $\{u_1, w_1, \ldots, w_{\Delta_f(G)-1}\}$ and v_2 to $\{u_2, \ldots, u_5, w_{\Delta_f(G)}, \ldots, w_{2\Delta_f(G)-6}\}$. Call the resultant graph by L. Let $f' : V(L) \longrightarrow \mathbb{N}$ be a function defined by

$$f'(z) = \begin{cases} f(z) & z \in V(G) \setminus \{v\}, \\ 1 & z \in \{v_1, v_2\}. \end{cases}$$

It is easy to see that L is connected, $\Delta_{f'}(L) = \Delta_f(G)$ and $V(L_{\Delta_{f'}}) = V(G_{\Delta_f}) \cup \{v_1\}$. Noting that $|N_{L_{\Delta_{f'}}}(v_1)| = 1$ and using Theorem 2.2, L has an f'-coloring with colors $\{1, \ldots, \Delta_{f'}(L)\}$, call θ. Now, define an f-coloring $c : E(G) \longrightarrow \{1, \ldots, \Delta_f(G)\}$ as follows. Let

$$\begin{cases} c(e) = \theta(e) & \text{for every } e \in E(G \setminus \{v\}) \\ c(vu_1) = \theta(u_1v_1) & \\ c(vu_i) = \theta(u_iv_2) & \text{for } i = 2, \ldots, 5 \\ c(vw_i) = \theta(v_1w_i) & \text{for } i = 1, \ldots, \Delta_f(G) - 1 \\ c(vw_i) = \theta(v_2w_i) & \text{for } i = \Delta_f(G), \ldots, 2\Delta_f(G) - 6. \end{cases}$$

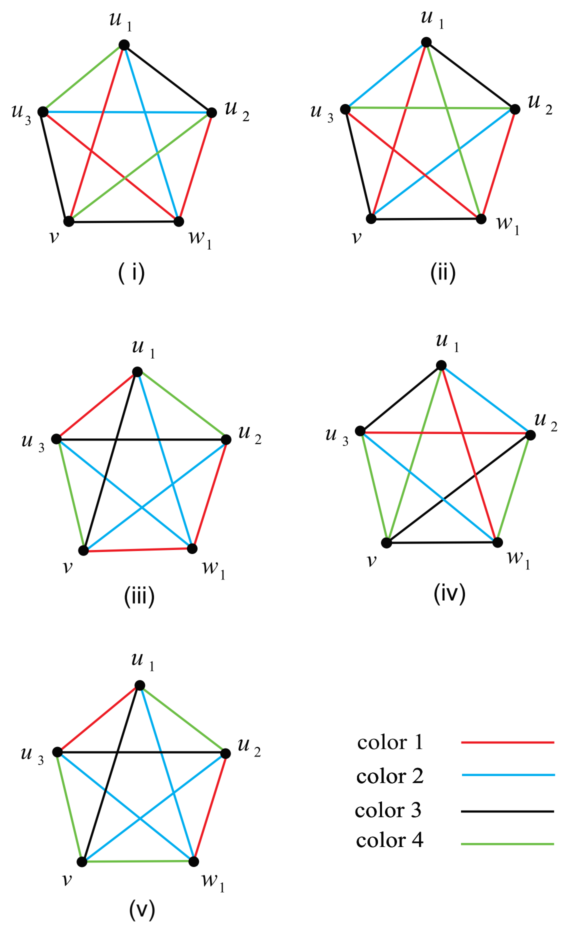

Figure 5: 4-edge coloring of $\langle\{v, u_1, u_2, u_3, w_1\}\rangle$

This implies that G is f-Class 1, a contradiction.

Case 3. $|N_{G_{\Delta_f}}(v)| = 6$.
First note that if there exists a vertex v with $f(v) = 2$ such that $|N_{G_{\Delta_f}}(v)| \leq 5$, then by Cases 1 and 2 we are done. Thus, we can suppose that for every vertex v with $f(v) = 2$, we have $|N_{G_{\Delta_f}}(v)| = 6$. Let $N_{G_{\Delta_f}}(v) = \{u_1, \ldots, u_6\}$ and $N_H(v) = \{w_1, \ldots, w_{2\Delta_f(G)-7}\}$. Since G is claw-free, every induced subgraph of order 3 of $\langle N_{G_{\Delta_f}}(v)\rangle$ has at least one edge. Thus $\langle N_{G_{\Delta_f}}(v)\rangle$ is disjoint union of two K_3. Without loss of generality, assume that

$$\langle\{u_1, u_2, u_3\}\rangle \simeq \langle\{u_4, u_5, u_6\}\rangle \simeq K_3. \tag{6}$$

Thus, one can assume that:

$$\text{for every vertex } x \text{ with } f(x) = 2,\ \langle N_{G_{\Delta_f}}(x)\rangle \text{ is the disjoint union of two } K_3. \tag{7}$$

Clearly, since $d_G(v) = 2\Delta_f(G) - 1 \geq 6$, we conclude that $\Delta_f(G) \geq 4$. Now, three cases may be considered:

(i) $\Delta_f(G) = 4$.
Clearly, $d_G(v) = 2\Delta_f(G) - 1 = 7$ and $N_H(v) = \{w_1\}$. We claim that $|N_{G_{\Delta_f}}(v) \cap$

$N_{G_{\Delta_f}}(w_1)| \geq 3$. Otherwise, $w_1u_{i_j} \notin E(G)$, for $j = 1, \dots, 4$, where $u_{i_j} \in N_{G_{\Delta_f}}(v)$. By (7), and without loss of generality, we can assume that $u_{i_1}u_{i_2} \notin E(G)$. Then $\langle\{v, w_1, u_{i_1}, u_{i_2}\}\rangle$ is a claw, a contradiction. Now, we divide the proof of this subcase into two parts:

• $|N_{G_{\Delta_f}}(v) \cap N_{G_{\Delta_f}}(w_1)| \geq 4$. Then, $d_G(w_1) \geq 5$ and since $\Delta_f(G) = 4$, we conclude that $f(w_1) \geq 2$ and by Claim 1 we find that $f(w_1) = 2$. Now, using (7), $\langle N_{G_{\Delta_f}}(w_1)\rangle$ is disjoint union of two K_3. Since $|N_{G_{\Delta_f}}(w_1) \cap \{u_1, u_2, u_3\}| \geq 1$ and $|N_{G_{\Delta_f}}(w_1) \cap \{u_4, u_5, u_6\}| \geq 1$, we conclude that $N_{G_{\Delta_f}}(w_1) = \{u_1, \dots, u_6\}$. Then, it is easy to see that G is the graph shown in the following figure which is colored with $\Delta_f(G) = 4$ colors and the proof of this subcase is complete.

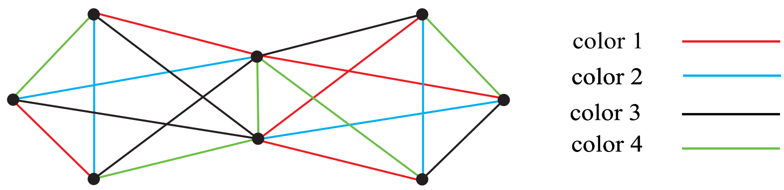

Figure 6: An f-coloring of G with 4 colors

• $|N_{G_{\Delta_f}}(v) \cap N_{G_{\Delta_f}}(w_1)| = 3$. Since $d_G(w_1) \geq 4$ and $w_1 \notin V(G_{\Delta_f})$, $f(w_1) = 2$. Using (7), without loss of generality we can assume that $N_{G_{\Delta_f}}(w_1) \cap N_{G_{\Delta_f}}(v) = \{u_1, u_2, u_3\}$, and there are three vertices, say $x_1, x_2, x_3 \in N_{G_{\Delta_f}}(w_1)$, such that $\langle\{x_1, x_2, x_3\}\rangle \simeq K_3$. Consider $L = G \setminus \{u_1, u_2, u_3, vw_1\}$. Let $f' : V(L) \longrightarrow \mathbb{N}$ be a function defined by $f'(z) = f(z)$, for every $z \in V(L)$. Now we want to prove the following claim which introduces a coloring of L with some properties.

Claim 2. L has an f'-coloring c with four colors $\{1, 2, 3, 4\}$ such that

$$|\{c(w_1x_1), c(w_1x_2), c(w_1x_3), c(vu_4), c(vu_5), c(vu_6)\}| = 4.$$

Proof of Claim 2. We consider two cases.

First assume that L is not connected. So, L has two connected components, one of them containing v and another containing w_1. It is easy to see that for every connected component I of L, $\Delta_{f'}(I) = \Delta_{f'}(L) = \Delta_f(G)$ and so $\Delta(I_{\Delta_{f'}}) = 2$. Now, since $f'(v) = f'(w_1) = 2$ and $d_L(v) = d_L(w_1) = 3$, by Theorem 1.9, every component of L is f'-class 1. Moreover, noting that $f'(v) = f'(w_1) = 2$, we obtain that there are at least two distinct colors appeared in the edges incident with v and also with w_1. Now, by a suitable permutation of colors on these edges in one of components, Claim 2 is proved.

Now, assume that L is connected. Consider $K = L \setminus \{w_1, x_1x_2, x_2x_3, x_1x_3\}$. Let $f'' : V(K) \longrightarrow \mathbb{N}$ be a function defined by

$$f''(z) = \begin{cases} f'(z) & z \in V(L) \setminus \{w_1\}, \\ 1 & z = v. \end{cases}$$

We want to show that K is f''-Class 1. It is not hard to see that every connected component of K has at least one of the three vertices $\{x_1, x_2, x_3\}$. Let J be a connected component of K. If $\Delta_{f''}(J) < \Delta_{f''}(K)$, then by Theorem 1.1, J has an f''-coloring with 4 colors. So, assume that $\Delta_{f''}(J) = \Delta_{f''}(K) = 4$. Now, since there exists $x_i \in V(J)$, for some $i \in \{1, 2, 3\}$ and noting that $d_J(x_i) = 1$, by Theorem 1.9, J is f''-Class 1 and so K has an f''-coloring with 4 colors $\{1, 2, 3, 4\}$, call θ. Let $N_K(x_1) = \{y_1\}$, $N_K(x_2) = \{y_2\}$ and $N_K(x_3) = \{y_3\}$. We can assume that

$$|\{\theta(x_1y_1), \theta(x_2y_2), \theta(x_3y_3)\}| \geq 2. \tag{8}$$

Because otherwise, we have $|\{\theta(x_1y_1), \theta(x_2y_2), \theta(x_3y_3)\}| = 1$. Now, since for every vertex $u \in V(G)$, $f(u) \leq 2$, we conclude that $|\{y_1, y_2, y_3\}| \geq 2$. Without loss of generality, one can suppose that y_1 is not adjacent to x_2 and x_3. Using (7), we find that $f(y_1) = 1$ and so $f''(y_1) = 1$. Thus since $d_K(y_1) = \Delta_{f''}(K) - 1 = 3$, there is a missed color call α in y_1 different from $\theta(x_1y_1)$. One can replace $\theta(x_1y_1)$ by α.

Now, without loss of generality, and noting that $f''(v) = 1$, one can assume that $\theta(vu_4) = 1$, $\theta(vu_5) = 2$, $\theta(vu_6) = 3$, $\theta(x_1y_1) = \alpha$, $\theta(x_2y_2) = \beta$ and $\theta(x_3y_3) = \gamma$. Now, to prove Claim 2, it suffices to extend the f''-coloring of K to an f'-coloring of L. To see this, in Figure 7 we introduce such a suitable coloring for $\langle\{w_1, x_1, x_2, x_3\}\rangle \cup \{x_1y_1, x_2y_2, x_3y_3\}$.

Note that if $\alpha = \beta = 1$ and $\gamma = 4$ and $f''(y_1) = 1$, then there is a missed color in y_1 different from 1. Now, by changing color w_1x_1 by this missed color, similar to one of the coloring of graphs shown in Figure 7. If $f''(y_1) = 2$, then $y_1 = y_2 = y_3$, by (7). So there is a color, say l, appeared in the neighbors y_1 once. Now, by changing the color w_1x_1 to l we obtain one of the cases given in Figure 7.

We can easily color $\langle\{v, u_1, u_2, u_3, w_1\}\rangle$ by colors $\{1, 2, 3, 4\}$ similar to one of the graphs in Figure 5. This implies that G is f-Class 1 and we are done.

(ii) $\Delta_f(G) = 5$.
By (6), $u_1u_4 \notin E(G)$. Thus $u_1w_1 \in E(G)$ or $u_4w_1 \in E(G)$. Without loss of generality, assume that $u_1w_1 \in E(G)$. Since two graphs $\langle\{v, u_1, u_4, w_2\}\rangle$ and $\langle\{v, u_1, u_4, w_3\}\rangle$ are not claws and $d_G(u_1) = 5$, with no loss of generality, we can suppose that $u_1w_2 \in E(G)$ and $u_4w_3 \in E(G)$. Moreover, since $\langle\{v, u_1, u_5, w_3\}\rangle$ and $\langle\{v, u_1, u_6, w_3\}\rangle$ are not claws and $N_G(u_1) = \{v, u_2, u_3, w_1, w_2\}$, we have $u_5w_3, u_6w_3 \in E(G)$. Now, we want to show that

$$u_iw_j \in E(G), \text{ for } i = 2, 3 \text{ and } j = 1, 2. \tag{9}$$

For a contradiction and with no loss of generality assume that $u_2w_1 \notin E(G)$. Then since $\langle\{v, u_2, u_i, w_1\}\rangle$ is not a claw, we have $u_iw_1 \in E(G)$, for $i = 4, 5, 6$. This implies that $d_G(w_1) \geq 5$ and since $\Delta_f(G) = 5$, we conclude that $f(w_1) = 2$. Now, by (7), $u_2w_1 \in E(G)$, a contradiction. Similarly, other cases of (9) hold.

Now, we would like to show that G is f-Class 1. Two cases may occur:

• $w_1w_2 \notin E(G)$.
Since $\langle\{v, u_4, w_1, w_2\}\rangle$ is not a claw, with no loss of generality, $u_4w_1 \in E(G)$ and so

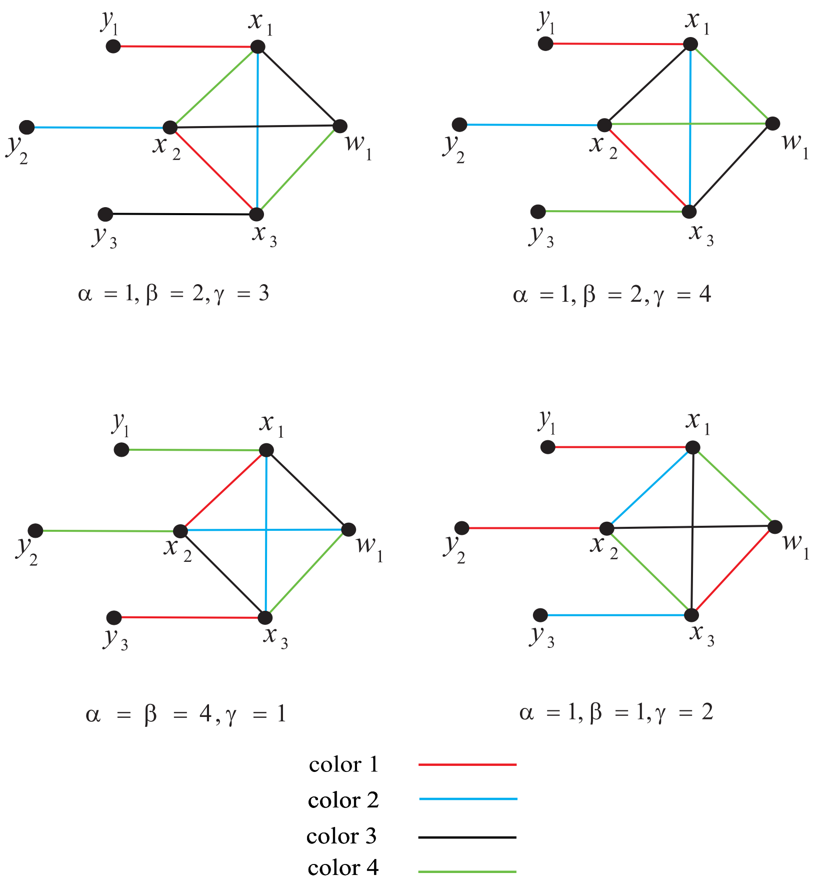

Figure 7: A 4-edge coloring of $\langle\{w_1, x_1, x_2, x_3\}\rangle \cup \{x_1y_1, x_2y_2, x_3y_3\}$

$d_G(w_1) \geq 5$, which implies that $f(w_1) = 2$ and by (7), $u_5w_1, u_6w_1 \in E(G)$. Since $d_G(w_1) = 9$, there exists a vertex $z \in N_G(w_1)\setminus\{v, u_1, \ldots, u_6, w_3\}$ and $\langle\{z, u_1, u_4, w_1\}\rangle$ is a claw, a contradiction and the proof of this case is complete.

- $w_1w_2 \in E(G)$.

Clearly, $\langle\{v, u_1, u_2, u_3, w_1, w_2\}\rangle \simeq K_6$ and so

$$\text{for every vertex } v \text{ with } f(v) = 2,\ v \text{ is contained in a } K_6. \tag{10}$$

Note that since $d_G(w_i) \geq 5$ and $w_i \notin V(G_{\Delta_f})$, by Claim 1 we conclude that $f(w_i) = 2$, for $i = 1, 2$. Let P be the induced subgraph on the union of vertices of all K_6 in G. First note that three vertices of each K_6 have degree 5 in G. This implies that every two K_6 have at most three vertices in common. Also, every two K_6 have not one vertex in common, because otherwise there exists a vertex of degree 10 in G. On the other hand, every two K_6 have not three vertices in common, because otherwise there exists a vertex $v \in V(P)$ such that $d_P(v) = 8$ and it is not hard to see that v is a center of a claw in G, a contradiction. Thus, the vertex set of every two K_6 have empty intersection or they have exactly two vertices in common. Hence each connected component of P is one of the graphs in Figure 8.

Define $f' : V(P) \longrightarrow \mathbb{N}$ as follows:

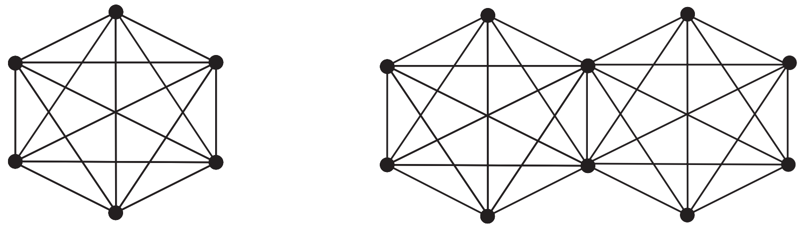

Figure 8: Every component of the graph P

$$f'(z) = \begin{cases} 1 & if \;\; d_P(z) = 5 \\ 2 & if \;\; d_P(z) = 9. \end{cases}$$

It is not hard to see that P has an f'-coloring with colors $\{1, \ldots, 5\}$.

Now, let $L = G \setminus E(P)$. We would like to prove the following claim.

Claim 3. $\chi'(L) = 5$.

If the claim is proved, then we color all edges of L and P by 5 colors to obtain an f-coloring of G. Since for every vertex v which are incident to some edges in L and P, we have $f(v) = 2$, we find an f-coloring of G using 5 colors.

Proof of Claim 3. Clearly, the maximum degree of each connected component of L is at most 5. If the maximum degree is less than 5, then by Vizing's Theorem we are done. Now, let I be a connected component of L such that $\Delta(I) = 5$. Note that $V(I_\Delta) \subseteq V(G_{\Delta_f})$ and $\Delta(I_\Delta) \leq 2$. Note that since G is connected, there exists a vertex $x \in V(I) \cap V(P)$ and so $d_I(x) \geq 1$. Since $\delta(P) = 5$ and $d_G(x) = 9$, we conclude that $d_I(x) = 4$. This implies that $f(x) = 2$ and by (7), it is not hard to see that $\mid N_I(x) \cap V(G_{\Delta_f}) \mid= 3$ and so there exists a vertex $y \in N_I(x)$ such that $d_I(y) = 4$. Let $N_I(x) \cap V(G_{\Delta_f}) = \{u, u', u''\}$. Obviously, since G is claw-free, $yu, yu', yu'' \in E(I)$.

Let $J = I \setminus \{x, y, uu', uu'', u'u''\}$. We show that J has a 5-edge coloring. If $\Delta(J) \leq 4$, then by Vizing's Theorem, J has a 5-edge coloring. Thus assume that $\Delta(J) = 5$ and so $\Delta(J_\Delta) \leq 2$ and $d_J(u) = d_J(u') = d_J(u'') = 1$. Hence by Vizing's Theorem and Theorem 1.6, every connected component of J has a 5-edge coloring. Let $N_J(u) = \{z\}, N_J(u') = \{z'\}$ and $N_J(u'') = \{z''\}$. We claim that there exists a 5-edge coloring of J in which the colors of edges $uz, u'z'$ and $u''z''$ are distinct. To see this, if $z = z' = z''$, then we are done. If $z \neq z' = z''$ and the colors of edges $uz, u'z'$ are the same and different from color of the edge $u''z''$, then since $d_J(z') = 4$, we conclude that there exists a missed color in z' which is different from the color of $u'z'$ and $u''z''$. Now, by substituting this missed color with the color of $u'z'$, we are done. Now, assume that z, z' and z'' are distinct. Then, remove three vertices u, u', u'' of J. Also, add a new vertex s, join s to the vertices z, z', z'' and call the resultant graph by K. Now, since $\Delta(K) = 5$, $\Delta(K_\Delta) \leq 2$ and $\delta(K) = 3$, by Theorem 1.6, K has a 5-edge coloring. Now, by a suitable extending this 5-edge coloring to a 5-edge coloring of J, we conclude that there exists a 5-edge coloring of J such that three distinct colors appear in edges $uz, u'z'$ and $u''z''$.

Now, we want to extend the 5-edge coloring of J to a 5-edge coloring of I to complete the proof of Claim 3. To see this, we show that there exists a 5-edge coloring for $Q = \langle\{u, u', u'', x, y\}\rangle$ such that three missed colors in u, u' and u'' are distinct. Add a new vertex q to Q and join q to u, u', u'' and call the resultant graph by R. Clearly, R is the subgraph of K_6 and so $\chi'(R) = 5$. Now, Claim 3 is proved.

(iii) $\Delta_f(G) \geq 6$.
Consider $G\setminus\{v\}$, add two new vertices v_1, v_2 to $G\setminus\{v\}$, and join v_1, v_2 to $\{u_1, w_1, \ldots, w_{\Delta_f(G)-1}\}$ and $\{u_2, \ldots, u_6, w_{\Delta_f(G)}, \ldots, w_{2\Delta_f(G)-7}\}$, respectively. Call the resultant graph L. Let $f' : V(L) \longrightarrow \mathbb{N}$ be a function defined by

$$f'(v) = \begin{cases} f(v) & v \in V(G) \setminus \{v, v_1, v_2\}, \\ 1 & v \in \{v_1, v_2\}. \end{cases}$$

It is easy to see that L is connected, $\Delta_{f'}(L) = \Delta_f(G)$ and $V(L_{\Delta_{f'}}) = V(G_{\Delta_f}) \cup \{v_1\}$. Note that $|N_{L_{\Delta_{f'}}}(v_1)| = 1$. Now, by Theorem 2.2, L has an f'-coloring with colors $\{1, \ldots, \Delta_{f'}(L)\}$; call it θ.

Now, define an f-coloring $c : E(G) \longrightarrow \{1, \ldots, \Delta_f(G)\}$ as follows. Let

$$\begin{cases} c(e) = \theta(e) & \text{for every } \ e \in E(G) \cap E(L) \\ c(u_1 v) = \theta(u_1 v_1) & \\ c(u_i v) = \theta(u_i v_2) & \text{for } \ i = 2, \ldots, 6 \\ c(v w_i) = \theta(v_1 w_i) & \text{for } \ i = 1, \ldots, \Delta_f(G) - 1 \\ c(v w_i) = \theta(v_2 w_i) & \text{for } \ i = \Delta_f(G), \ldots, 2\Delta_f(G) - 7. \end{cases}$$

Thus G is f-Class 1, a contradiction, and the proof of the theorem is complete. □

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