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Triangles in uniquely 3-colorable graphs on surfaces

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Abstract

A graph G is *uniquely k-colorable* if the chromatic number of G is k and every two k-colorings of G produce the same partition of the vertex set into k independent subsets (color classes). In this paper, we investigate the existence of triangles in uniquely 3-colorable graphs on surfaces. It is proved by Chartrand and Geller in 1969 that any uniquely 3-colorable planar graph with at least four vertices contains at least two triangles, and by Aksionov in 1977 that if the number of vertices is at least five, then the uniquely 3-colorable planar graph contains at least three triangles. On the other hand, for surfaces F^2 with non-positive Euler characteristic, there exist uniquely 3-colorable graphs on F^2 without any triangle, which are constructed by Chao and Chen in 1993. We prove that any uniquely 3-colorable graph on the projective plane contains at least one triangle. Furthermore, we report the finiteness of uniquely 3-colorable graphs on surfaces with high girth and a sufficient condition for the uniquely 3-colorability of graphs on surfaces.

1 Introduction

In this paper, we only deal with finite undirected simple graphs unless otherwise mentioned, and let K_n be a complete graph with n vertices. In particular, K_3 is called a *triangle*.

A *k-coloring* of a graph G is a map $c : V(G) \to \{1, 2, \ldots, k\}$ such that for any edge $uv \in E(G)$, $c(u) \neq c(v)$, where $V(G)$ and $E(G)$ are the set of vertices and edges of G, respectively. A graph G is *k-colorable* if there exists a k-coloring of G, and a *chromatic number* of G, denoted by $\chi(G)$, is the minimum number k such that G is k-colorable. Moreover, a graph G with $\chi(G) = k$ is called a *k-chromatic* graph.

In this paper, we consider a special vertex coloring, called a *unique coloring.* A graph G is *uniquely k-colorable* if $\chi(G) = k$ and G has only one k-coloring up

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to permutation of the colors, where the coloring is called a *unique k-coloring*. In other words, any uniquely k-colorable graph G has only one partition of $V(G)$ into k independent subsets. We denote the set of uniquely k-colorable graphs by UC_k. Clearly, $K_n \in UC_n$.

Uniquely colorable graphs were defined and studied by Harary and Cartwright [17] and Harary et al. [18], and the following necessary condition was given (for its proof, see also [4, 34]). For two distinct colors $i, j \in \{1, 2, \ldots, k\}$ in a k-coloring c of a graph G, define $G_{i,j}$ to be the subgraph of G induced by $c^{-1}(i) \cup c^{-1}(j)$.

Theorem 1.1 ([17]). *If $c : V(G) \to \{1, 2, \ldots, k\}$ is a unique k-coloring of $G \in UC_k$, then the graph $G_{i,j}$ is connected for all $i \neq j$ $(i, j \in \{1, 2, \ldots, k\})$.*

It is easy to see that for a graph $G \in UC_k$, $k = 1$ if and only if $E(G) = \emptyset$, and $k = 2$ if and only if G is bipartite and connected.

For a k-chromatic graph G, we define

$$\Lambda(G) = |E(G)| - |V(G)|(k-1) + \binom{k}{2}.$$

This value is introduced by Bollobás [4], and he proposed the problem concerning the minimum number for uniquely colorable graphs. It is independently proved in [7, 34, 36] that $\Lambda(G) \geq 0$. We intuitively think that if a uniquely k-colorable graph G has a small number of edges, then G has to contain a large clique. In fact, Xu [36] conjectured that any uniquely k-colorable graph G contains K_k if $\Lambda(G) = 0$. However, this conjecture is disproved by Akbari et al. [1] using a computer. Furthermore, Truszczyński [34] and Chao and Chen [5, 6] independently investigated the minimum number of vertices of uniquely k-colorable graphs without K_k. (Harary et al. [18] made mention of those constructions of counterexamples for Xu's conjecture, but their construction is not correct.) See also [9, 10] for other topics about constructions of uniquely colorable graphs with small clique number, where the *clique number* of a graph G is the maximum number k such that G contains K_k as its subgraph.

As above, one challenging problem is to construct a uniquely k-colorable graph G with small $\Lambda(G)$ since such graphs satisfy some sparsity condition as well as a large girth or a small clique number. Applying embeddability condition to graphs, we can impose not only upper bounds on the number of edges but also structural restrictions concerning topology (or geometry). So, by focusing on uniquely colorable graphs on surfaces, it is expected that we can obtain good results being different from those of abstract graphs.

For the spherical case, we can easily verify that any uniquely 3-colorable graph contains a triangle as an exercise: Let G be a uniquely 3-colorable graph on the sphere. Suppose that G contains no triangle. Since the boundary walk of any face of G is at least 4, we have $2|E(G)| \geq 4|F(G)|$, where $F(G)$ is the set of faces of G. By Euler's formula $(|V(G)| - |E(G)| + |F(G)| = 2)$, we have $2|V(G)| - 4 \geq |E(G)|$. However, this contradicts to $\Lambda(G) \geq 0$.

Chartrand and Geller [8] proved a stronger result than the above and basic facts for uniquely colorable planar graphs.

Theorem 1.2 ([8]).

(a) Any uniquely 3*-colorable planar graph with* $|V(G)| \geq 4$ *contains at least two triangles.*

(b) An outerplanar graph G *with* $|V(G)| \geq 3$ *is uniquely* 3*-colorable if and only if it is maximal outerplanar.*

(c) Every maximal planar graph G *with* $\chi(G) = 3$ *(i.e., an Eulerian triangulation) is uniquely* 3*-colorable.*

(d) Every uniquely 4*-colorable planar graph is maximal planar (i.e., a triangulation).*

Aksionov [2] improved Theorem 1.2 (a) and gave a constructive characterization of uniquely 3-colorable planar graphs with exactly three triangles. Let H be a plane graph shown in Figure 1. Note that $H \in UC_3$ and it has exactly three triangles.

Theorem 1.3 ([2]). *Any uniquely* 3*-colorable planar graph with* $|V(G)| \geq 5$ *contains at least three triangles.*

Theorem 1.4 ([2]). *If* G *is a uniquely* 3*-colorable planar graph with* $|V(G)| \geq 5$ *and exactly three triangles, then* G *is obtained from* H *by adding a vertex* v *of degree* 2 *preserving the planarity, so that the two neighbors of* v *belong to two distinct color classes.*

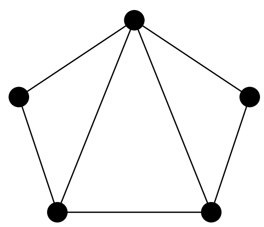

Figure 1: The graph H

In [2], the following two conjectures are proposed:

- Any uniquely 3-colorable planar graph with at least four vertices contains two triangles sharing an edge.
- If a graph G is *edge-critical* uniquely 3-colorable planar graph, i.e., $G-e \notin UC_3$ for any edge $e \in E(G)$, then $\Lambda(G) = 0$, i.e., $|E(G)| = 2|V(G)| - 3$.

Observe that the converse of the second statement is clearly true, however both conjectures are disproved by Mel'nikov and Steinberg [28]. Then the study for the minimum number of edges of an edge-critical uniquely 3-colorable planar graph begins; the first non-trivial upper bound is obtained in [26] and it is recently improved by Li et al. [24].

On the other hand, there exists a uniquely 3-colorable graph on the torus without any triangle shown in Figure 2, which is constructed by Chao and Chen [5] (note that the graph can be also embedded into the Klein bottle). Based on this graph, for any surface F^2 with non-positive Euler characteristic, we can make a uniquely 3-colorable graph without any triangle which can be embedded into F^2, by adding vertices of degree 2 suitably.

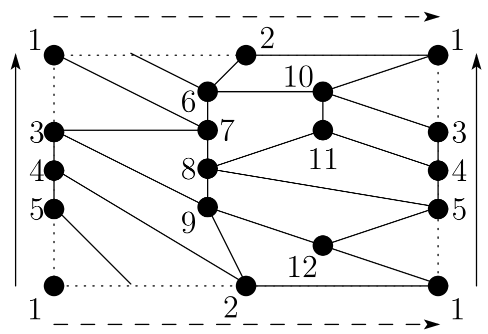

Figure 2: A uniquely 3-colorable graph on the torus without any triangle

The following is our main result in this paper. As described above, the theorem is best possible in the sense that for any other surface F^2 with non-positive Euler characteristic, there exists a uniquely 3-colorable graph on F^2 without any triangle.

Theorem 1.5. *Any uniquely* 3*-colorable graph on the projective plane contains a triangle.*

This paper is organized as follows: In the next section, we prepare terminologies and propositions together with introduction of known results for 3-colorability of graphs on surfaces. In Section 3, we shall prove Theorem 1.5. After that, in Section 4, we provide a sufficient condition that a graph on a surface is uniquely 3-colorable, and show the finiteness of a subclass of the uniquely 3-colorable graphs on surfaces. In the final section, we give some remarks for uniquely 4-colorable graphs on surfaces.

2 Preliminaries

A *surface* means a compact 2-manifold without a boundary. In what follows, a graph G on a surface F^2 means a 2-cell embedding of an abstract graph G on F^2. A *face* of G on a surface F^2 is a connected component of $F^2 - G$, and a *region* of G is the union of several faces. A *k-face* (respectively, a *k-region*) is a face (respectively, a 2-cell region) bounded by a k-cycle (respectively, a closed k-walk), and the boundary cycle (or a closed walk) of a face (or a region) f is denoted by ∂f. A cycle C in a graph on a surface F^2 is *trivial* (respectively, *essential*) if C bounds (respectively, does not bound) a 2-cell region on F^2.

In the literature, Grötzsch [15] proved that every planar graph without any triangle is 3-colorable. After that, Grünbaum [16] improved the Grötzsch's theorem so that any planar graph G is 3-colorable if G has at most three triangles. Note that this result is best possible since a complete graph K_4 on the sphere has four triangles. Thomassen [32] proved that every graph on the torus or the projective plane with girth at least 5 is 3-colorable. In particular, Gimbel and Thomassen [13] characterized 3-chromatic graphs on the projective plane as follows. A *quadrangulation* on a surface F^2 is a connected graph G on F^2 such that each face of G is quadrilateral, and G is *non-bipartite* if $\chi(G) > 2$.

Theorem 2.1 ([13]). *Let G be a graph on the projective plane such that each trivial cycle of G has length at least four. Then G is 3-colorable if and only if G does not contain a non-bipartite quadrangulation as its subgraph.*

In the above theorem, the "only if" part immediately follows from the following famous result by Youngs [37], implying the interesting fact that there eximplying ists no 3-chromatic quadrangulation on the projective plane.

Theorem 2.2 ([37]). *If G is a non-bipartite quadrangulation on the projective plane, then $\chi(G) = 4$.*

We introduce a graph operation, called a *face-contraction*, which is defined as a local operation in quadrangulations [29]. Let $f = abcd$ be a 4-face of a graph on a surface F^2. A *face-contraction* of f at $\{a, c\}$ is to identify a and c and replace two pairs of multiple edges $\{ab, bc\}$ and $\{cd, da\}$ with two single edges, respectively, as shown in Figure 3. If this operation breaks the simplicity of graphs, then we do not apply it. Moreover, the inverse operation is called a *face-splitting*.

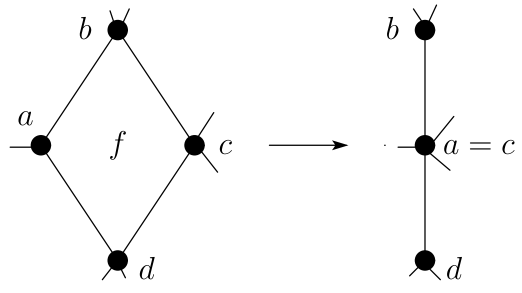

Figure 3: A face-contraction of f at $\{a, c\}$

Lemma 2.1. *Let G be a uniquely 3-colorable graph on a surface F^2 with a 4-face f and let G' be a graph obtained from G by a single face-contraction of f. If G' is 3-colorable, then G' is uniquely 3-colorable.*

Proof. If G' is not uniquely 3-colorable, then there exist two distinct 3-colorings c and c' of G'. In this case, we can extend these 3-colorings to those of G by a single face-splitting, a contradiction to $G \in UC_3$. □

In the end of this section, we introduce the following lemma which will be used for almost all of the case-by-case argument in the proof of Theorem 1.5. For a region R of a graph G on a surface, an *inner* vertex of R is a vertex in R but not on ∂R. A *k-vertex* is a vertex of degree exactly k, and we denote the set of neighbors of a vertex v by $N(v)$. For a cycle C, a *chord* e of C is an edge connecting two vertices on C but $e \notin E(C)$.

Lemma 2.2. *Let D be a bipartite plane graph with the infinite face bounded by a $2k$-cycle $C = v_0v_1v_2 \dots v_{2k-1}$ with $k \geq 2$. Let B and W be the bipartition of D, i.e., $V(D) = B \cup W$ and $B \cap W = \emptyset$, and let $c : V(D) \to \{1, 2, 3\}$ be a 3-coloring of D. If all vertices in $B \cap V(C)$ are colored by the same color, then we have one of the following:*

(1) *D contains no inner vertex (i.e., D contains only chords in its interior).*

(2) *All inner vertices are in B.*

(3) *D has two distinct 3-colorings extended by the 3-coloring of C.*

Proof. It suffices to prove the statement (3) when D contains an inner vertex in W. Without loss of generality, suppose that $c(v) = 1$ for each vertex $v \in B \cap V(C)$. Then we construct two distinct 3-colorings f and g as follows:

$c(u) = f(u) = g(u)$ for each vertex $u \in V(C)$.

$f(v) = g(v) = 1$ for each vertex $v \in B \setminus V(C)$.

$f(w) = 2$ and $g(w) = 3$ for each vertex $w \in W \setminus V(C)$.

By the bipartiteness of D, f and g are proper, i.e., any two adjacent vertices have distinct colors. Moreover, f cannot be obtained from g solely by permutations of the colors by the coloring of vertices in W, and thus, they are distinct. □

3 Proof of Theorem 1.5

We shall prove Theorem 1.5 using the above lemmas.

Proof of Theorem 1.5. We can easily check that any uniquely 3-colorable graph on the projective plane with at most five vertices has a triangle. Hence, let G be a minimal counterexample to Theorem 1.5 with at least six vertices, that is, G has no triangle, $G \in UC_3$ and G is embedded on the projective plane. Let $c : V(G) \to \{1, 2, 3\}$ be a unique 3-coloring of G. We first show the following claims. We denote by $\Delta(G)$ and $\delta(G)$ the maximum degree and the minimum degree of G, respectively.

Claim 3.1. *G satisfies the following:*

(*i*) $|E(G)| = 2|V(G)| - 3$.

(*ii*) $G_{i,j}$ *is a tree for each two distinct colors* $i, j \in \{1, 2, 3\}$.

(*iii*) $\delta(G) \geq 3$ *and the number of* 3*-vertices is at least six.*

Proof. Since $G \in UC_3$, $|E(G)| \geq 2|V(G)| - 3$. Moreover, $|E(G)| \leq 2|V(G)| - 2$ by Euler's formula since G has no triangle. Observe that if $|E(G)| = 2|V(G)| - 2$ (i.e., $2|E(G)| = 4|F(G)|$), then G is a quadrangulation. Since no 3-chromatic quadrangulation on the projective plane exists by Theorem 2.2, statement (*i*) holds and also (*ii*) holds by Theorem 1.1.

We next show (*iii*). Since $G \in UC_3$, $\delta(G) \geq 2$. We first suppose that G has a 2-vertex v. Let $G' = G - v$ be a graph obtained from G by removing v. Note that G' has no triangle and is uniquely 3-colorable (cf. [18]), which contradicts the minimality of G. Hence, $\delta(G) \geq 3$. Let p_k be the number of k-vertices in G and then $|V(G)| = \sum_{i=3}^{\Delta(G)} p_i$ and $2|E(G)| = \sum_{i=3}^{\Delta(G)} ip_i$. By (*i*), we have

$$6 = \sum_{i=3}^{\Delta} p_i(4 - i) = p_3 - p_5 - 2p_6 - \cdots - (\Delta - 4)p_\Delta.$$

Therefore statement (*iii*) holds since $p_3 \geq 6$. ■

Claim 3.2. *Let* $Q = v_1v_2v_3v_4$ *be a* 4*-face in* G *with* $c(v_1) = c(v_3)$ *and* $c(v_2) \neq c(v_4)$. *Then there exists a path* v_1xyv_3 *with* $\{x, y\} \cap \{v_2, v_4\} = \emptyset$.

Proof. By Claim 3.1 (*ii*), there exists no path v_1uv_3 with $u \notin \{v_2, v_4\}$ since otherwise G has a cycle induced by two color sets, $v_1uv_3v_2$ or $v_1uv_3v_4$. Furthermore, if no desired path v_1xyv_3 exists in G, then by applying a face-contraction of Q at $\{v_1, v_3\}$, we can obtain a smaller counterexample by Lemma 2.1, a contradiction. ■

By Claim 3.1 (*i*), G has exactly one 6-face and each other face is quadrilateral, or it has exactly two 5-faces and each other face is quadrilateral. So each vertex of G is shared by at least one 4-face.

Claim 3.3. *G has no* 3*-vertex shared by exactly three* 4*-faces.*

Proof. For contradiction, we suppose that G has a 3-vertex v shared by exactly three 4-faces $vx_1y_1x_2, vx_2y_2x_3$ and $vx_3y_3x_1$. By symmetry, we may suppose that $c(v) = c(y_1) = c(y_3) = 1$, $c(x_2) = c(x_3) = 2$ and $c(x_1) = c(y_2) = 3$ by Claim 3.1 (*ii*). By Claim 3.2, there exist two paths y_1ux_3v and $y_3u'x_2v$ with $c(u) = c(u') = 3$. If $u = u'$, it contradicts to Claim 3.1 (*ii*) since $uy_1x_1y_3$ is a cycle in $G_{1,3}$. So we have $u \neq u'$, and both cycles $vx_1y_1ux_3$ and $vx_1y_3u'x_2$ are essential. Thus G has a 6-region R_6 and a 8-region R_8 with $\partial R_6 = y_1x_2u'y_3x_3u$ and $\partial R_8 = x_1y_1ux_3y_2x_2u'y_3$, respectively.

Observe that all inner faces in one of these two regions are quadrilateral even if G has two 5-faces, otherwise, by cutting along the boundary cycle of R_6 and embedding R_6 on the plane, the resulting plane graph has exactly one 5-face, which contradicts to the Handshaking Lemma. Furthermore, all inner faces of R_6 cannot be quadrangular since now each antipodal pair of ∂R_6 has the same color [3]. (This fact can be proved by the *Winding Number*; see also [21, 37]). Thus we may suppose that each face of R_8 is quadrangular.

Since R_8 can be regarded as a bipartite plane graph and $c(y_1) = c(y_3) = 1$, $c(x_2) = c(x_3) = 2$ and $c(x_1) = c(y_2) = c(u) = c(u') = 3$, we apply Lemma 2.2 to R_8, where $B = \{x_1, y_2, u, u'\}$ is arranged in Lemma 2.2. Since $G \in UC_3$, Lemma 2.2 (3) does not occur. If R_8 has a chord in the interior or all inner vertices are in B, then G has a triangle or $G_{i,j}$ has a cycle for some $i, j \in \{1, 2, 3\}$ since each vertex of G is of degree at least 3, a contradiction. ■

Now we shall complete the proof by considering the following two cases.

Case 1. There exists a 6-face f in G.

Let $\partial f = u_0u_1u_2u_3u_4u_5$. By Claim 3.3, all 3-vertices lie on ∂f, and hence, G has exactly six 3-vertices and others are of degree exactly 4 (by the equality of the proof of Claim 3.1 (*iii*)). Let $u'_k \in N(u_k) \setminus V(\partial f)$ for each k, and then there exists a 4-face $u_iu_{i+1}u'_{i+1}u'_i$ for each i, where the subscripts are modulo 6. Note that ∂f has no chord (i.e., $u_i \neq u'_j$ for each pair i, j) since G has no triangle and the 3-coloring of ∂f is cyclic, where a 3-coloring of a cycle with length $3k$ ($k \geq 1$) is *cyclic* if three colors cyclically appear on the cycle, i.e., $1, 2, 3, 1, 2, 3$ and so on (if the 3-coloring of ∂f is not cyclic, then we can obtain a 3-chromatic quadrangulation only by adding an edge to f). Moreover, $u'_i \neq u'_j$ for each pair $i, j \in \{0, 1, \dots, 5\}$ with $i \neq j$ since G has no triangle and $u'_0u'_1 \cdots u'_5$ is cyclically colored in this order by the coloring of ∂f. Thus G has the 6-cycle $C = u'_0u'_1u'_2u'_3u'_4u'_5$ along ∂f.

We can repeatedly apply the above argument to the non-2-cell region bounded by C. Therefore, we can infinitely find new 6-cycles with a cyclic 3-coloring as above, a contradiction to the finiteness of G.

Case 2. There exist two 5-faces f and f' in G.

By Claims 3.1 (*iii*) and 3.3, we may suppose that G has a 3-vertex v which is shared by two 4-faces $vx_1y_1x_2, vx_3y_2x_1$ and a 5-face $f = vx_2z_1z_2x_3$. Without loss of generality, we suppose that $c(v) = c(y_1) = 1, c(x_1) = 2, c(x_2) = 3$, and let $L(v) = \{x_1, x_2, x_3, y_1, y_2, z_1, z_2\}$.

We show that there exist two paths $P_1 = y_1d_1x_3v$ and $P_2 = y_2d_2x_2v$, where d_i may be in $L(v)$ for $i = 1, 2$. By Claim 3.2, P_1 exists. Moreover, if $c(y_2) = 1$, then P_2 also exists. So, we may suppose that $c(y_2) = 3$, and then $c(x_3) = 2$.

Let G' be the graph obtained from G by a face-contraction of $vx_3y_2x_1$ at $\{v, y_2\}$, and we suppose that G' has no triangle. If $\chi(G') = 3$, then G' is also uniquely 3-

colorable by Lemma 2.1, a contradiction to the minimality of G. Thus $\chi(G') = 4$ and then G' contains a non-bipartite quadrangulation as its subgraph by Theorem 2.1.

Let Q be a *maximal* non-bipartite quadrangulation in G', that is, G' contains no non-bipartite quadrangulation Q' with $V(Q) \subsetneq V(Q')$ and $E(Q) \subsetneq E(Q')$. Observe that $x_2y_2 \in E(Q)$ (otherwise, Q is also a subgraph of G). So, there exist two 4-faces of Q sharing x_2y_2, and hence, the 4-face $x_1y_1x_2y_2$ of G' is also a face of Q by the maximality of Q. Let $f_Q = x_2s_1s_2y_2$ be the other 4-face. Since G has P_1, we have $s_2 = x_3$, and so, $x_3y_2 \in E(Q)$ (otherwise, G has a triangle $x_3s_2y_2$ or $y_1x_2s_1$). Thus, we have a non-bipartite quadrangulation Q_G which is a subgraph of G, as follows.

$$V(Q_G) = V(Q) \cup \{v\} \quad \text{and} \quad E(Q_G) = (E(Q) \setminus \{x_2y_2\}) \cup \{vx_1, vx_2, vx_3\}.$$

Observe that Q can be obtained from Q_G by the face-contraction of $vx_3y_2x_1$ at $\{v, y_2\}$, and hence, $\chi(G) \geq 4$ since a face-contraction preserves the bipartiteness of quadrangulations [29], a contradiction to $\chi(G) = 3$. Thus, G contains both P_1 and P_2.

Similarly to the proof of Claim 3.3, G has one of the following structures:

(1) $d_i \notin L(v)$ for $i = 1, 2$ and $d_1 = d_2$.

(2) $d_i \notin L(v)$ for $i = 1, 2$ and $d_1 \neq d_2$.

(3) $d_i \in L(v)$ for $i = 1, 2$, i.e., $d_1 = z_2$ and $d_2 = z_1$.

(4) Exactly one of d_i's belongs to $L(v)$.

In the structure (1), since $c(v) = c(y_1) = 1, c(x_1) = 2, c(x_2) = 3$, we have $c(d_1) = 2, c(x_3) = 3$ and $c(y_2) = 1$, and then $G_{1,2}$ has a cycle, a contradiction. For any other cases (2), (3) and (4), two cycles $vx_1y_1d_1x_3$ and $vx_1y_2d_2x_2$ must be essential. Thus, similarly to the proof of Claim 3.3, G has a 6-region R_6 outside of the 2-cell region bounded by $L(v)$ in each case. Moreover, all inner faces of R_6 are quadrangular by the Handshaking Lemma. However, since the 3-coloring of ∂R_6 is cyclic or satisfies the condition in Lemma 2.2 (i.e., all vertices on ∂R belonging to one partite set B are colored by the same color), we have a contradiction similarly to Case 1 or the proof of Claim 3.3. □

4 Sufficient conditions and finiteness

As described in the Introduction, there are several necessary conditions for uniquely colorable graphs. Similarly, a number of sufficient conditions for the graphs exist. In the literature, Osterweil [30] gave a sufficient condition for uniquely 3-colorable graphs, using a construction with a *six clique ring* which is obtained from six complete graphs by adding several edges to them forming a 6-cycle or a 6-walk. A few years later, Bollobás [4] provided two non-trivial sufficient conditions, as follows, and he also proved that both results are best possible.

Theorem 4.1 ([4])**.** *Let G be a k-colorable graph with n vertices. If $\delta(G) > \frac{3k-5}{3k-2}n$, then G is uniquely k-colorable.*

Theorem 4.2 ([4])**.** *Let G be a graph with n vertices. If G has a k-coloring in which the induced subgraph by the union of any two color classes is connected, then $\delta(G) > \left(1 - \frac{1}{k-1}\right) n$ which implies that G is uniquely k-colorable.*

From another viewpoint, Hillar and Windfeldt [19] gave an algebraic characterization of uniquely colorable graphs. However, as far as we know, there exists no sufficient condition for uniquely k-colorable graphs concerning the embeddability of graphs. So we provide a new sufficient condition for uniquely 3-colorable graphs on surfaces.

Theorem 4.3. *Let G be a 3-colorable simple graph on a surface F^2. If G has at least $2|V(G)| - 2\epsilon(F^2) - 3$ triangular faces, where $\epsilon(F^2)$ is the Euler characteristic of F^2, then G is uniquely 3-colorable.*

Proof. Let G be a 3-colorable graph on a surface F^2 with n vertices and $t \geq 2n - 2\epsilon(F^2) - 3$ triangular faces, and we may suppose that $n \geq 4$ and $t \geq 1$. If $t = 2n - 2\epsilon(F^2)$, then G must be a triangulation. It is known that a triangulation on a surface is 3-colorable if and only if it is Eulerian, and every Eulerian triangulation is uniquely 3-colorable (cf. [22, 23]). (Note that even if an Eulerian triangulation has multiple edges, it is uniquely 3-colorable.) So we suppose that $t \leq 2n - 2\epsilon(F^2) - 1$. Observe that we can transform G into a triangulation (which might have multiple edges) only by adding edges to non-triangular faces. Conversely, we obtain G by removing edges of some triangulation on F^2 (which might have multiple edges). Further, removing edge e of a graph reduces the number of triangular faces by exactly two if the edge e is shared by two triangular faces. Thus, there exists no graph with exactly $t = 2n - 2\epsilon(F^2) - 1$ triangular faces since the removing an edge from a triangulation produces a graph with exactly $2n - 2\epsilon(F^2) - 2$ triangular faces. So it suffices to consider the following two cases.

Case 1. $t = 2n - 2\epsilon(F^2) - 2$.

As described above, G has exactly one 4-face f bounded by a 4-cycle $u_0u_1u_2u_3$. For contradiction, we suppose that G is not uniquely 3-colorable. There exist three types of a 3-colorings c of G:

(i) $c(u_0) = c(u_2)$ and colors of other two vertices are distinct.

(ii) $c(u_1) = c(u_3)$ and colors of other two vertices are distinct.

(iii) $c(u_0) = c(u_2)$ and $c(u_1) = c(u_3)$.

Let c and c' be two distinct 3-colorings of G. If both c and c' have the type (i) (respectively, (ii)), then by adding the edge u_1u_3 (respectively, u_0u_2), we can obtain an Eulerian triangulation (which might have multiple edges) with two distinct 3-colorings, a contradiction. (It is well-known that every 3-colorable triangulation on a surface is Eulerian and uniquely 3-colorable [22]). If they have the type (iii), then similar to the above, we have a contradiction by adding a 4-vertex v into f with $N(v) = \{u_0, u_1, u_2, u_3\}$ since v can be colored by the third color. If they have distinct types, then we can obtain a 3-colorable triangulation which is not Eulerian (i.e., the graph has a vertex of odd degree) since we can transform G into the 3-colorable triangulation by two distinct operations in the above three operations, u_0u_2, u_1u_3 or a 4-vertex v, a contradiction (since one of resultant triangulations is not Eulerian).

Case 2. $t = 2n - 2\epsilon(F^2) - 3$.

By the assumption of t, G is obtained from a graph on F^2 with exactly $2n - 2\epsilon(F^2) - 2$ triangular faces by removing an edge shared by a triangular face and a quadrangular face, that is, G has exactly one 5-face f bounded by a 5-cycle $u_0u_1u_2u_3u_4$. Observe that for any 3-coloring of G, exactly one of five vertices on ∂f is colored by a color which is not used for any other vertex on ∂f. Let c and c' be two distinct 3-colorings of G, and let p and p' be vertices with a unique color on ∂f as above. If $p = p'$, say $p = u_0$ by symmetry, then we can obtain an Eulerian triangulation (which might have multiple edges) with two distinct 3-colorings by adding two edges u_0u_2 and u_0u_3, a contradiction. Otherwise, say $p = u_0$ and $p' = u_1$ (or u_2), u_0, u_1 and u_4 are of even degree since we have an Eulerian triangulation (which might have multiple edges) by adding two edges u_0u_2 and u_0u_3. However, similar to Case 1, we can obtain a 3-colorable triangulation which is not Eulerian by adding u_1u_3 and u_1u_4 (or u_2u_4 and u_2u_0), contradicting Theorem 1.2 (c). □

Now we introduce a construction of graphs which imply the sharpness of Theorem 4.3. We prepare a 3-colorable graph H on a surface F^2 with $2m - 2\epsilon(F^2) - 2$ triangular faces and exactly one quadrangular face f bounded by a cycle $abcd$, where $m = |V(H)|$. Let D be a plane graph obtained from the double wheel by removing an edge on the rim, where a 4-cycle $xyzw$ bounds the infinite face (see Figure 4). Suppose that a and c have the same color in any 3-coloring of H. Let G be the 3-chromatic graph with n vertices obtained from H by embedding D into f so that $a = x$ and $c = z$, that is, G has exactly two 4-faces $awcd$ and $abcy$, where $n = m + |V(D)| - 2$. Observe the number of triangular faces of G is

$$2m - 2\epsilon(F^2) - 2 + 2|V(D)| - 4 - 2 = 2n - 2\epsilon(F^2) - 4.$$

Furthermore, since there exists a 3-coloring of H such that a and c have the same color, we can obtain two distinct 3-colorings of G based on the coloring of H by the permutation of colors of vertices in $V(D) \setminus \{x, z\}$. Therefore, the bound of Theorem 4.3 is best possible.

We conclude this section by considering the number of uniquely colorable graphs with restrictions. In general, there are infinitely many uniquely colorable graphs on a surface without any additional restriction, but we guess that the embeddability of graphs into a fixed surface F^2 limits the number of uniquely colorable graphs G to finite since the number of vertices (also edges) of such a graph is bounded by two powerful inequalities: Euler's formula and $\Lambda(G) \geq 0$. For this problem, it is known

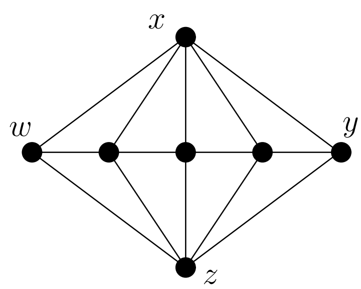

Figure 4: The graph D

that for any integer $k \geq 5$, the number of uniquely k-colorable graphs on a fixed surface F^2 is finite [27]. As far as we know, no result for the finiteness of uniquely 3-colorable graphs on surfaces has been proved. We shall show the finiteness of the set of uniquely 3-colorable graphs with high girth, where the *girth* of a graph G is the length of a shortest cycle in G.

Theorem 4.4. *For any surface F^2, the number of uniquely 3-colorable graphs on F^2 with girth at least 5 is finite.*

Proof. Let F^2 be a surface with Euler characteristic $\epsilon(F^2)$, and let G be a uniquely 3-colorable graph on F^2 with girth at least 5. Then we have $|E(G)| \geq 2|V(G)| - 3$ and $2|E(G)| \geq 5|F(G)|$. Thus, by Euler's formula, we have

$$5|V(G)| - 3|E(G)| \geq 5|V(G)| - 5|E(G)| + 5|F(G)| = 5\epsilon(F^2),$$

$$5|V(G)| - 5\epsilon(F^2) \geq 3|E(G)| \geq 6|V(G)| - 9, \text{ and hence,}$$

$$9 - 5\epsilon(F^2) \geq |V(G)|.$$

Therefore, since the number of vertices of G is bounded by a constant depending only on $\epsilon(F^2)$, the number of such graphs G is finite. $\square$

5 Note on uniquely 4-colorable graphs on surfaces

Remember that $\Lambda(G) \geq 0$ for any uniquely k-colorable graph G, and any uniquely 4-colorable graph with $n \geq 3$ vertices has at least $3n - 6$ edges. This means that any uniquely 4-colorable graph on the sphere must be a triangulation on the sphere since for every planar graph G with n vertices, $|E(G)| \leq 3n - 6$ with equality if and only if G is a triangulation on the sphere. On the other hand, for any other surface F^2, a uniquely 4-colorable graph on F^2 is not necessarily a triangulation on F^2. For example, the projective plane admits the embedding of K_4 with each face quadrilateral. Hence the following problem remains, but it seems to be difficult.

Problem 5.1. *Give a characterization of uniquely 4-colorable graphs on a fixed surface.*

For the spherical case, a famous conjecture is proposed in [11] (or see [20, pp. 48–49]): Any uniquely 4-colorable planar graph can be obtained from K_4 on the sphere by repeated applications of adding a vertex of degree 3 into a triangular face so that the vertex is adjacent to all vertices on the boundary of the corresponding face. Fowler [12] gave a positive solution for the conjecture by using a computer; however, the conjecture has not been completely solved yet.

The dual of a uniquely 4-colorable plane graph is a uniquely 3-edge-colorable cubic planar graph, where a graph G is *uniquely* 3*-edge-colorable* if the chromatic index of G is three and every two 3-edge-colorings of G produce the same partition of $E(G)$ into three independent subsets (matchings). The above conjecture for uniquely 4-colorable planar graphs is equivalent to the fact that any uniquely 3-edge-colorable cubic planar graph contains a triangle. Furthermore, Cantoni's Conjecture [35] implies that a cubic planar graph G is uniquely 3-edge-colorable if and only if G has exactly three Hamiltonian cycles [33, Problem 2.16], where Thomason [31] verified that the condition of planarity cannot be removed. It is very interesting to prove the above statements directly using properties of uniquely 3-edge-colorable graphs. For more details of uniquely edge-colorable graphs, see [14, 25].

Incidentally, by a similar proof to Theorem 4.4, we can easily obtain the following.

Theorem 5.1. *For any surface F^2, the number of uniquely* 4*-colorable graphs on F^2 with girth at least* 4 *is finite.*

As a corollary, any uniquely 4-colorable graph on a fixed surface contains a triangle with finite exceptions. We conclude the paper with the following conjecture according to Theorem 1.5.

Conjecture 5.1. *Any uniquely* 4*-colorable graph on the projective plane contains K_4 as its subgraph.*

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