AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **73(2)** (2019), Pages 313–333

On k-total dominating graphs

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Abstract

For a graph G, the k-total dominating graph $D_k^t(G)$ is the graph whose vertices correspond to the total dominating sets of G that have cardinality at most k; two vertices of $D_k^t(G)$ are adjacent if and only if the corresponding total dominating sets of G differ by either adding or deleting a single vertex. The graph $D_k^t(G)$ is used to study the reconfiguration problem for total dominating sets: a total dominating set can be reconfigured to another by a sequence of single vertex additions and deletions, such that the intermediate sets of vertices at each step are total dominating sets, if and only if they are in the same component of $D_k^t(G)$. Let $d_0(G)$ be the smallest integer ℓ such that $D_k^t(G)$ is connected for all $k \geq \ell$.

We investigate the realizability of graphs as total dominating graphs. For k the upper total domination number $\Gamma_t(G)$, we show that any graph without isolated vertices is an induced subgraph of a graph G such that $D_k^t(G)$ is connected. We obtain the bounds $\Gamma_t(G) \leq d_0(G) \leq n$ for any connected graph G of order $n \geq 3$, characterize the graphs for which either bound is realized, and determine $d_0(C_n)$ and $d_0(P_n)$.

* Supported by the Natural Sciences and Engineering Research Council of Canada.

ISSN: 2202-3518

1 Introduction

A *dominating set* of a graph $G = (V, E)$ is a set $D \subseteq V(G)$ such that every vertex of $G - D$ is adjacent to a vertex in D. If no proper subset of D is a dominating set, then D is *a minimal dominating set* of G. The *domination number* $\gamma(G)$ of G the minimum cardinality of a dominating set of G, while the *upper domination number* $\Gamma(G)$ is the maximum cardinality of a minimal dominating set. We denote the *independence number* (the maximum cardinality of an independent set) of G by $\alpha(G)$.

A *total dominating set* (*TDS*) of a graph G without isolated vertices is a set $S \subseteq V(G)$ such that every vertex of G is adjacent to a vertex in S. If no proper subset of S is a TDS of G, then S is *a minimal TDS* (*an MTDS*) of G. Every graph without isolated vertices has a TDS, since $S = V(G)$ is such a set. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS. The *upper total domination number* of G, denoted by $\Gamma_t(G)$, is the maximum cardinality of an MTDS. A TDS of size γ_t is called γ_t*-set* of G, and an MTDS of size $\Gamma_t(G)$ is called a Γ_t*-set*; γ*-sets* and Γ*-sets* are defined similarly. Since every total dominating set of G is a dominating set, $\gamma(G) \leq \gamma_t(G)$ for any graph G without isolated vertices. However, not every minimal total dominating set is minimal dominating, hence there is no similar comparison of $\Gamma(G)$ and $\Gamma_t(G)$. For example, for the star $K_{1,n},\ n \geq 3$,

$$\gamma(K_{1,n}) = 1 < \gamma_t(K_{1,n}) = \Gamma_t(K_{1,n}) = 2 < \Gamma(K_{1,n}) = n.$$

On the other hand, consider the path P_n of order n. It is well known that $\gamma(P_n) = \lceil \frac{n}{3} \rceil$ and $\Gamma(P_n) = \alpha(P_n) = \lceil \frac{n}{2} \rceil$. As mentioned in Observation 3.1 and Proposition 3.2, $\gamma_t(P_n) = \frac{n}{2} + 1$ if $n \equiv 2 \pmod 4$ and $\gamma_t(P_n) = \lceil \frac{n}{2} \rceil$ otherwise, and $\Gamma_t(P_n) = 2 \lfloor \frac{n+1}{3} \rfloor$. Therefore, for n large enough,

$$\gamma(P_n) < \Gamma(P_n) \leq \gamma_t(P_n) < \Gamma_t(P_n).$$

For a given threshold k, let S and S' be total dominating sets of order at most k of G. The *total dominating set reconfiguration (TDSR) problem* asks whether there exists a sequence of total dominating sets of G starting with S and ending with S', such that each total dominating set in the sequence is of order at most k and can be obtained from the previous one by either adding or deleting exactly one vertex.

This problem is similar to the *dominating set reconfiguration (DSR) problem*, which is PSPACE-complete even for planar graphs, bounded bandwidth graphs, split graphs, and bipartite graphs, while it can be solved in linear time for cographs, trees, and interval graphs [12].

The DSR problem naturally leads to the concept of the k-dominating graph introduced by Haas and Seyffarth [10] as follows. If G is a graph and k a positive integer, then the *k-dominating graph* $D_k(G)$ of G is the graph whose vertices correspond to the dominating sets of G that have cardinality at most k, two vertices of $D_k(G)$ being adjacent if and only if the corresponding dominating sets of G differ by either adding or deleting a single vertex. The DSR problem therefore simply asks whether two given vertices of $D_k(G)$ belong to the same component of $D_k(G)$.

The Haas-Seyffarth paper [10] stimulated the work of Alikhani, Fatehi and Klavžar [2], Mynhardt, Roux and Teshima [17], Suzuki, Mouawad and Nishimura [18], as well as their own follow-up paper [11]. Haas and Seyffarth [10] gave conditions for $D_k(G)$ to be connected; this is the case, for example, when $k \geq \gamma(G)+\Gamma(G)$, or when $k \geq \Gamma(G)+1$ and G is bipartite or chordal. They also posed the question of whether $D_{\Gamma(G)+1}(G)$ is connected for all graphs G. However, Suzuki et al. [18] found an infinite class of graphs G such that $D_{\Gamma(G)+1}(G)$ is disconnected and $D_{\Gamma(G)+2}(G)$ is connected. Mynhardt et al. [17] then presented constructions of graphs G with arbitrary upper domination number $\Gamma(G) \geq 3$ and arbitrary domination number $2 \leq \gamma(G) \leq \Gamma(G)$ that show that the smallest integer k such that $D_\ell(G)$ is connected for all $\ell \geq k$ can be as high as $\Gamma(G)+\gamma(G)-1$, or even $\Gamma(G)+\gamma(G)$ provided $\gamma(G) < \Gamma(G)$.

The study of k-dominating graphs was further motivated by similar studies of graph colourings and other graph problems, such as independent sets, cliques and vertex covers—see e.g. [3, 4, 5, 6, 15, 16]—and by a general goal to further understand the relationship between the dominating sets of a graph. Motivated by definition of k-dominating graph, we define the k-total dominating graph of G as follows.

Definition 1 *The* k-total dominating graph $D_k^t(G)$ *of* G *is the graph whose vertices correspond to the total dominating sets of* G *that have cardinality at most* k. *Two vertices of* $D_k^t(G)$ *are adjacent if and only if the corresponding total dominating sets of* G *differ by either adding or deleting a single vertex. For* $r \geq 0$, *we abbreviate* $D^t_{\Gamma_t(G)+r}(G)$ *to* $D^t_{\Gamma_t+r}(G)$, *and* $D^t_{\gamma_t(G)+r}(G)$ *to* $D^t_{\gamma_t+r}(G)$.

In studying the TDSR problem, it is therefore natural to determine conditions for $D_k^t(G)$ to be connected. We begin the study of this problem in Section 2. To this purpose we define $d_0(G)$ to be the smallest integer ℓ such that $D_k^t(G)$ is connected for all $k \geq \ell$, and note that $d_0(G)$ exists for all graphs G without isolated vertices because $D^t_{|V(G)|}(G)$ is connected.

If G is a graph without isolated vertices and $k \geq \gamma_t(G)$, then both the k-dominating graph $D_k(G)$ and the k-total dominating graph $D_k^t(G)$ are defined, and the latter is an induced subgraph of the former. Nevertheless, because of the incomparability of the upper parameters $\Gamma(G)$ and $\Gamma_t(G)$, it is natural to expect that the two graphs can be very different, and we will see that this is indeed the case.

We introduce our notation in Section 1.1 and provide background material on total domination in Section 1.2. For instance, we characterize graphs G such that $\Gamma_t(G) = |V(G)|-1$. In Section 3 we determine $d_0(C_n)$ and $d_0(P_n)$; interestingly, it turns out that $d_0(C_8) = \Gamma_t(C_8)+2$, making C_8 the only known graph for which $D^t_{\Gamma_t+1}(G)$ is disconnected. The main result for cycles requires four lemmas, which we state in Section 3 but only prove in Section 5 to improve the flow of the exposition. In Section 4 we study the realizability of graphs as total dominating graphs. We show that the hypercubes Q_n and stars $K_{1,n}$ are realizable for all $n \geq 2$, that C_4, C_6, C_8 and C_{10} are the only realizable cycles, and that P_1 and P_3 are the only realizable paths. Section 6 contains a list of open problems and questions for future consideration.

1.1 Notation

For domination related concepts not defined here we refer the reader to [13]. The monograph [14] by Henning and Yeo is a valuable resource on total domination.

For vertices u, v of a graph G, we write $u \sim v$ if u and v are adjacent. A vertex v such that $u \sim v$ for all $u \in V(G) - \{v\}$ is a *universal vertex*. We refer to a vertex of G of degree 1 as a *leaf* and to the unique neighbour of a leaf in $G \ncong K_2$ as a *stem*, and denote the number of leaves and stems of G by $\lambda(G)$ and $\sigma(G)$, respectively. As usual, for $u, v \in V(G)$, $d(u, v)$ denotes the distance from u to v.

A set of cardinality n is also called an *n-set*. A subset of cardinality k of a set A is called a *k-subset* of A. The *hypercube* Q_n is the graph whose vertices are the 2^n subsets of an n-set, where two vertices are adjacent if and only if one set is obtained from the other by deleting a single element.

The disjoint union of r copies of a graph H is denoted by rH. The *corona* $G \circ K_1$ *of a graph* G is the graph obtained by joining each vertex of G to a new leaf. A *generalized corona* of G is a graph obtained by joining each vertex of G to one or more new leaves. For a graph G and a subset U of $V(G)$, we denote by $G[U]$ the subgraph of G induced by U.

Remark 1.1 *The set of stems of a graph G is a subset of any TDS of G, otherwise some leaf is not totally dominated. Hence $\gamma_t(G) \geq \sigma(G)$.*

The *open neighbourhood* of a vertex v is $N(v) = \{u \in V(G) : u \sim v\}$ and the *closed neighbourhood* of v is $N[v] = N(v) \cup \{v\}$. Let $S \subseteq V(G)$. The *closed neighbourhood* of S is $N[S] = \bigcup_{s \in S} N[s]$. The *open private neighbourhood of a vertex $s \in S$ relative to S*, denoted $\mathrm{OPN}(s, S)$, consists of all vertices in the open neighbourhood of s that do not belong to the open neighbourhood of any $s' \in S - \{s\}$, that is, $\mathrm{OPN}(s, S) = N(s) - \bigcup_{s' \in S - \{s\}} N(s')$. A vertex in $\mathrm{OPN}(s, S)$ may belong to S, in which case it is called an *internal private neighbour of s relative to S*, or it may belong to $V(G) - S$, in which case it is called an *external private neighbour of s relative to S*. The set of internal (external, respectively) private neighbours of s relative to S are denoted by $\mathrm{IPN}(s, S)$ ($\mathrm{EPN}(s, S)$, respectively). Hence $\mathrm{OPN}(s, S) = \mathrm{IPN}(s, S) \cup \mathrm{EPN}(s, S)$. These sets play an important role in determining whether a TDS is an MTDS or not.

1.2 Preliminary results

Cockayne, Dawes and Hedetniemi [7] characterize minimal total dominating sets as follows.

Proposition 1.2 [7] *A TDS S of a graph G is an MTDS if and only if $\mathrm{OPN}(s, S) \neq \varnothing$ for every $s \in S$.*

We restate Proposition 1.2 in a more convenient form for our purposes.

Corollary 1.3 *Let S be a TDS of a graph G, H the subgraph of G consisting of the components of $G[S]$ of order at least 3, and X the set of stems of H. Then S is an MTDS if and only if $\mathrm{EPN}(s, S) \neq \varnothing$ for each $s \in V(H) - X$.*

Proof. Since each vertex in S is dominated by another vertex in S, $G[S]$ has no isolated vertices. A vertex $s' \in S$ belongs to $\mathrm{IPN}(s, S)$ if and only if $s' \sim s$ and $\deg_{G[S]}(s') = 1$. Therefore, if $G[\{s, s'\}]$ is a K_2 component of $G[S]$, then $s' \in \mathrm{IPN}(s, S) \subseteq \mathrm{OPN}(s, S)$ and $s \in \mathrm{IPN}(s', S) \subseteq \mathrm{OPN}(s', S)$. Further, if x is a stem of H, then x is adjacent to a vertex $x' \in S$ such that $\deg_{G[S]}(x') = 1$, hence $x' \in \mathrm{IPN}(x, S)$. By Proposition 1.2, therefore, S is an MTDS if and only if $\mathrm{OPN}(s, S) \neq \varnothing$ for every $s \in V(H) - X$. But for any $s \in V(H) - X$, each vertex in $N(s) \cap S$ is adjacent to at least two vertices in S, hence $\mathrm{IPN}(s, S) = \varnothing$, which implies that $\mathrm{OPN}(s, S) \neq \varnothing$ if and only if $\mathrm{EPN}(s, S) \neq \varnothing$. ■

Cockayne et al. [7] also established an upper bound on the total domination number, while Favaron and Henning [9] established an upper bound on the upper total domination number.

Proposition 1.4 *If G is a connected graph of order $n \geq 3$, then*

(i) [7] $\gamma_t(G) \leq \frac{2n}{3}$, *and*

(ii) [9] $\Gamma_t(G) \leq n-1$; *furthermore, if G has minimum degree $\delta \geq 2$, then $\Gamma_t(G) \leq n - \delta + 1$, and this bound is sharp.*

We now characterize graphs G such that $\Gamma_t(G) = |V(G)| - 1$.

Proposition 1.5 *A connected graph G of order $n \geq 3$ satisfies $\Gamma_t(G) = n-1$ if and only if n is odd and G is obtained from $\frac{n-1}{2}K_2$ by joining a new vertex to at least one vertex of each K_2.*

Proof. It is clear that $\Gamma_t(G) = n-1$ for any such graph G. For the converse, assume G is a connected graph of order $n \geq 3$ such that $\Gamma_t(G) = n-1$ and let S be a Γ_t-set of G. Suppose H is a component of $G[S]$ of order at least 3. If $\delta(H) \geq 2$, then H has no stems. If $\delta(H) = 1$, then H has at least as many leaves as stems, so that H has at least two vertices that are not stems. In either case Corollary 1.3 implies that at least two vertices in S has nonempty external private neighbourhoods, which is impossible since $|S| = n-1$. Therefore each component of H is a K_2. Since only one vertex of G does not belong to S and G is connected, the result follows. ■

2 Connectedness of $D_k^t(G)$

Haas and Seyffarth [10] showed that $D_{\Gamma(G)}(G)$ is disconnected whenever $E(G) \neq \varnothing$. In contrast, we show that any graph without isolated vertices is an induced subgraph of a graph G such that $D^t_{\Gamma_t}(G)$ is connected. We obtain the bounds $\Gamma_t(G) \leq d_0(G) \leq n$ for any connected graph G of order $n \geq 3$, and characterize graphs that satisfy equality in either bound.

We begin with some definitions and basic results. For $k \geq \gamma_t(G)$ and A, B total dominating sets of G of cardinality at most k, we write $A \overset{k}{\leftrightarrow} B$, or simply $A \leftrightarrow B$ if k is clear from the context, if there is a path in $D^t_k(G)$ connecting A and B. The binary relation $\leftrightarrow$ is clearly symmetric and transitive. Any superset of a TDS is a TDS. Hence if $A \subseteq B$ and $b \in B - A$, then $A \cup \{b\}$ is a TDS. Repeating this argument shows that $A \leftrightarrow B$. Therefore, if $A \overset{k}{\leftrightarrow} A'$ for all MTDSs A, A' of cardinality at most k, then $D^t_k(G)$ is connected. We state these facts explicitly for referencing.

Observation 2.1 *Let A, B, C be total dominating sets of a graph G of cardinality at most $k \geq \gamma_t(G)$.*

(i) If $A \subseteq B$, then $A \overset{k}{\leftrightarrow} B$.

(ii) If $A \cup B \subseteq C$, then $A \overset{k}{\leftrightarrow} C \overset{k}{\leftrightarrow} B$.

(iii) If $A \overset{k}{\leftrightarrow} A'$ for all MTDSs A, A' of cardinality at most k, then $D^t_k(G)$ is connected.

As in the case of dominating sets, the connectedness of $D^t_k(G)$ however does not guarantee the connectedness of $D^t_{k+1}(G)$. For example, consider the tree $T = S_{2,2,2}$ (the spider with three legs of length 2 each) in Figure 1. This figure shows $D^t_6(T)$, where vertices are represented by copies of T, and the total dominating sets are indicated by the solid circles. The unique Γ_t-set is an isolated vertex in $D^t_\Gamma(T)$, so $D^t_{\Gamma_t}(T) = D^t_{n-1}(T)$ is disconnected.

In the case of dominating sets $D_{\Gamma(G)}(G)$ is disconnected whenever G has at least one edge (and hence at least two minimal dominating sets): let H be a nontrivial component of G and D a minimal dominating set of H. Then $V(H) - D$ is a dominating set of H and therefore contains a minimal dominating set D' disjoint from D. It follows that G has at least two minimal dominating sets. Let X be a Γ-set and Y any other minimal dominating set of G. By the minimality of X, Y is not a subset of X, and no superset of X belongs to $D_{\Gamma(G)}(G)$. Hence X and Y belong to different components of $D_{\Gamma(G)}(G)$.

For total domination the situation is not quite as simple. A fundamental difference between domination and total domination is that every graph with at least one edge has at least two different minimal dominating sets, whereas there are many graphs with a unique MTDS. Consider, for example, the double star $S(r, t)$, which consists of two adjacent vertices u and v such that u is adjacent to r leaves and v is adjacent to t leaves. By Remark 1.1, u and v belong to any TDS of $S(r, t)$. Since $\{u, v\}$ is an MTDS, it is the **only** MTDS of $S(r, t)$. Therefore $D^t_{\Gamma_t}(S(r, t)) = K_1$, which is connected. We show that the stems of G determine whether $D^t_{\Gamma_t}(G)$ is connected or not.

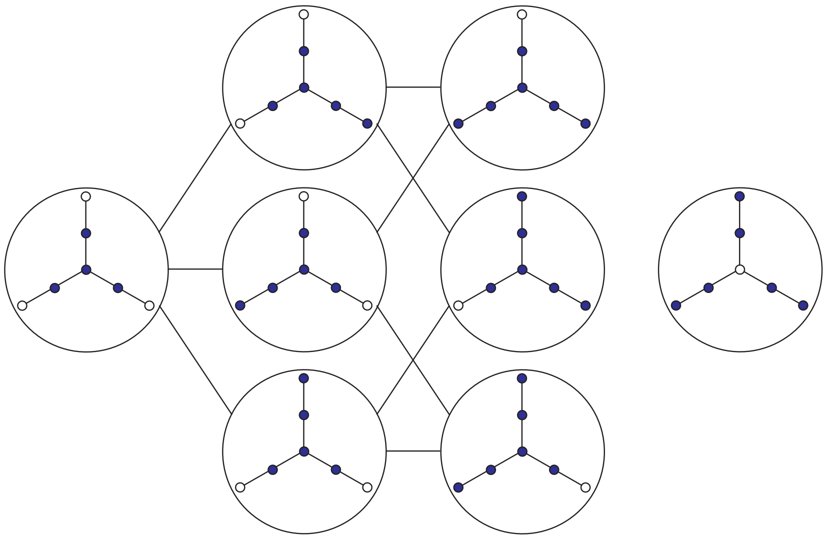

Figure 1: The graph $D_6^t(S_{2,2,2})$

Theorem 2.2 *Let G be a connected graph of order $n \geq 3$ with $\Gamma_t(G) = k$. Denote the set of stems of G by X. Then $D_k^t(G)$ is connected if and only if X is a TDS of G.*

Proof. Let S be any Γ_t-set of G. Then no subset of S is a TDS and no superset of S is a vertex of $D_k^t(G)$, hence S is an isolated vertex of $D_k^t(G)$. Therefore

$$D_k^t(G) \text{ is connected if and only if } S \text{ is the only MTDS of } G. \tag{1}$$

Suppose X is a TDS of G. Any $x \in X$ is adjacent to a leaf, hence $X - \{x\}$ does not dominate G. Therefore X is an MTDS. This implies that no superset of X is an MTDS. But by Remark 1.1, X is contained in any TDS of G. Consequently, X is the only MTDS of G, so $\gamma_t(G) = \Gamma_t(G) = \sigma(G)$ and $D_k^t(G) = K_1$.

Conversely, suppose X is not a TDS of G. We show that G has at least two MTDSs. First assume that X dominates G. Then $G[X]$ has an isolated vertex, say x, which is adjacent to a leaf $\ell_x \notin X$. Now

$$Y = X \cup \{\ell_x : x \text{ is an isolated vertex of } G[X]\}$$

is an MTDS of G. For another MTDS of G, let T' be a spanning tree of G, let T be the subtree of T' obtained by deleting all leaves of T' and let $Z = V(T)$. If $|Z| = 1$, then $T = K_1$ and T' is a star. Say $Z = \{z\}$. Then z is a universal vertex of G and $X = \{z\}$. Since $n \geq 3$ there exists a vertex $y \in N(z) - \ell_z$, and $\{y, z\}$ is an MTDS of G distinct from Y. On the other hand, if $|Z| \geq 2$, then Z is a TDS that does not contain any leaves of G. Hence Z contains an MTDS distinct from Y.

Now assume that X does not dominate G and let S be any MTDS of G. Then there exists a vertex $v \in S - X$. Since $v \notin X$, $\deg_G(u) \geq 2$ for each $u \in N(v)$. Let $S' = V(G) - \{v\}$. Since G is connected, every vertex not adjacent to v has a neighbour in S', and every vertex in $N(v)$ has at least one other neighbour in S'. Thus S' is a TDS, and since $v \notin S'$, S' contains an MTDS distinct from S. The result follows from (1). ■

The class of graphs whose stems form a TDS includes (but is not limited to) the generalized coronas of graphs without isolated vertices. Hence:

Corollary 2.3 *Any graph without isolated vertices is an induced subgraph of a graph G such that $D^t_{\Gamma_t}(G)$ is connected.*

The first paragraph of the proof of Theorem 2.2 implies the following result.

Corollary 2.4 *The graph $D^t_{\Gamma_t}(G)$ is disconnected if and only if G has at least two MTDSs.*

If the set of stems of G is a TDS, then it is the unique MTDS of G, hence we also have the following corollary.

Corollary 2.5 *If G is a connected graph of order $n \geq 3$ whose set of stems is a TDS, then $D^t_{\gamma_t}(G)$ is connected.*

The converse of Corollary 2.5 does not hold—for the spider $S(2,2,2)$ in Figure 1, $D^t_4(S(2,2,2)) = K_1$, which is connected, but the stems form an independent set of cardinality 3, which is not a TDS.

Since any TDS of cardinality greater than Γ_t contains a TDS of cardinality Γ_t, the following result is immediate from Observation 2.1(i) (and similar to [10, Lemma 4]).

Lemma 2.6 *If $k \geq \Gamma_t(G)$ and $D^t_k(G)$ is connected, then $D^t_{k+1}(G)$ is connected.*

We now know that

$$\Gamma_t(G) \leq d_0(G) \leq n$$

for any connected graph of order $n \geq 3$, and that the first inequality is strict if and only if the stems of G do not form a TDS. Equality in the upper bound is realized by graphs with upper total domination number equal to $n-1$, as characterized in Proposition 1.5, because all these graphs also have an MTDS of cardinality $\frac{n-1}{2}+1$ different from the Γ_t-set described in the proof, so $D^t_{n-1}(G)$ is disconnected. We next show that if $\Gamma_t(G) < n-1$, then $d_0(G) \leq n-1$.

Theorem 2.7 *If G is a connected graph of order $n \geq 3$ such that $\Gamma_t(G) < n-1$, then $d_0(G) \leq \min\{n-1, \Gamma_t(G)+\gamma_t(G)\}$.*

Proof. Let X be the set of stems of G. Suppose first that G has a unique MTDS S, so that $d_0(G) = \Gamma_t(G)$ by Corollary 2.4. By Remark 1.1, X is the unique MTDS of G, hence $|X| \geq 2$. But each vertex of X is adjacent to a leaf, hence $n \geq 2|X| \geq |X|+2$. Therefore

$$d_0(G) = \Gamma_t(G) \leq \left\lfloor \frac{n}{2} \right\rfloor \leq \left\{ \begin{array}{l} n-1 \\ \Gamma_t(G) + \gamma_t(G) \end{array} \right. .$$

Assume therefore that G has at least two MTDSs and let A and B be any two MTDSs of G. If $|A \cup B| \leq n-1$, then $A \overset{n-1}{\leftrightarrow} B$ by Observation 2.1(ii), hence assume $|A \cup B| = n$. By the hypothesis, $\Gamma_t(G) \leq n-2$, hence there exist distinct vertices $a_1, a_2 \in A - B$ and $b_1, b_2 \in B - A$. By Remark 1.1, $\{a_1, a_2, b_1, b_2\} \cap X = \varnothing$. Consider the four pairs $a_i, b_j,\ i, j = 1, 2$. Suppose first that for one of these pairs a_i, b_j, every vertex adjacent to both a_i and b_j has degree at least 3. Since we also have that $a_i, b_j \notin X$, $G - a_i - b_j$ has no isolated vertices. This implies that $V(G) - \{a_i, b_j\}$, $V(G) - \{a_i\}$ and $V(G) - \{b_j\}$ are TDS's of G, and we have

$$A \overset{n-1}{\leftrightarrow} V(G) - \{b_j\} \overset{n-1}{\leftrightarrow} V(G) - \{a_i, b_j\} \overset{n-1}{\leftrightarrow} V(G) - \{a_i\} \overset{n-1}{\leftrightarrow} B.$$

Hence assume that for each pair $a_i, b_j,\ i, j = 1, 2$, there exists a vertex $c_{i,j}$ such that $N(c_{i,j}) = \{a_i, b_j\}$. Then in $G_1 = G - a_1 - c_{1,1}$, $b_1 \sim c_{2,1}$, and $c_{1,2} \sim b_2$. Since $\deg(c_{1,1}) = 2$, a_1 and $c_{1,1}$ have no common neighbours except possibly b_1 (which is adjacent to $c_{2,1}$). Therefore G_1 has no isolated vertices, which means that $V(G) - \{a_1, c_{1,1}\}$ is a TDS of G. Similarly, $V(G) - \{b_1, c_{1,1}\}$ and $V(G) - \{c_{1,1}\}$ are TDS's of G. Now

$$\begin{aligned} A \quad & \overset{n-1}{\leftrightarrow} V(G) - \{b_1\} \overset{n-1}{\leftrightarrow} V(G) - \{b_1, c_{1,1}\} \overset{n-1}{\leftrightarrow} V(G) - \{c_{1,1}\} \\ & \overset{n-1}{\leftrightarrow} V(G) - \{a_1, c_{1,1}\} \overset{n-1}{\leftrightarrow} V(G) - \{a_1\} \overset{n-1}{\leftrightarrow} B. \end{aligned}$$

By Observation 2.1(iii), $d_0(G) \leq n-1$.

Now let C be any fixed γ_t-set and B any MTDS of G. Then $|C \cup B| \leq |C| + |B| \leq \gamma_t(G) + \Gamma_t(G)$. By Observation 2.1($ii$), $C \overset{\gamma_t+\Gamma_t}{\leftrightarrow} B$. By transitivity, $A \overset{\gamma_t+\Gamma_t}{\leftrightarrow} B$ for all MTDSs A, B of G, so by Observation 2.1(iii), $d_0(G) \leq \Gamma_t(G) + \gamma_t(G)$. ■

To summarise, in this section we showed that

- for any connected graph G of order $n \geq 3$,

$$\Gamma_t(G) \leq d_0(G) \leq n. \tag{2}$$

- The lower bound in (2) is realized if and only if G has exactly one MTDS, i.e., if and only if the stems of G form a TDS.

- The upper bound in (2) is realized if and only if $\Gamma_t(G) = n-1$, i.e., if and only if n is odd and G is obtained from $\frac{n-1}{2}K_2$ by joining a new vertex to at least one vertex of each K_2.

3 Determining d_0 for paths and cycles

Our aim in this section is to show (in Theorem 3.9) that $d_0(C_8) = \Gamma_t(C_8) + 2$ and $d_0(C_n) = \Gamma_t(C_n) + 1$ if $n \neq 8$. Similar techniques can be used to show that $d_0(P_2) = \Gamma_t(P_2) = d_0(P_4) = \Gamma_t(P_4) = 2$ and $d_0(P_n) = \Gamma_t(P_n) + 1$ if $n = 3$ or $n \geq 5$. We need four lemmas (Lemmas 3.5 – 3.8) to obtain the result for cycles. To enhance the logical flow of the paper, we only state the lemmas in this section and defer their proofs to Section 5.

It is easy to determine the total domination numbers of paths and cycles.

Observation 3.1 [14, Observation 2.9] *For $n \geq 3$,*

$$\gamma_t(P_n) = \gamma_t(C_n) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \equiv 2 \pmod{4} \\ \left\lceil \frac{n}{2} \right\rceil & \text{otherwise.} \end{cases}$$

The upper total domination number for paths was determined by Dorbec, Henning and McCoy [8].

Proposition 3.2 [8] *For any $n \geq 2$, $\Gamma_t(P_n) = 2\left\lfloor \frac{n+1}{3} \right\rfloor$.*

A proof of the following proposition on the upper total domination number of cycles can be found in the appendix of [1], which is a preprint of this work.

Proposition 3.3 *For any $n \geq 3$,*

$$\Gamma_t(C_n) = \begin{cases} 2\left\lfloor \frac{n}{3} \right\rfloor & \text{if } n \equiv 2 \pmod{6} \\ \left\lfloor \frac{2n}{3} \right\rfloor & \text{otherwise.} \end{cases}$$

Let $C_n = (v_0, v_1, \ldots, v_{n-1}, v_0)$. When discussing subsets of $\{v_0, v_1, \ldots, v_{n-1}\}$ the arithmetic in the subscripts is performed modulo n. We mention some obvious properties of minimal total dominating sets of C_n.

Remark 3.4 *Let S be an MTDS of C_n. Then*

(*i*) *each component of $C_n[S]$ is either P_2, P_3 or P_4;*

(*ii*) *each P_3 or P_4 component is preceded and followed by exactly two consecutive vertices of $C_n - S$;*

(*iii*) *$C_n - S$ does not contain three consecutive vertices of C_n.*

Using the next four lemmas, we show in Theorem 3.9 that, with the single exception of $n = 8$, $d_0(C_n) = \Gamma_t(C_n) + 1$. We only state the lemmas here; their proofs are given in Section 5. The first lemma concerns MTDSs that induce P_3 or P_4 components.

Lemma 3.5 *Let $n \geq 10$.*

(*i*) *If S is an MTDS such that $C_n[S]$ contains a P_4 component, then S is connected, in $D^t_{|S|+1}(C_n)$, to an MTDS without P_4 components.*

(*ii*) *If S is an MTDS such that $C_n[S]$ contains two consecutive P_3 components, then S is connected, in $D^t_{|S|+1}(C_n)$, to an MTDS with fewer P_3 components.*

(*iii*) *If S is an MTDS such that $C_n[S]$ contains at least one P_3 and at least one P_2 component but no P_4 components, then S is connected, in $D^t_{\Gamma_t+1}(C_n)$, to an MTDS that has no P_3 components.*

The next lemma concerns MTDSs that induce only P_2 components. For brevity we refer to such an MTDS as a P_2-*MTDS*. For a P_2-MTDS S, each P_2 component is followed by one or two vertices not belonging to S. We refer to these P_2 components as $P_2\overline{P}_1$ and $P_2\overline{P}_2$ *components*, respectively. An MTDS S is called a *maximum* P_2-MTDS if S is a P_2-MTDS of maximum cardinality.

Lemma 3.6 *Let S be a P_2-MTDS of C_n, $n \geq 10$.*

(*i*) *S is a maximum P_2-MTDS if and only if $C_n[S]$ has at most two $P_2\overline{P}_2$ components.*

(*ii*) *If $C_n[S]$ has at least one $P_2\overline{P}_1$ component and S' is any P_2-MTDS such that $|S| \leq |S'| \leq |S| + 2$, then $S \overset{|S|+3}{\leftrightarrow} S'$.*

We next consider C_n, $n \equiv 0 \pmod 4$.

Lemma 3.7 *Suppose $n \geq 8$ and $n \equiv 0 \pmod 4$; say $n = 4k$. (By Observation 3.1, $\gamma_t(C_n) = 2k$.) Then*

(*i*) *$D^t_{2k+1}(C_n)$ is disconnected;*

(*ii*) *if $n \geq 12$, then C_n has a P_2-MTDS S^* such that $|S^*| = 2k + 2$ and $C_n[S^*]$ has four $P_2\overline{P}_1$ components;*

(*iii*) *all γ_t-sets belong to the same component of $D^t_{2k+2}(C_n)$ and all P_2-MTDSs of cardinality $2k$ or $2k + 2$ belong to the same component of $D^t_{2k+3}(C_n)$.*

Our final lemma concerns small cycles.

Lemma 3.8 *If $3 \leq n \leq 9$ and $n \neq 8$, then $d_0(C_n) = \Gamma_t(C_n) + 1$.*

Theorem 3.9 *For $n = 8$, $d_0(C_8) = \Gamma_t(C_8) + 2$. In all other cases, $d_0(C_n) = \Gamma_t(C_n) + 1$.*

Proof. Since $\gamma_t(C_8) = \Gamma_t(C_8) = 4$, Lemma 3.7($i$) implies that $D^t_{\Gamma_t+1}(C_8)$ is disconnected and then the first part of Lemma 3.7(iii) implies that $d_0(C_8) = \Gamma_t(C_8) + 2$. By Lemma 3.8, the theorem is true for $3 \leq n \leq 7$ and $n = 9$. Hence assume $n \geq 10$.

Let S be any MTDS of C_n. By Lemma 3.5 (possibly applied several times), if $C_n[S]$ has a P_3 or P_4 component, then there exists a P_2-MTDS S^* such that $S \overset{\Gamma_t+1}{\leftrightsquigarrow} S^*$. Thus we may assume that S is a P_2-MTDS. If $n \not\equiv 0 \pmod 4$, then S has at least one $P_2\overline{P}_1$ component. If $n \equiv 0 \pmod 4$, then $n \geq 12$ and, by Lemma 3.7, all P_2-MTDSs of cardinality $\frac{n}{2}$ or $\frac{n}{2} + 2$ belong to the same component of $D^t_{2k+3}(C_n)$. Moreover, any P_2-MTDS of cardinality $\frac{n}{2} + 2$ has a $P_2\overline{P}_1$ component. In either case repeated application of Lemma 3.6(ii) shows that all P_2-MTDSs belong to the same component of $D^t_{\Gamma_t+1}(C_n)$. The result follows from Observation 2.1(iii), Corollary 2.4 and Lemma 2.6. ■

The proof of the following result for paths is similar and omitted. Note that for $n \equiv 0 \pmod 4$, $\Gamma_t(P_n) = \Gamma_t(C_n) + 2$, which explains the difference between $d_0(P_8)$ and $d_0(C_8)$. The result is trivial for $P_2 = K_2$, while the result for P_4 follows from Corollary 2.5.

Theorem 3.10 *$d_0(P_2) = \Gamma_t(P_2) = d_0(P_4) = \Gamma_t(P_4) = 2$ and $d_0(P_n) = \Gamma_t(P_n) + 1$ if $n = 3$ or $n \geq 5$.*

4 Realizability of graphs as total dominating graphs

One of the main problems in the study of k-total dominating graphs is determining which graphs are total dominating graphs. Since $D^t_k(G) = H$ if and only if $D^t_{k+2}(G \cup K_2) = H$, in studying graphs G such that $D^t_k(G) = H$ for a given graph H we restrict our investigation to graphs G without K_2 components (and also without isolated vertices, so that $\gamma_t(G)$ is defined).

As noted in [2, 17] for the k-dominating graph $D_k(G)$ of a graph G of order n, the k-total dominating graph $D^t_k(G)$ is similarly a subgraph of the hypercube Q_n (provided $k \geq \gamma_t(G)$ and G has no isolated vertices) and is therefore bipartite. Since any subset of $V(K_n)$ of cardinality at least 2 is a TDS of K_n and since Q_n is vertex transitive, $D^t_n(K_n) \cong Q_n - N[v]$ for some $v \in V(Q_n)$. We show in Corollary 4.2(i) that Q_n itself is realizable as the k-total dominating graph of several graphs, and in Corollary 4.2(ii) that stars $K_{1,n}$, $n \geq 2$, are realizable. Again the set of stems plays an important role.

In the last two results of the section we determine the realizability of paths and cycles.

Theorem 4.1 *Let H be any graph of order r, $2 \leq r \leq n$, without isolated vertices and let G be a generalized corona of H having exactly n leaves. For each ℓ such that $0 \leq \ell \leq n$, $D^t_{r+\ell}(G)$ is the subgraph of Q_n corresponding to the collection of all k-subsets, $0 \leq k \leq \ell$, of an n-set.*

Proof. Every vertex of H is a stem of G. By Remark 1.1, $X = V(H)$ is contained in any TDS of G. Since H has no isolated vertices, X is an MTDS of G. As shown in the proof of Theorem 2.2, X is the only MTDS of G. For any set L of leaves of G, $X \cup L$ is a TDS of G. Moreover, for any sets L_1 and L_2 of leaves, $X \cup L_1 \overset{r+\ell}{\leftrightarrow} X \cup L_2$ if and only if each $|L_i| \leq \ell$ and L_1 is obtained from L_2 by adding or deleting exactly one vertex. The result now follows from the definitions of Q_n and $D^t_{r+\ell}(G)$. ■

Corollary 4.2 *Let H be any graph of order r, $2 \leq r \leq n$, without isolated vertices and let G be a generalized corona of H having exactly n leaves. For every integer $n \geq 2$,*

(*i*) $D^t_{r+n}(G) \cong Q_n$

(*ii*) $D^t_{r+1}(G) \cong K_{1,n}$.

Proof. (*i*) By Theorem 4.1, $D^t_{r+n}(G)$ is the subgraph of Q_n corresponding to the collection of all subsets of an n-set. Hence $D^t_{r+n}(G) \cong Q_n$.

(*ii*) By Theorem 4.1, $D^t_{r+1}(G)$ is the subgraph of Q_n corresponding to the empty set and all singleton subsets of an n-set. Hence $D^t_{r+1}(G) \cong K_{1,n}$. ■

We mentioned above that for a graph G without isolated vertices and $\gamma_t(G) \leq k$, $D^t_k(G)$ is a subgraph of Q_n. The strategy used in the proof of Theorem 4.1 enables us to be a little more specific in many cases.

Proposition 4.3 *Let G be a connected graph of order $n \geq 3$ having $\sigma(G)$ stems. For any $k \geq \gamma_t(G)$, $D^t_k(G)$ is a subgraph of $Q_{n-\sigma(G)}$.*

In particular, $D^t_n(G)$ is a subgraph of $Q_{n-\sigma(G)}$ in which the vertex corresponding to $V(G)$ has degree $\Delta(D^t_n(G)) = n - \sigma(G) = \Delta(Q_{n-\sigma(G)})$.

Proof. Let X be the set of stems of G. By Remark 1.1, X is contained in any TDS of G. Hence all TDS's of G are subsets of $V(G)$ that contain X, and there are $2^{n-\sigma(G)}$ such sets. This shows that $D^t_k(G)$ is a subgraph of $Q_{n-\sigma(G)}$ for any $k \geq \gamma_t(G)$.

Now consider $D^t_n(G)$. For $v \in V(G)$, $G - v$ has an isolated vertex if and only if $v \in X$. Therefore $V(G) - \{u\}$ is a TDS of G if and only if $u \in V(G) - X$, which implies that $\deg(V(G)) = n - \sigma(G)$ in $D^t_n(G)$. Let S be any TDS of G; necessarily, $X \subseteq S$. There are at most $n - |S|$ supersets of S of cardinality $|S| + 1$ that are TDS's, and at most $|S| - \sigma(G)$ subsets of S of cardinality $|S| - 1$ that are TDS's. Hence in $D^t_n(G)$, $\deg(S) \leq n - |S| + |S| - \sigma(G) = n - \sigma(G)$. ■

Concerning the realizability of cycles, it is easily seen that $D^t_4(P_4) \cong C_4$, $D^t_3(C_4) \cong D^t_5(P_6) \cong C_8$, $D^t_4(C_5) \cong C_{10}$ and, if G is the graph obtained by joining two leaves of $K_{1,3}$, then $D^t_3(G) \cong D^t_3(K_{1,3}) \cong C_6$. We show that C_{2r}, $r \in \{2, 3, 4, 5\}$, are the only cycles realizable as k-total domination graphs.

Proposition 4.4 *(i) There is no graph G of order $n > 6$ such that $D_k^t(G) \cong C_m$ for some integer k.*

(ii) For $m > 10$, there is no graph G such that $D_k^t(G) \cong C_m$ for some integer k.

Proof. (i) Suppose to the contrary that $D_k^t(G) \cong C_m$. Let S be a γ_t-set of G. Then $\deg(S) = 2$ in $D_k^t(G)$. Since each superset of S is a TDS of G, S has exactly two supersets of cardinality $|S|+1$. This implies that $n-|S| = 2$, i.e., $n-\gamma_t(G) = 2$. But we know that $\gamma_t(G) \leq \frac{2n}{3}$ (Proposition 1.4(i)) and so $n \leq 6$, which is a contradiction.

(ii) Now suppose that $D_k^t(G) \cong C_m$, where $m > 10$. By (i), G has order $n \leq 6$ and $n - \gamma_t(G) = 2$. Say $D_k^t(G)$ is the cycle $(S_1, S_2, \ldots, S_m, S_1)$. Since $n = \gamma_t(G) + 2$, each S_i has cardinality $\gamma_t(G)$, $\gamma_t(G) + 1$ or $\gamma_t(G) + 2$.

First assume $|S_i| = \gamma_t(G) + 2 = n$ for some i. Then $n \geq 4$ and we also have $k = n$. Since S_i has degree 2 in $D_k^t(G)$, G has exactly two TDS's of cardinality $n-1$. By Remark 1.1, G has exactly two vertices that are not stems. Since $n \geq 4$ (and G has no K_2 components), G consists of two stems and two leaves, i.e., $G = P_4$. But $D_4^t(P_4) \cong C_4$, contradicting $m > 10$.

We may therefore assume that $k = n - 1$ and $n - 2 \leq |S_i| \leq n - 1$ for each i. But then, by definition of adjacency in $D_k^t(G)$, $|S_i| = n - 1$ for $\frac{m}{2}$ values of i and $|S_i| = n - 2$ for the other $\frac{m}{2}$ values of i. Since $m > 10$, $V(G)$ has at least six subsets of cardinality $n - 1$, which implies that $n \geq 6$. Therefore $n = 6$, $\gamma_t(G) = 4$, $k = 5$ and $m = 12$, and each of the six 5-subsets of $V(G)$ is a TDS. By Remark 1.1, G has no stems and hence no leaves. Let v be a vertex of G such that $\deg(v) = \Delta(G)$. If $\deg(v) = 5$, then $\{u, v\}$ is a TDS for any $u \in V(G) - \{v\}$, which contradicts $\gamma_t(G) = 4$. If $\deg(v) = 4$, let u be the unique vertex nonadjacent to v and let w be any vertex adjacent to u. Then $G[\{u, v, w\}] \cong P_3$, so $\{u, v, w\}$ is a TDS, also a contradiction. If $\deg(v) = 3$, let u_1 and u_2 be the vertices nonadjacent to v. Since G has no leaves, each u_i is adjacent to a vertex $w_i \in N(v)$. Hence $\{v, w_1, w_2\}$ is a TDS of cardinality at most 3, again a contradiction. Therefore G is 2-regular. But if $G = 2K_3$, then G has $\binom{3}{2}^2 = 9 > 6$ TDS's of cardinality 4, and if $G = C_6$, then any vertex of $D_5^t(G)$ corresponding to five consecutive vertices of C_6 has degree 3. With this final contradiction the proof is complete. ■

The realizability of paths is somewhat similar to that of cycles in that only a small number of paths are k-total dominating graphs. Since $D_2^t(K_2) \cong P_1$ and $D_3^t(P_3) \cong D_3^t(P_4) \cong D_4^t(P_5) \cong P_3$, P_1 and P_3 are realizable. Indeed, they are the only realizable paths, as we show next.

Proposition 4.5 *For $m \neq 1, 3$, there is no graph G such that $D_k^t(G) \cong P_m$ for some integer k.*

Proof. Suppose G is a graph of order n such that $D_k^t(G) \cong P_m$ for some integer k. Say $P_m = (S_1, S_2, \ldots, S_m)$, where each S_i is a TDS of G. It is easy to examine all graphs of order at most 3, hence assume $n \geq 4$.

If S_1 is a γ_t-set of G, then exactly one superset of S_1 of cardinality $|S_1| + 1$, namely S_2, is a TDS. Since every superset of S_1 is a TDS, $\gamma_t(G) = n - 1 \leq \frac{2n}{3}$ (by

Proposition 1.4(i)), hence $n \leq 3$, contrary to the assumption above. Thus we may assume that S_1 and (similarly) S_m have cardinality at least $\gamma_t(G)+1$. Therefore S_i, for some $1 < i < m$, is a γ_t-set. Exactly as in the proof of Proposition 4.4 we obtain that $n \leq 6$, $\gamma_t = n-2$ and each S_i has cardinality $\gamma_t(G)$, $\gamma_t(G)+1$ or $\gamma_t(G)+2$. If $|S_i| = \gamma_t(G)+2$, then, as shown in the proof of Proposition 4.4, $G = P_4$ and $D_4^t(P_4) \cong C_4 \not\cong P_m$. Therefore $|S_1| = |S_m| = \gamma_t(G)+1 = n-1 = k$. Now, S_2 is a γ_t-set of cardinality $n-2$, hence S_1 and S_3 are the only supersets of S_2 of cardinality $n-1$, and S_2, in turn, is the only subset of S_1 that is a TDS. Therefore

$$\mathrm{OPN}(v, S_1) = \varnothing \text{ for exactly one vertex } v \in S_1. \tag{3}$$

Let $G_1 = G[S_1]$. Since $4 \leq n \leq 6$, $3 \leq |S_1| \leq 5$. Suppose G_1 contains a triangle, say (a_1, a_2, a_3, a_1). Then by (3) there exists a $b_i \in \mathrm{OPN}(a_i, S_1)$ for $i = 1, 2$ (without loss of generality), where $b_1 \neq b_2$ and $\{b_1, b_2\} \cap \{a_1, a_2, a_3\} = \varnothing$. Since $|S_1| = n-1$, b_1 or b_2 belongs to S_1.

Say $b_1 \in S_1$. Then $b_1 \in \mathrm{IPN}(a_1, S_1)$ and we also have from (3) that $\mathrm{OPN}(b_1, S_1) \neq \varnothing$ or $\mathrm{OPN}(a_3, S_1) \neq \varnothing$. But if $\mathrm{OPN}(a_3, S_1) \neq \varnothing$, then $n = 6$ and $\{a_1, a_2, a_3\}$ is a TDS of G of cardinality $n-3$, which is not the case. Therefore there exists $c_1 \in \mathrm{OPN}(b_1, S_1)$. But then $V(G) = \{a_1, a_2, a_3, b_1, b_2, c_1\}$ and $\{a_1, a_2, b_1\}$ is a TDS of G, again a contradiction. We conclude that G_1 is triangle-free.

If G_1 contains an r-cycle, $r \geq 4$, then (3) implies that at least $r-1$ vertices of the cycle have private neighbours not on the cycle. But then $n \geq 7$, a contradiction. Therefore G_1 is acyclic. If G_1 has two K_2 components, then, by the restrictions on the order of G_1, $G_1 = 2K_2$ and $\mathrm{IPN}(v, S_1) \neq \varnothing$ for each $v \in S_1$, contrary to (3).

Suppose G_1 has a path component $P_r = (u_1, u_2, \ldots, u_r)$, $r \geq 3$. Then neither leaf of P_r has an internal private neighbour, so, by (3), one of them has an external private neighbour. Say u_1 has external private neighbour w_1. Since $|S_1| = n-1$, $V(G) = S_1 \cup \{w_1\}$, and since $w_1 \in \mathrm{EPN}(u_1, S_1)$, $\deg_G(w_1) = 1$. Now, if S_1 is disconnected, then $G_1 = K_2 \cup P_r$, and since $\deg_G(w_1) = 1$, K_2 is also a component of G. Hence $G = K_2 \cup P_4$, so $D_5^t(G) = D_3^t(P_4) = P_3$. On the other hand, if S_1 is connected, then G is isomorphic to P_4, P_5 or P_6, in which case $D_{n-1}^t(G)$ is P_3 or C_8.

Finally, suppose G_1 is a tree but not a path. Then G_1 has at least three leaves. By (3), two of them have external private neighbours, contrary to $|S_1| = n-1$. ■

A *full subgraph* of Q_n is a subgraph that corresponds to all subsets of cardinality at most k of an n-set, for some integer k such that $0 \leq k \leq n$. In this section we showed that

- all full subgraphs of Q_n, $n \geq 2$, are realizable as k-total dominating graphs.
- In particular, Q_n and $K_{1,n}$ are realizable for all $n \geq 2$.

We also showed that

- C_4, C_6, C_8 and C_{10} are the only realizable cycles, and
- P_1 and P_3 are the only realizable paths.

5 Proofs of lemmas in Section 3

This section contains the proofs of the lemmas stated in Section 3. To simplify our discussion of total dominating sets of C_n, we encode each TDS S using an n-tuple (or part of an n-tuple) of the symbols $\circ$ and $\bullet$, where $\bullet$ in position i indicates that $v_{i-1} \in S$, while $\circ$ in position i indicates that $v_{i-1} \notin S$. For example, the MTDS $S = \{v_0, v_1, v_4, v_5\}$ of C_8 is written as $S = (\bullet\bullet\circ\circ\bullet\bullet\circ\circ)$. By Remark 3.4($ii$), every P_3 or P_4 component belongs to a code of the form $(\cdots\circ\circ\bullet\bullet\bullet\circ\circ\cdots)$ or $(\cdots\circ\circ\bullet\bullet\bullet\bullet\circ\circ\cdots)$, respectively. The n-tuples are often compressed by writing the number of consecutive occurrences of $\circ$ or $\bullet$ above the symbol; for example, we may write $(\cdots\circ\circ\bullet\bullet\bullet\circ\circ\cdots)$ and $(\cdots\circ\circ\bullet\bullet\bullet\bullet\circ\circ\cdots)$ as $(\cdots\overset{2}{\circ}\overset{3}{\bullet}\overset{2}{\circ}\cdots)$ and $(\cdots\overset{2}{\circ}\overset{4}{\bullet}\overset{2}{\circ}\cdots)$, respectively. When a P_2-component of S can be followed by one of two vertices of $C_n - S$, we write $(\cdots\overset{2}{\bullet}\circ\cdots)$ without indicating a number above $\circ$.

When adding vertices to a TDS, for example, when adding a vertex to $(\cdots\overset{2}{\circ}\overset{2}{\bullet}\overset{2}{\circ}\cdots)$ to form $(\cdots\overset{2}{\circ}\overset{3}{\bullet}\overset{1}{\circ}\cdots)$, we write $(\cdots\overset{2}{\circ}\overset{2}{\bullet}\overset{2}{\circ}\cdots) \to (\cdots\overset{2}{\circ}\overset{3}{\blacklozenge}\overset{1}{\lozenge}\cdots)$ to emphasize the position of the addition. Conversely, when deleting a vertex from $(\cdots\overset{2}{\circ}\overset{3}{\bullet}\overset{1}{\circ}\cdots)$ (say) to form $(\cdots\overset{2}{\circ}\overset{2}{\bullet}\overset{2}{\circ}\cdots)$, we write $(\cdots\overset{2}{\circ}\overset{3}{\bullet}\overset{1}{\circ}\cdots) \to (\cdots\overset{2}{\circ}\overset{2}{\blacklozenge}\overset{2}{\lozenge}\cdots)$.

We restate the lemmas for convenience.

Lemma 3.5 *Let $n \geq 10$.*

(i) *If S is an MTDS such that $C_n[S]$ contains a P_4 component, then S is connected, in $D^t_{|S|+1}(C_n)$, to an MTDS without P_4 components.*

(ii) *If S is an MTDS such that $C_n[S]$ contains two consecutive P_3 components, then S is connected, in $D^t_{|S|+1}(C_n)$, to an MTDS with fewer P_3 components.*

(iii) *If S is an MTDS such that $C_n[S]$ contains at least one P_3 and at least one P_2 component but no P_4 components, then S is connected, in $D^t_{\Gamma_t+1}(C_n)$, to an MTDS that has no P_3 components.*

Proof. (i) By Remark 3.4(i), S is of the form $(\cdots\overset{2}{\circ}\overset{4}{\bullet}\overset{2}{\circ}\cdots)$. Consider the TDS S' with $|S'| = |S|$ obtained by

$$(\cdots\overset{2}{\circ}\overset{4}{\bullet}\overset{2}{\circ}\cdots) \to (\cdots\overset{2}{\circ}\overset{5}{\blacklozenge}\overset{1}{\lozenge}\cdots) \to (\cdots\overset{2}{\circ}\overset{2}{\blacklozenge}\overset{1}{\lozenge}\overset{2}{\blacklozenge}\overset{1}{\circ}\cdots) = S'.$$

If S' is an MTDS, let $S'' = S'$. If S' is not an MTDS, then S' is of the form $(\cdots\overset{2}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\overset{3}{\bullet}\cdots)$ or $(\cdots\overset{2}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\overset{4}{\bullet}\cdots)$. In the former case, let $S'' = (\cdots\overset{2}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\bullet}\overset{2}{\lozenge}\overset{2}{\blacklozenge}\cdots)$ and in the later case let $S'' = (\cdots\overset{2}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\bullet}\overset{2}{\lozenge}\overset{3}{\blacklozenge}\cdots)$. In all cases, S'' is an MTDS having fewer P_4 components than S such that $S \overset{|S|+1}{\leftrightarrow} S''$ and $|S''| \leq |S|$. The result follows by repeating this procedure.

(*ii*) The result follows from the operations

$$(\cdots\overset{2}{\circ}\overset{3}{\bullet}\overset{2}{\circ}\overset{3}{\bullet}\overset{2}{\circ}\cdots)\to(\cdots\overset{2}{\circ}\overset{3}{\bullet}\overset{1}{\lozenge}\overset{4}{\blacklozenge}\overset{2}{\circ}\cdots)\to(\cdots\overset{2}{\circ}\overset{2}{\blacklozenge}\overset{2}{\lozenge}\overset{4}{\bullet}\overset{2}{\circ}\cdots)\to(\cdots\overset{2}{\circ}\overset{2}{\bullet}\overset{1}{\lozenge}\overset{5}{\blacklozenge}\overset{2}{\circ}\cdots)\to(\cdots\overset{2}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\blacklozenge}\overset{1}{\lozenge}\overset{2}{\blacklozenge}\overset{2}{\circ}\cdots).$$

(*iii*) If $C_n[S]$ contains at least one P_3 and at least one P_2 component but no P_4 components, then $C_n[S]$ contains either (a) two consecutive P_3 components or (b) a P_3 component preceded and followed by a P_2 component. In the former case, $C_n[S]$ is of the form $(\cdots\overset{2}{\circ}\overset{3}{\bullet}\overset{2}{\circ}\overset{3}{\bullet}\overset{2}{\circ}\cdots)$ and, by (*ii*), $S \overset{|S|+1}{\leftrightarrow} S'$ where $S' = (\cdots\overset{2}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\bullet}\overset{2}{\circ}\cdots)$. Moreover, $|S'| = |S|$. In the latter case, $C_n[S]$ is of the form $(\cdots\circ\overset{2}{\bullet}\overset{2}{\circ}\overset{3}{\bullet}\overset{2}{\circ}\overset{2}{\bullet}\circ\cdots)$, and $S'' = (\cdots\circ\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\bullet}\circ\cdots)$ is an MTDS of larger cardinality having fewer P_3 components than S. Hence S is not a Γ_t-set. The operations

$$(\cdots\circ\overset{2}{\bullet}\overset{2}{\circ}\overset{3}{\bullet}\overset{2}{\circ}\overset{2}{\bullet}\circ\cdots)\to(\cdots\circ\overset{2}{\bullet}\overset{1}{\lozenge}\overset{4}{\blacklozenge}\overset{2}{\circ}\overset{2}{\bullet}\circ\cdots)\to(\cdots\circ\overset{2}{\bullet}\overset{1}{\circ}\overset{5}{\blacklozenge}\overset{1}{\lozenge}\overset{2}{\bullet}\circ\cdots)\to(\cdots\circ\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\blacklozenge}\overset{1}{\lozenge}\overset{2}{\blacklozenge}\overset{1}{\circ}\overset{2}{\bullet}\circ\cdots)=S''$$

show that $S \overset{|S''|+1}{\leftrightarrow} S'$, hence $S \overset{\Gamma_t+1}{\leftrightarrow} S''$. By repeating the operations for (a) and (b) as necessary we obtain the desired result. ■

Lemma 3.6 *Let S be a P_2-MTDS of C_n, $n \geq 10$.*

(*i*) *S is a maximum P_2-MTDS if and only if $C_n[S]$ has at most two $P_2\overline{P}_2$ components.*

(*ii*) *If $C_n[S]$ has at least one $P_2\overline{P}_1$ component and S' is any P_2-MTDS such that $|S| \leq |S'| \leq |S| + 2$, then $S \overset{|S|+3}{\leftrightarrow} S'$.*

Proof. (*i*) Suppose $C_n[S]$ has p components, q of which are $P_2\overline{P}_2$ components. Then $|S| = 2p$ and $n = 4q + 3(p - q)$. The result follows by comparing these numbers to the formula for $\Gamma_t(C_n)$ in Proposition 3.3.

(*ii*) First note that if S is of the form $(\cdots\circ\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\bullet}\overset{2}{\circ}\cdots\overset{2}{\bullet}\overset{2}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\cdots)$, then repeating the operations

$$(\cdots\circ\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\bullet}\overset{2}{\circ}\cdots\overset{2}{\bullet}\overset{2}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\cdots)\to(\cdots\circ\overset{2}{\bullet}\overset{1}{\circ}\overset{3}{\blacklozenge}\overset{1}{\lozenge}\cdots\overset{2}{\bullet}\overset{2}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\cdots)\to(\cdots\circ\overset{2}{\bullet}\overset{2}{\lozenge}\overset{2}{\blacklozenge}\overset{1}{\circ}\cdots\overset{2}{\bullet}\overset{2}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\cdots)$$

as necessary shows that S is connected in $D^t_{|S|+1}(C_n)$ to an MTDS of the same cardinality, hence with the same number of both types of components, in which all the components of each type appear consecutively.

Thus, assume S is of the form $(\underline{\bullet}\bullet\overset{1}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\cdots\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\bullet}\overset{2}{\circ}\overset{2}{\bullet}\overset{2}{\circ}\cdots)$, where all the components of each type appear consecutively and $\underline{\bullet}$ is a marked vertex to indicate the position of the first $P_2\overline{P}_1$ component. The operations

$$
\begin{aligned}
&(\underline{\bullet}\,\bullet\,\overset{1}{\circ}\,\overset{2}{\bullet}\,\overset{1}{\circ}\,\overset{2}{\bullet}\,\overset{1}{\circ}\cdots\overset{2}{\bullet}\,\overset{1}{\circ}\,\overset{2}{\bullet}\,\overset{2}{\circ}\cdots\overset{2}{\bullet}\,\overset{2}{\circ}) \to (\underline{\bullet}\overset{4}{\blacklozenge}\,\overset{1}{\circ}\,\overset{2}{\bullet}\,\overset{1}{\circ}\cdots\overset{2}{\bullet}\,\overset{1}{\circ}\,\overset{2}{\bullet}\,\overset{2}{\circ}\cdots\overset{2}{\bullet}\,\overset{2}{\circ})\\
&\to (\underline{\lozenge}\,\overset{4}{\bullet}\,\overset{1}{\circ}\,\overset{2}{\bullet}\,\overset{1}{\circ}\cdots\overset{2}{\bullet}\,\overset{1}{\circ}\,\overset{2}{\bullet}\,\overset{2}{\circ}\cdots\overset{2}{\bullet}\,\overset{2}{\circ}) \to (\underline{\circ}\overset{7}{\blacklozenge}\,\overset{1}{\circ}\cdots\overset{2}{\bullet}\,\overset{1}{\circ}\,\overset{2}{\bullet}\,\overset{2}{\circ}\cdots\overset{2}{\bullet}\,\overset{2}{\circ})\\
&\to (\underline{\circ}\overset{2}{\blacklozenge}\overset{1}{\lozenge}\overset{4}{\blacklozenge}\,\overset{1}{\circ}\cdots\overset{2}{\bullet}\,\overset{1}{\circ}\,\overset{2}{\bullet}\,\overset{2}{\circ}\cdots\overset{2}{\bullet}\,\overset{2}{\circ}) \to \cdots \to (\underline{\circ}\,\overset{2}{\bullet}\,\overset{1}{\circ}\,\overset{2}{\bullet}\,\overset{1}{\circ}\,\overset{2}{\bullet}\cdots\overset{4}{\blacklozenge}\,\overset{2}{\circ}\cdots\overset{2}{\bullet}\,\overset{2}{\circ})\\
&\to (\underline{\circ}\,\overset{2}{\bullet}\,\overset{1}{\circ}\,\overset{2}{\bullet}\,\overset{1}{\circ}\,\overset{2}{\bullet}\cdots\overset{5}{\blacklozenge}\overset{1}{\lozenge}\cdots\overset{2}{\bullet}\,\overset{2}{\circ}) \to (\underline{\circ}\,\overset{2}{\bullet}\,\overset{1}{\circ}\,\overset{2}{\bullet}\,\overset{1}{\circ}\,\overset{2}{\bullet}\cdots\overset{2}{\blacklozenge}\overset{1}{\lozenge}\overset{2}{\blacklozenge}\,\overset{1}{\circ}\cdots\overset{2}{\bullet}\,\overset{2}{\circ})\\
&\to \cdots \to (\underline{\circ}\,\overset{2}{\bullet}\,\overset{1}{\circ}\,\overset{2}{\bullet}\,\overset{1}{\circ}\,\overset{2}{\bullet}\cdots\overset{2}{\bullet}\,\overset{1}{\circ}\,\overset{2}{\bullet}\,\overset{1}{\circ}\cdots\overset{2}{\lozenge}\overset{2}{\blacklozenge}\overset{1}{\lozenge})
\end{aligned}
$$

produce a P_2-MTDS S'' which can also be obtained from S by a rotation $v_i \to v_{i+1}$ for each i. Thus $S \overset{|S|+1}{\leftrightarrow} S''$. These operations can be repeated to show that $S \overset{|S|+1}{\leftrightarrow} S_3$ for each rotation S_3 of S. By the above and transitivity, for each P_2-MTDS S' such that $|S| = |S'|$, $S \overset{|S|+1}{\leftrightarrow} S'$ and hence $S \overset{|S|+3}{\leftrightarrow} S'$.

Now assume that S' is any P_2-MTDS such that $|S'| = |S| + 2$. Then S is not a maximum P_2-MTDS and hence, by (i), S has at least three $P_2\overline{P}_2$ components. As shown above we may assume all $P_2\overline{P}_2$ components of $C_n[S]$ are consecutive. Hence S is of the form $(\overset{2}{\bullet}\overset{2}{\circ}\overset{2}{\bullet}\overset{2}{\circ}\overset{2}{\bullet}\overset{2}{\circ}\cdots\overset{2}{\bullet}\overset{1}{\circ}\cdots)$, and the addition of three vertices in succession produces a TDS of the form $(\overset{2}{\bullet}\overset{1}{\lozenge}\overset{7}{\blacklozenge}\,\overset{2}{\circ}\cdots\overset{2}{\bullet}\,\overset{1}{\circ}\cdots)$. Then the operations

$$
(\overset{2}{\bullet}\overset{1}{\circ}\overset{7}{\bullet}\overset{2}{\circ}\cdots\overset{2}{\bullet}\overset{1}{\circ}\cdots) \to (\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\blacklozenge}\overset{1}{\lozenge}\overset{4}{\blacklozenge}\overset{2}{\circ}\cdots\overset{2}{\bullet}\overset{1}{\circ}\cdots) \to (\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\overset{5}{\blacklozenge}\overset{1}{\lozenge}\cdots\overset{2}{\bullet}\overset{1}{\circ}\cdots) \to (\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\blacklozenge}\overset{1}{\lozenge}\overset{2}{\blacklozenge}\overset{1}{\lozenge}\cdots\overset{2}{\bullet}\overset{1}{\circ}\cdots) = S_1
$$

produce a P_2-MTDS S_1 such that $|S_1| = |S| + 2$ and $S_1 \overset{|S|+3}{\leftrightarrow} S$. However, we have already shown above that $S_1 \overset{|S_1|+1}{\leftrightarrow} S'$, i.e. $S_1 \overset{|S|+3}{\leftrightarrow} S'$. By transitivity, $S \overset{|S|+3}{\leftrightarrow} S'$. ■

Lemma 3.7 *Suppose $n \geq 8$ and $n \equiv 0 \pmod 4$; say $n = 4k$. (By Observation 3.1, $\gamma_t(C_n) = 2k$.) Then*

(i) $D^t_{2k+1}(C_n)$ is disconnected;

(ii) if $n \geq 12$, then C_n has a P_2-MTDS S^* such that $|S^*| = 2k + 2$ and $C_n[S^*]$ has four $P_2\overline{P}_1$ components;

(iii) all γ_t-sets belong to the same component of $D^t_{2k+2}(C_n)$ and all P_2-MTDSs of cardinality $2k$ or $2k + 2$ belong to the same component of $D^t_{2k+3}(C_n)$.

Proof. Any γ_t-set S of C_n is a P_2-MTDS, hence of the form $(\cdots\overset{2}{\circ}\overset{2}{\bullet}\overset{2}{\circ}\overset{2}{\bullet}\overset{2}{\circ}\cdots)$.

(i) By symmetry the addition of any single vertex v to S results in $S' = (\cdots\overset{2}{\circ}\overset{3}{\blacklozenge}\overset{1}{\lozenge}\overset{2}{\bullet}\overset{2}{\circ}\cdots)$, and by Remark 3.4$(iii)$, v is the only vertex whose deletion from S' produces a TDS, namely S. However, by symmetry, C_n has four γ_t-sets.

(ii) If $n \geq 12$, then by Observation 3.1 and Proposition 3.3, $\Gamma_t(C_n) \geq \gamma_t(C_n)+2$. Hence C_n has a P_2-MTDS S^* such that $|S^*| = 2k+2$ and $C_n[S^*]$ has $k+1$ components. Elementary calculations show that four components are $P_2\overline{P}_1$ components.

(iii) By adding two vertices in succession, then deleting a (different) vertex, we obtain

$$(\cdots \circ\, \circ\, \underline{\bullet}\, \bullet\, \overset{2}{\circ}\, \overset{2}{\bullet}\, \overset{2}{\circ}\, \overset{2}{\bullet} \cdots) \to (\cdots \overset{2}{\circ}\, \underline{\bullet} \overset{2}{\blacklozenge} \overset{1}{\lozenge} \overset{3}{\blacklozenge} \overset{1}{\lozenge}\, \overset{2}{\bullet} \cdots) \to (\cdots \overset{2}{\circ}\, \underline{\bullet}\, \overset{2}{\bullet}\, \overset{2}{\lozenge} \overset{2}{\blacklozenge}\, \overset{1}{\circ}\, \overset{2}{\bullet} \cdots),$$

where $\underline{\bullet}$ is a marker to indicate a specific vertex. Adding and deleting another vertex, we obtain

$$(\cdots \overset{2}{\circ}\, \underline{\bullet}\, \overset{2}{\bullet}\, \overset{2}{\circ}\, \overset{2}{\bullet}\, \overset{1}{\circ}\, \overset{2}{\bullet} \cdots) \to (\cdots \overset{2}{\circ}\, \underline{\bullet}\, \overset{2}{\bullet}\, \overset{2}{\circ}\, \overset{2}{\bullet}\, \overset{1}{\circ} \overset{3}{\blacklozenge} \cdots) \to (\cdots \overset{2}{\circ}\, \underline{\bullet}\, \overset{2}{\bullet}\, \overset{2}{\circ}\, \overset{2}{\bullet}\, \overset{2}{\lozenge} \overset{2}{\blacklozenge} \cdots).$$

Continuing the process, we eventually obtain the TDS $(\cdots \overset{2}{\blacklozenge} \overset{1}{\lozenge} \underline{\bullet}\, \overset{2}{\bullet}\, \overset{2}{\circ}\, \overset{2}{\bullet}\, \overset{2}{\circ}\, \overset{2}{\bullet} \cdots)$ of cardinality $\gamma_t(C_n) + 1$, and one last step—a vertex deletion—produces the γ_t-set

$$S' = (\cdots \overset{2}{\bullet}\, \circ \underline{\lozenge}\, \overset{2}{\bullet}\, \overset{2}{\circ}\, \overset{2}{\bullet}\, \overset{2}{\circ}\, \overset{2}{\bullet} \cdots).$$

Hence S' is obtained from S by a rotation $v_i \to v_{i+1}$ for each i. Repeating the procedure twice more shows that S is connected to each of the three other γ_t-sets of C_n.

Now let S^* be a P_2-MTDS of cardinality $2k + 2$. Then $n \geq 12$ and as shown in Lemma 3.6(ii) we may assume that the four $P_2\overline{P}_1$ components of $C_n[S^*]$ occur consecutively. Hence S^* is of the form $(\overset{2}{\bullet}\, \overset{1}{\circ}\, \overset{2}{\bullet}\, \overset{1}{\circ}\, \overset{2}{\bullet}\, \overset{1}{\circ}\, \overset{2}{\bullet}\, \overset{1}{\circ}\, \overset{2}{\bullet}\, \overset{2}{\circ} \cdots)$. The operations

$$(\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\bullet}\overset{2}{\circ}\cdots) \to (\overset{2}{\bullet}\overset{1}{\circ}\overset{5}{\blacklozenge}\overset{1}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\bullet}\overset{2}{\circ}\cdots) \to (\overset{2}{\bullet}\overset{2}{\lozenge}\overset{4}{\blacklozenge}\overset{1}{\circ}\overset{2}{\bullet}\overset{1}{\circ}\overset{2}{\bullet}\overset{2}{\circ}\cdots) \to (\overset{2}{\bullet}\overset{2}{\circ}\overset{7}{\blacklozenge}\overset{1}{\circ}\overset{2}{\bullet}\overset{2}{\circ}\cdots) \to (\overset{2}{\bullet}\overset{2}{\circ}\overset{2}{\blacklozenge}\overset{2}{\lozenge}\overset{2}{\blacklozenge}\overset{2}{\lozenge}\overset{2}{\bullet}\overset{2}{\circ}\cdots)$$

show that S^* belongs to the same component of $D^t_{2k+3}(C_n)$ as a γ_t-set of C_n. The result follows by transitivity. ■

Lemma 3.8 *If* $3 \leq n \leq 9$ *and* $n \neq 8$*, then* $d_0(C_n) = \Gamma_t(C_n) + 1$.

Proof. The result is obvious for $n \in \{3, 4, 5\}$ because $D^t_3(C_3) \cong K_{1,3}$, $D^t_3(C_4) \cong C_8$ and $D^t_4(C_5) \cong C_{10}$. All MTDSs of C_6 are of the form $(\overset{2}{\bullet}\, \overset{1}{\circ}\, \overset{2}{\bullet}\overset{1}{\circ})$ or $(\overset{4}{\bullet}\overset{2}{\circ})$, and one easily obtains that $d_0(C_6) = 5 = \Gamma_t(C_6) + 1$. All MTDSs of C_7 are of the form $(\overset{2}{\bullet}\, \overset{1}{\circ}\, \overset{2}{\bullet}\overset{2}{\circ})$ and the result is easy to check. Finally, all MTDSs of C_9 are of the form $(\overset{3}{\bullet}\, \overset{2}{\circ}\, \overset{2}{\bullet}\overset{2}{\circ})$ and $(\overset{2}{\bullet}\, \overset{1}{\circ}\, \overset{2}{\bullet}\, \overset{1}{\circ}\, \overset{2}{\bullet}\overset{1}{\circ})$ and again the result follows easily. ■

6 Problems for future work

Problem 6.1 Determine $d_0(G)$ for other classes of graphs.

As mentioned in the introduction, Mynhardt et al. [17] obtained constructions showing that the smallest integer k such that the domination graph $D_\ell(G)$ is connected for all $\ell \geq k$ can be as high as $\Gamma(G) + \gamma(G) - 1$, or $\Gamma(G) + \gamma(G)$ provided $\gamma(G) < \Gamma(G)$. For total domination graphs, C_8 is the only known graph for which $D^t_{\Gamma_t+1}(G)$ is disconnected. We therefore state the following open problem.

Problem 6.2 Construct classes of graphs G_α such that the difference satisfies $d_0(G_\alpha) - \Gamma_t(G_\alpha) \geq \alpha \geq 2$ (or show that the difference is bounded).

Problem 6.3 Find more classes of graphs that can/cannot be realized as k-total domination graphs.

As mentioned in [10] for domination graphs, $D_2(K_{1,n}) \cong K_{1,n}$, and the authors stated finding other graphs G such that $D_k(G) \cong G$ for some k as an open problem. We formulate a similar problem. Problem 6.5 was also addressed in [10].

Problem 6.4 Note that $D_3^t(P_3) \cong P_3$. Determine other graphs G such that $D_k^t(G) \cong G$ for some k.

Problem 6.5 Determine the complexity of deciding whether two MTDSs of G are in the same component of $D_k^t(G)$, or of $D_{\Gamma(G)+1}^t(G)$.

Problem 6.6 Determine conditions that imply that $D_k^t(G)$ is Hamiltonian.

Acknowledgements

The authors are indebted to the referees of the original version of the paper for their helpful corrections and suggestions for improving the paper.

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(Received 18 Jan 2018; revised 15 Oct 2018)