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Generalized vector space partitions*

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Abstract

A *vector space partition* $\mathcal{P}$ in $\mathbb{F}_q^v$ is a set of subspaces such that every 1-dimensional subspace of $\mathbb{F}_q^v$ is contained in exactly one element of $\mathcal{P}$. Replacing "1-dimensional" by "t-dimensional", we generalize this notion to vector space t-partitions and study their properties. There is a close connection to subspace codes and some problems are interesting and unsolved even for the set case $q = 1$.

1 Introduction

A vector space partition of $\mathbb{F}_q^v$ consists of subspaces such that every 1-dimensional subspace is covered exactly once. As a natural extension we consider sets of subspaces

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such that every t-dimensional subspace is covered exactly once. For $t \geq 2$ this leads to questions that even remain unsolved for the set case $q = 1$. For $q \geq 2$ there is a close relation to constant-dimension or mixed-dimension subspace codes with respect to the subspace distance. Having such a code at hand, intersecting every codeword with a hyperplane gives an object that cannot be described as a mixed-dimension subspace code in terms of the minimum subspace distance directly. However, our generalization of a vector space partition captures this situation and yields non-trivial upper bounds for constant-dimension subspace codes.

More precisely, let $q > 1$ be a prime power and v a positive integer. A *vector space partition* $\mathcal{P}$ of $\mathbb{F}_q^v$ is a set of subspaces with the property that every non-zero vector is contained in a unique member of $\mathcal{P}$. If $\mathcal{P}$ contains m_d subspaces of dimension d, then $\mathcal{P}$ is of *type* $v^{m_v} \dots 1^{m_1}$. We may leave out dimensions with $m_d = 0$. Subspaces of dimension 1 in $\mathcal{P}$ are called *holes*. The vector space partition consisting only of holes and the vector space partition $\{\mathbb{F}_q^v\}$ are called *trivial*.

Here, we give a natural generalization of this notion. For a positive integer t, we call a set $\mathcal{P}$ of subspaces of $\mathbb{F}_q^v$ a *vector space t-partition*, if all elements of $\mathcal{P}$ are of dimension at least t and every t-dimensional subspace is contained in a unique element of $\mathcal{P}$. Ordinary vector space partitions are precisely the vector space t-partitions with $t = 1$. Besides the simplicity of the proposed generalization, there are some similarities that promise interesting applications. The class of vector space t-partitions contains the q-analogs of a Steiner systems, which are given by the cases where all elements of $\mathcal{P}$ have the same dimension. As a further generalization we mention the possibility of replacing "contained in a unique member of $\mathcal{P}$" by "contained in exactly λ elements of $\mathcal{P}$", which would include subspace designs (q-analogs of combinatorial designs). In the case $t = 1$, this has been considered in [13].

Let $\mathcal{P}$ be a non-trivial vector space partition of $\mathbb{F}_q^v$ with a non-empty set $\mathcal{N}$ of holes, and k the second smallest dimension of the elements of $\mathcal{P}$. Then, we have $\#\mathcal{N} \equiv \#\{N \in \mathcal{N} : N \leq H\} \pmod{q^{k-1}}$ for each hyperplane H of $\mathbb{F}_q^v$. This condition allows to conclude restrictions on $\#\mathcal{N}$ independently of the dimension v of the ambient space. Exploiting this congruence condition yields a series of improvements [33, 38, 34] for the maximum size of a partial k-spread, which is the set of the k-dimensional elements of a vector space partition of type $k^{m_k}1^{m_1}$. The underlying techniques can possibly be best explained using the language of projective q^{k-1}-divisible linear codes and the linear programming method, see [28].

In our more general setting of a non-trivial vector space t-partition $\mathcal{P}$ of $\mathbb{F}_q^v$, the set $\mathcal{N}$ of its t-dimensional subspaces will play the role of the holes. If $\mathcal{N} \neq \emptyset$ and $k > t$ is the second smallest dimension of the elements of $\mathcal{P}$, we will prove $\#\mathcal{N} \equiv \#\{N \in \mathcal{N} : N \leq H\} \pmod{q^{k-t}}$ for each hyperplane H of $\mathbb{F}_q^v$. Similarly, we will study restrictions on $\#\mathcal{N}$ independently of the dimension v of the ambient space. Again some kind of linear programming method will be applied and partially solved analytically.

The analog of partial k-spreads of maximum size are vector space t-partitions of type $k^{m_k}t^{m_t}$ with maximum m_k (for given parameters v and t), which have been previously studied under the name (optimal) constant-dimension codes. Denoting the maximum possible m_k by $A_q(v, 2k - 2t + 2; k)$, our main motivation for the introduction of vector space t-partitions are indeed the recent improvements of $257 \le A_2(8,6;4) \le 289$ to $257 \le A_2(8,6;4) \le 272$ [22] and finally $A_2(8,6;4) = 257$ [23]. These parameters play a rather prominent role for constant-dimension codes, since, besides $A_2(6,4;3) = 77 < 81$ [27], all other known upper bounds for $A_q(v,d;k)$, where $d \notin \{2k, 2v-2k\}$, are obtained via the so-called Johnson bound and the existence of divisible codes, see [30]. While the result of [22] is based on more than 1000 hours computing time, we will apply similar techniques in order to computationally show $m_3 \le 240$ for all vector space 2-partitions of $\mathbb{F}_2^7$ of type $4^{17}3^{m_3}2^{m_2}$ in less than seventy hours. For a vector space 2-partition of $\mathbb{F}_2^8$ of type $4^{m_4}2^{m_2}$ that contains 17 4-dimensional elements in a common hyperplane or passing through a common point, this directly implies $m_4 \le 257$, i.e., meeting the lower bound of $A_2(8,6;4)$. In the remaining cases, a direct counting argument gives $A_2(8,6;4) \le 272$.

While the mentioned result is based on explicit computer computations for fixed parameters, we have some hope that a more thorough study of vector space 2-partitions may lead to an improvement of the currently best known bound $A_q(8,6;4) \le (q^4+1)^2$ or of other parameters in general. To that end we will present the first preliminary results on the existence of vector space t-partitions and the possible cardinalities of the corresponding set $\mathcal{N}$ of t-dimensional subspaces satisfying $\#\mathcal{N} \equiv \#\{N \in \mathcal{N} : N \le H\} \pmod{q^{k-t}}$ for each hyperplane H of $\mathbb{F}_q^v$.

There is another connection with constant-dimension codes. Let $\mathcal{C}$ be the set of k-dimensional elements of a vector space $(t+1)$-partition of $\mathbb{F}_q^{2k}$ of type $(k+t)^1k^\star(t+1)^\star$, where $k > t+1 \ge 1$. Replacing each element in $\mathcal{C}$ by its dual, we obtain a constant-dimension code in $\mathbb{F}_q^{2k}$ with minimum subspace distance $2k-2t$ and cardinality $\#\mathcal{C}$ such that every codeword is disjoint from a $(k-t)$-dimensional subspace.

Vector space t-partitions of type $k^{m_k}t^{m_t}$ are also of interest for the set case, i.e., $q = 1$. In other words, we are considering sets of k-subsets of $\{1,2,\ldots,v\}$ such that every t-set is contained in exactly one k-set (or contained in at most one k-set, if we anticipate the possible completion with t-sets). These structures are equivalent to binary constant-weight codes with length v and minimum Hamming distance $d \ge 2k-2t+2$. See e.g. [1, 2] for upper bounds on m_k.

The classification of the possible types of vector space t-partitions is also an interesting problem for $q = 1$. While it is trivial for $t = 1$ it is not completely resolved for $t = 2$. In the latter case one speaks of pairwise balanced designs (with index 1) or linear spaces, see e.g. [8, 11, 36]. In [41] in has been shown that there is no set of triples covering each pair exactly once except a single uncovered pair.[1] For more results in that direction we refer to [26].

The remaining part of this article is structured as follows. In Section 2 we introduce the preliminaries before we study the existence of vector space t-partitions in Section 3. As a contained substructure, q^r-divisible sets of t-subspaces are introduced and studied in Section 4. We close with several open problems and a conclusion in Section 5.

[1] There exist e.g. 6 triples and 6 quadruples of an 11-set leaving exactly one pair uncovered and 12 triples, 3 quadruples, and a quintuple of a 12-set leaving exactly two intersecting pairs uncovered.

2 Preliminaries

We briefly call k-dimensional subspaces of $\mathbb{F}_q^v$ *k-subspaces*. 1-subspaces are called points, 2-subspaces are called lines, 3-subspaces are called planes, 4-subspaces are called solids, and $(v-1)$-subspaces are called hyperplanes. The number of k-subspaces in $\mathbb{F}_q^v$ is given by the Gaussian binomial coefficient $\begin{bmatrix} v \\ k \end{bmatrix}_q := \prod_{i=0}^{k-1} \frac{q^{v-i}-1}{q^{k-i}-1}$.

Definition 2.1 *Let $t \in \mathbb{N}_{>0}$. A* vector space t-partition *of $\mathbb{F}_q^v$ is a set $\mathcal{P}$ of subspaces of $\mathbb{F}_q^v$ such that every t-subspace of $\mathbb{F}_q^v$ is contained in exactly one element of $\mathcal{P}$ and all elements of $\mathcal{P}$ have dimension at least t (so that they are incident with at least one t-subspace). We call $\mathcal{P}$* trivial *if all elements either have dimension t or v. If $\mathcal{P}$ contains m_i elements of dimension $t \le i \le v$ we call $v^{m_v}(v-1)^{m_{v-1}} \ldots t^{m_t}$ the* type *of $\mathcal{P}$, where i^{m_i} can also be omitted if $m_i = 0$.*

As an example we consider vector space 2-partitions of $\mathbb{F}_2^{13}$ of type $3^{1597245}$, which correspond to 2-Steiner systems of planes in $\mathbb{F}_2^{13}$, whose existence has been proved in [5]. The existence of a vector space 2-partition of $\mathbb{F}_2^7$ of type 3^{381} is equivalent to the existence of a binary q-analog of the Fano plane. If it exists it has an automorphism group of order at most two [31, 6], the number of incidences between the blocks and other k-subspaces are known [32], and not all sets of blocks incident with a point can correspond to a Desarguesian line spread [20, 42]. Possible substructures of a q-analog of the Fano plane presently trigger a lot of research, see e.g. [7, 15] and the references therein. The maximum known value of m_3 of a vector space 2-partition of $\mathbb{F}_2^7$ of type $3^{m_3}2^{m_2}$ is $m_3 = 333$ [25]. For general results on the existence of vector space t-partitions of $\mathbb{F}_q^v$ of type $s^{m_s}t^{m_t}$, also known as (partial) (s,t)-spreads, we refer the reader to e.g. [10, 37].

For two k-subspaces U, W in $\mathbb{F}_q^v$ the *subspace distance* is given by $d_S(U,W) = \dim(U+W) - \dim(U \cap W) = \dim(U) + \dim(W) - 2\dim(U \cap W) = 2k - 2\dim(U \cap W)$.

Definition 2.2 *A constant-dimension code $\mathcal{C}$ of $\mathbb{F}_q^v$ of constant dimension k and minimum subspace distance d is a set of k-subspaces such that the dimension of the intersection of any pair of k-subspaces is at most $\lfloor k - d/2 \rfloor$. By $A_q(v,d;k)$ we denote the corresponding maximum size, i.e., the number of k-subspaces.*

Each vector space t-partition $\mathcal{P}$ of $\mathbb{F}_q^v$ of type $k^{m_k}t^{m_t}$ is in 1-to-1-correspondence to a constant-dimension code $\mathcal{C} = \{U \in \mathcal{P} : \dim(U) = k\}$ with minimum distance at least $2k - 2t + 2$, so that $m_k \le A_q(v, 2k-2t+2; k)$. Note that by duality we have $A_q(v,d;k) = A_q(v,d;v-k)$. For known bounds, we refer to the online table `http://subspacecodes.uni-bayreuth.de` [24]. As an example for constant-dimension codes we would like to mention lifted maximum rank distance (MRD) codes, see [9, 16, 39].

Theorem 2.3 *(see [40]) For positive integers k, d, v with $k \le v$, $d \le 2\min\{k, v-k\}$, and $d \equiv 0 \pmod 2$, the size of a lifted MRD code $\mathcal{C}$ of k-subspaces in $\mathbb{F}_q^v$ with minimum distance at least d is given by*

$$M(q,k,v,d) := q^{\max\{k,v-k\}\cdot(\min\{k,v-k\}-d/2+1)}.$$

Moreover, there exists a $(v-k)$-dimensional subspace U of $\mathbb{F}_q^v$ such that every element of $\mathcal{C}$ has trivial intersection with U. The set of $(\min\{k, v-k\} - d/2 + 1)$-subspaces that is disjoint to U is perfectly covered by the codewords.

Corollary 2.4 *For non-negative integers k, t, v with $k \ge t + 2$ and $v \ge 2k - t + 1$, there exists a vector space $(t+1)$-partition of $\mathbb{F}_q^v$ of type $(v-k+t)^1 k^m (t+1)^\star$, where $\log_q m = \max\{k, v-k\} \cdot (\min\{k, v-k\} - k + t + 1)$.*

Proof. Consider a lifted MRD code $\mathcal{C}$ of k-subspaces in $\mathbb{F}_q^v$ with minimum distance $d = 2k - 2t$. Let U be the $(v-k)$-subspace that has trivial intersection with the elements from $\mathcal{C}$. Add a $(v-k+t)$-subspace containing U, and complete the construction by adding uncovered $(t+1)$-subspaces. □

We remark that the construction also works for $v = 2k - t$, where we obtain a vector space $(t+1)$-partition of $\mathbb{F}_q^v$ of type $k^{m+1}(t+1)^\star$ with $m = q^k$.

3 Existence of vector space t-partitions

In this section we will study the possible types of vector space t-partitions of $\mathbb{F}_q^v$ for small dimensions v. Here we will assume $t \ge 2$ and refer to the survey [19] for the case $t = 1$. From that paper we also transfer the first conditions on the parameters m_i of a vector space t-partition $\mathcal{P}$ of $\mathbb{F}_q^v$ of type $k^{m_k} \dots t^{m_t}$. Since every t-subspace is contained in a unique element in $\mathcal{P}$, we have

$$\sum_{i=t}^{k} m_i \cdot \begin{bmatrix} i \\ t \end{bmatrix}_q = \begin{bmatrix} v \\ t \end{bmatrix}_q, \tag{1}$$

which is called *packing condition* in [19] for $t = 1$. This equation allows us to suppress the precise value of m_t as done in Corollary 2.4. Due to the dimension formula $\dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap V)$, for any two subspaces U and V of $\mathbb{F}_q^v$, we have

$$m_i \le 1 \text{ if } 2i > v + t - 1 \quad \text{and} \quad m_i m_j = 0 \text{ if } i + j > v + t - 1 \tag{2}$$

for $t \le i < j \le v$. The specialization to $t = 1$ is called *dimension condition* in [19].

Known constructions of vector space t-partitions are given by lifted MRD codes. If $\mathcal{P}$ is a vector space t-partition of $\mathbb{F}_q^v$, U an element of $\mathcal{P}$, and $\mathcal{P}'$ a vector space t-partition of U, then $(\mathcal{P} \backslash U) \cup \mathcal{P}'$ is also a vector space t-partition of $\mathbb{F}_q^v$. We call $\mathcal{P}'$ *derived* from $\mathcal{P}$, matching the definition of a derived vector space partition for $t = 1$.

From equations (1) and (2) we conclude that for $t \le v \le t + 1$ each vector space t-partition $\mathcal{P}$ of $\mathbb{F}_q^v$ is trivial, i.e., either $\mathcal{P} = \{\mathbb{F}_q^v\}$ – type v^1 – or $\mathcal{P}$ is given by the $\begin{bmatrix} v \\ t \end{bmatrix}_q$ t-subspaces of $\mathbb{F}_q^v$. In the following we will consider the non-trivial vector space t-partitions only. For $v = t + 2$ the dimension condition allows $m_{t+1} = 1$ only, so that the packing condition gives type $(t+1)^1 t^{\begin{bmatrix} t+2 \\ t \end{bmatrix}_q - \begin{bmatrix} t+1 \\ t \end{bmatrix}_q}$. Here, $\mathcal{P}$ consists of an arbitrary $(t+1)$-subspace U and all t-subspaces not contained in U. So far, all discussed cases are unique up to isomorphism. For $v = t + 3$ we get $m_{t+2} \le 1$ and $m_{t+2}m_{t+1} = 0$ so that we can have type $(t+2)^1 t^{\left[{t+3 \atop t}\right]_q - \left[{t+2 \atop t}\right]_q}$ or type $(t+1)^{m_{t+1}} t^{m_t}$. In the latter case we have $m_{t+1} \le A_q(t+3, 4; t+1) = A_q(t+3, 4; 2)$. The corresponding objects to $m_{t+1} = A_q(t + 3, 4; t + 1) = A_q(t + 3, 4; 2)$ are so-called (partial) line spreads of maximum size. If t is odd, then $A_q(t + 3, 4; 2) = (q^{t+3} - 1)/(q^2 - 1)$, and $A_q(t + 3, 4; 2) = (q^{t+3} - q^2(q-1) - 1)/(q^2 - 1)$ otherwise, see e.g. [3]. Here, there are several isomorphism types in general. So, using derived vector space t-partitions, in $\mathbb{F}_q^{t+3}$ there exist vector space t-partitions of type $(t+1)^i t^{\left[{t+3 \atop t}\right]_q - i\left[{t+1 \atop t}\right]_q}$ for all $0 \le i \le A_q(t + 3, 4; 2)$. For $v = t + 4$ we conclude from the dimension condition that only the types $(t+3)^1 t^\star$, $(t+2)^1(t+1)^a t^\star$, and $(t+1)^b t^\star$ might be possible for a non-trivial vector space t-partition. In the latter case we have $b \le A_q(t+4, 4; t+1) = A_q(t + 4, 4; 3)$. Since the current knowledge on $A_q(t + 4, 4; 3)$ is rather limited, we mention the known bounds for $t = 2$ only: $A_2(6, 4; 3) = 77$ with precisely 5 attaining isomorphism types and $q^6 + 2q^2 + 2q + 1 \le A_q(6, 4; 3) \le (q^3 + 1)^2 = q^6 + 2q^3 + 1$ for $q \ge 3$, see [27]. For type $(t + 2)^1(t + 1)^a t^\star$ Corollary 2.4 gives a construction with $a = q^{2t+2}$, which is tight for $t = 2$.

Lemma 3.1 *If $\mathcal{P}$ is a vector space 2-partition of $\mathbb{F}_q^v$ of type $(v - k + 1)^1 k^a 2^\star$, where $k \ge 3$ and $v \ge 2k$, then $a \le q^{2(v-k)}$.*

Proof. Let U be the unique $(v-k+1)$-subspace of $\mathcal{P}$. The number of lines disjoint from U is given by

$$\begin{bmatrix} v \\ 2 \end{bmatrix}_q - \begin{bmatrix} v-k+1 \\ 2 \end{bmatrix}_q - \frac{1}{q} \cdot \begin{bmatrix} v-k+1 \\ 1 \end{bmatrix}_q \cdot \left(\begin{bmatrix} v \\ 1 \end{bmatrix}_q - \begin{bmatrix} v-k+1 \\ 1 \end{bmatrix}_q \right)$$

$$= q^{2(v-k)} \cdot \frac{q^{2k-1} - q^{k+1} - q^k + q^2}{(q^2-1)(q-1)}.$$

Since each k-subspace K of $\mathcal{P}$ intersects U in exactly one point and the number of lines in K disjoint from a given point is given by

$$\begin{bmatrix} k \\ 2 \end{bmatrix}_q - \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q = \frac{q^{2k-1} - q^{k+1} - q^k + q^2}{(q^2-1)(q-1)},$$

we have $a \le q^{2(v-k)}$. □

So the size of the construction from Corollary 2.4 is met for all cases where $t \in \{0, 1\}$. For $t = 0$ the upper bound follows from counting the k-subspaces disjoint to a $(v - k)$-subspace. Removing the $(v - k)$-subspace gives precisely the lifted MRD codes with corresponding parameters. For $t = 1$ the construction of Corollary 2.4 is far from being unique. We remark that the five isomorphism types of constant-dimension codes meeting the upper bound $A_2(6, 4; 3) = 77$ each contain subsets of 64 codewords that intersect a fixed solid in precisely a point. Moreover, there are exactly four isomorphism types of 64 planes that intersect a fixed solid in precisely a point. The three ones that do not equal the lifted MRD code have automorphism groups of orders 24, 16, and 12, respectively, and all can be extended to a constant-dimension code of cardinality 77. Lemma 3.1 is also valid for the set case $q = 1$, where it is tight.

For $v \geq t + 5$ the situation gets rather involved, so that we assume $t = 2$ and $v = 7$ in the remaining part of this section. The dimension condition allows just the following types: $6^1 2^\star$, $5^1 3^{\tilde{m}_3} 2^\star$, $4^{m_4} 3^{m_3} 2^\star$, and $3^{\bar{m}_3} 2^\star$, where $\bar{m}_3 \leq A_q(7, 4; 3)$ with e.g. $333 \leq A_2(7, 4; 3) \leq 381$ and $6978 \leq A_3(7, 4; 3) \leq 7651$, see [24]. For the other parameterized cases we have $m_4 \leq A_q(7, 6; 4) = A_q(7, 6; 3) = q^4 + 1$ and $\tilde{m}_3 \leq q^8$, which is tight, see Corollary 2.4 and Lemma 3.1. Now, let us first look at constructions for the two maximal values for m_4.

Lemma 3.2 *For each prime power $q \geq 2$ there exist vector space 2-partitions of $\mathbb{F}_q^7$ of type $4^{m_4} 3^{m_3} 2^\star$ with*

$$(m_4, m_3) = \left(q^4 + 1, q^8 - q^4\right) \text{ and } (m_4, m_3) = \left(q^4, q^8 - q^4 + q^2 + q + 1\right).$$

Proof. Let $\mathcal{C}_8$ be a lifted MRD code of q^8 solids in $\mathbb{F}_q^8$ with minimum distance 6 and U be the unique solid having trivial intersection to the elements from $\mathcal{C}_8$. For an arbitrary hyperplane H of $\mathbb{F}_q^8$ that does not contain U we set $\mathcal{C}_7 := \{V \cap H : V \in \mathcal{C}_8\}$, so that $\mathcal{C}_7$ consists of q^4 solids and $q^8 - q^4$ planes. If S is an arbitrary solid in H that contains $U \cap H$, then $\mathcal{C}_7 \cup S$ together with the uncovered lines of H gives a vector space 2-partition of H with type $4^{q^4+1} 3^{q^8-q^4} 2^\star$. For the other case, consider $r = \begin{bmatrix}3\\1\end{bmatrix}_q < \begin{bmatrix}4\\1\end{bmatrix}_q$ arbitrary solids $S_1, \ldots, S_r$ in H containing $U \cap H$. Denoting the $\begin{bmatrix}3\\2\end{bmatrix}_q = r$ lines contained in $U \cap H$ by $L_1, \ldots, L_r$, we choose r planes $E_1, \ldots, E_r$ such that $L_i \subseteq E_i \subseteq S_i$. With this, $\mathcal{C}_7 \cup \{E_i : 1 \leq i \leq r\}$ can be completed by the uncovered lines to a vector space 2-partition of H of type $4^{m_4} 3^{m_3} 2^\star$ with $(m_4, m_3) = (q^4, q^8 - q^4 + q^2 + q + 1)$. □

With respect to upper bounds for m_3 we consider the objects of $\mathcal{P}$ that are incident to a given point P. Modulo P we obtain vector space partitions of $\mathbb{F}_2^6$ of type $3^{\bar{m}_3} 2^{\bar{m}_2} 1^\star$. The possible types have been completely classified, see e.g. [19]. If $\bar{m}_3 = 3j + r$ with $j \in \mathbb{N}$ and $r \in \{0, 1, 2\}$, then $\bar{m}_2 \leq 21 - 5r + r^2 - 7j =: f(\bar{m}_3)$.

Lemma 3.3 *If $\mathcal{P}$ is a vector space 2-partition of $\mathbb{F}_2^7$ of type $4^{m_4} 3^{m_3} 2^\star$, then $m_3 \leq 240$ if $m_4 = 17$ and $m_3 \leq 276$ if $m_4 = 16$.*

Proof. Let $\mathcal{S}$ be a set of 16 or 17 solids in $\mathbb{F}_2^7$ pairwise intersecting in a point. By dualization we obtain a set of 16 or 17 planes in $\mathbb{F}_2^7$ with trivial intersection. Those configurations have been classified up to symmetry in [29]. Given all possible choices for $\mathcal{S}$, we develop an integer linear programming formulation for the maximization of m_3. For each plane E in $\mathbb{F}_2^7$ we introduce a variable $x_E \in \{0, 1\}$ with $E \in \mathcal{P}$ iff $x_E = 1$, so that $m_3 = \sum_{E \leq \mathbb{F}_2^7} x_E$. If L is a line of $\mathbb{F}_2^7$ that is contained in an element of $\mathcal{S}$, then we have $\sum_{L \leq E \leq \mathbb{F}_2^7} x_E = 0$ and $\sum_{L \leq E \leq \mathbb{F}_2^7} x_E \leq 1$ otherwise. The LP relaxation of the current formulation can be further improved by adding $\sum_{P \leq E \leq \mathbb{F}_2^7} x_E \leq f(\tau(P))$, where P is a point in $\mathbb{F}_2^7$ and $\tau(P)$ counts the number of elements of $\mathcal{S}$ that contain P. Given $\mathcal{S}$ we denote the corresponding integer linear programming formulation by $ILP_{\mathcal{S}}$ and its LP relaxation by $LP_{\mathcal{S}}$.

For $\#\mathcal{S} = 17$ it took 7 minutes to compute the 715 linear programs $LP_{\mathcal{S}}$. Except 10 cases, all of them have a target value strictly less than 240. In exactly one case a target value of 240 can be attained for $ILP_{\mathcal{S}}$, which took less than 66 hours to verify computationally. For $\#\mathcal{S} = 16$ we computed the 14445 instances $LP_{\mathcal{S}}$ leaving just 28 cases with a target value of at least 247. It took 6 h to verify that $ILP_{\mathcal{S}}$ has a target value of at most 276 for these 28 instances. After 99 h there remain just 7 instances which may yield a target value strictly greater than 247, i.e., the lower bound given by Lemma 3.2. □

Let a_i denote the number of points of $\mathbb{F}_2^7$ that are contained in exactly i solids of $\mathcal{S}$. We remark that for $\#\mathcal{S} = 17$, we can easily deduce $a_1 = 7$, $a_2 = 112$, and $a_3 = 8$, so that $m_3 \le \frac{7\cdot f(1)+112\cdot f(2)+8\cdot f(3)}{7} = 273$. For $\#\mathcal{S} < 17$ even less information on the a_i is sufficient to establish a competitive upper bound for m_3.

Lemma 3.4 *If $\mathcal{P}$ is a vector space 2-partition of $\mathbb{F}_2^7$ of type $4^{m_4}3^{m_3}2^{\star}$, then* $m_3 \le 381 - \left\lceil \frac{m_4(61-m_4)}{7} \right\rceil$.

PROOF. Let a_i denote the number of points of $\mathbb{F}_2^7$ that are contained in exactly i of the m_4 solids of $\mathcal{P}$. Counting points gives $\sum_{i\ge 0} a_i = \begin{bmatrix}7\\1\end{bmatrix}_2 = 127$ and $\sum_{i\ge 0} i a_i = \begin{bmatrix}4\\1\end{bmatrix}_2 m_4 = 15m_4$. Since every pair of solids of $\mathcal{P}$ intersects in exactly one point, we additionally have $\sum_{i\ge 0} i(i-1)a_i = m_4(m_4-1)$. With this and the definition of the function f, $\left\lfloor \frac{1}{7}\sum_{i\ge 0} f(i)\cdot a_i \right\rfloor$ is an upper bound for m_3. Next we maximize $\sum_{i\ge 0} f(i)\cdot a_i$ for non-negative integers a_i satisfying the three equations stated above. Since Lemma 3.3 gives a stronger bound than $m_3 \le 274$ for $m_4 = 17$, we can assume $m_4 \le 16$ in the following. From the last two equations we conclude

$$a_1 = m_4(16-m_4) + \sum_{i\ge 3} i(i-2)a_i \ge \sum_{i\ge 3}(2i-3)a_i,$$

so that $a_1 \ge 2l-3$ if $a_l \ge 1$ for some $l \ge 3$. We claim that $a_i = 0$ for all $i \ge 3$ in an optimal solution. Assume $a_l \ge 1$ for some $l \ge 3$. Now, we modify the given a_i-vector by decreasing a_l by 1, increasing a_{l-1} by 1, increasing a_2 by $l-1$, decreasing a_1 by $2l-3$ and increasing a_0 by $l-2$. The resulting vector $(a_0', a_1', \dots)$ satisfies the three equations and has non-negative integer entries. By this operation the value of $\sum_{i\ge 0} f(i)\cdot a_i$ increases by $f(l-1) - f(l) + 2l - 6 \ge f(l-1) - f(l) \ge 1$. Thus, the optimal solution is given by $a_2 = \binom{m_4}{2}$, $a_1 = m_4(16-m_4)$, and $a_0 = 127 - \frac{m_4(31-m_4)}{2}$ with

$$\left\lfloor \frac{1}{7}\sum_{i\ge 0} f(i)\cdot a_i \right\rfloor = \left\lfloor \frac{1}{7}\cdot\left(m_4^2 - 61m_4 + 2667\right)\right\rfloor = 381 - \left\lceil \frac{m_4(61-m_4)}{7}\right\rceil .$$

□

We remark that Lemma 3.4 gives $m_3 \leq 278$ for $m_4 = 16$. Summarizing the binary case $q = 2$, we have the following bounds for $\max m_3$:

m_4	17	16	15	14	13	12	11	10	9
$\max m_3$	240	$247 \ldots 276$	$248 \ldots 282$	$249 \ldots 287$	$252 \ldots 291$	$273 \ldots 297$	$274 \ldots 302$	$275 \ldots 308$	$276 \ldots 314$
m_4	8	7	6	5	4	3	2	1	0
$\max m_3$	$284 \ldots 320$	$285 \ldots 327$	$286 \ldots 333$	$287 \ldots 341$	$291 \ldots 348$	$297 \ldots 356$	$300 \ldots 364$	$312 \ldots 372$	$333 \ldots 381$

The upper bounds are obtained from Lemma 3.3 and Lemma 3.4. Lemma 3.2 gives constructions for $m_4 \in \{16, 17\}$. The construction for $m_4 = 0$ is taken from [25]. For $m_4 \in \{1, 2, 3, 4, 8, 12, 13\}$ the stated lower bounds are obtained from an integer linear programming formulation with prescribed subgroups of the automorphism group, i.e., the Kramer–Mesner approach. All other lower bounds are obtained by replacing a solid by a plane contained in the solid.

4 q^r-divisible sets of t-subspaces

Besides the conditions of Equation (1) and Equation (2), there is another technique for excluding the existence of (ordinary) vector space partitions, which just takes into account the second smallest occurring dimension. To this end, let $\mathcal{P}$ be a non-trivial vector space partition of $\mathbb{F}_q^v$, $\mathcal{N} \neq \emptyset$ be its set of holes[2], i.e., 1-dimensional elements, and k be the second smallest dimension of the elements of $\mathcal{P}$. Then, we have $\#\mathcal{N} \equiv \#\{N \in \mathcal{N} : N \leq H\} \pmod{q^{k-1}}$ for each hyperplane H of $\mathbb{F}_q^v$. Assigning a weight $w(H) \in \mathbb{N}$ to every hyperplane H via $w_{\mathcal{N}}(H) := \#\mathcal{N} - \#\{N \in \mathcal{N} : N \leq H\}$, we can say that the weights of the hyperplanes are divisible by q^{k-1}. So, we also call the set $\mathcal{N}$ of points q^{k-1}-divisible. The possible cardinalities of q^r-divisible sets of points, or equivalently the length of q^r-divisible linear codes, see [28], are quite restrictive. This approach allows to exclude the existence of vector space partitions without knowing the precise values of the m_i or the dimension v of the ambient space. The asymptotic result on the maximal cardinality of partial spreads from [38] can e.g. be obtained using q^r-divisible sets of points, see [34]. However, there are some rare cases where the existence of a vector space partition was excluded with more involved techniques, see e.g. [12] for the exclusion of a vector space partition of type $4^{13}3^{6}2^{6}$ in $\mathbb{F}_2^8$. Nevertheless, the classification of all possible cardinalities of q^r-divisible sets of points is an important relaxation. So far, in the binary case, the classification is complete for $r \leq 2$ only, see [28, Theorem 13], while there is a single open case for $r = 3$. A general result for *small* cardinalities but arbitrary parameters q and r was obtained in [28, Theorem 12], see Theorem 5.1. For each pair of parameters there is a largest integer $\mathrm{F}(q, r)$, called *Frobenius number*, such that no q^r-divisible set of points of cardinality $\mathrm{F}(q, r)$ exists, see e.g. [21] for some bounds. For q^r-divisible multisets of points the possible cardinalities have been completely characterized in [30].

The aim of this section is to generalize the notion of q^r-divisible sets of points to q^r-divisible sets of t-subspaces and to deduce restrictions for the possible cardinalities of such sets.

[2]In, e.g., [18] the author speaks of the tail of the vector space partition and considers lower bounds for its length, i.e., the cardinality of $\mathcal{N}$.

Definition 4.1 *Let $\mathcal{C}$ be a set of t-subspaces in $\mathbb{F}_q^v$. We call $\mathcal{C}$ q^r-divisible if $\#\mathcal{C} \equiv \#\{C \in \mathcal{C} : C \le H\} \pmod{q^r}$ for all hyperplanes H of $\mathbb{F}_q^v$.*

The link between q^r-divisible sets of t-subspaces and vector space t-partitions is given by:

Proposition 4.2 *Let $\mathcal{P}$ be a non-trivial vector space t-partition of $\mathbb{F}_q^v$ with $m_i = 0$ for all $t < i < k$, then the set $\mathcal{N}$ of t-subspaces of $\mathcal{P}$ is q^{k-t}-divisible.*

Proof. Using the convention $\begin{bmatrix} l-1 \\ 0 \end{bmatrix}_q = 1$, we have $\begin{bmatrix} l \\ t \end{bmatrix}_q - \begin{bmatrix} l-1 \\ t \end{bmatrix}_q = \begin{bmatrix} l-1 \\ t-1 \end{bmatrix}_q \cdot q^{l-t}$, which is divisible by q^{k-t} for all $l \ge k$. Note that we have $v > k$ since $\mathcal{P}$ is non-trivial. Counting the t-subspaces of $\mathbb{F}_q^v$ gives $\sum_{i=k}^{v-1} m_i \begin{bmatrix} i \\ t \end{bmatrix}_q + \#\mathcal{N} = \begin{bmatrix} v \\ t \end{bmatrix}_q$. Now, let H be an arbitrary hyperplane of $\mathbb{F}_q^v$, $\mathcal{N}'$ be the set of elements of $\mathcal{N}$ that are contained in H, and $\mathcal{P}' := \{U \cap H : U \in \mathcal{P}, \dim(U) \ge k\} \cup \mathcal{N}'$ be a vector space t-partition of H of type $(v-1)^{m'_{v-1}} \ldots (k-1)^{m'_{k-1}} (t)^{\#\mathcal{N}'}$, where we allow $t = k-1$, slightly abusing notation. With this, we have $\sum_{i=k-1}^{v-1} m'_i \begin{bmatrix} i \\ t \end{bmatrix}_q + \#\mathcal{N}' = \begin{bmatrix} v-1 \\ t \end{bmatrix}_q$. By subtracting both equations we conclude that q^{k-t} divides $\#\mathcal{N} - \#\mathcal{N}'$ since each i-subspace in $\mathcal{P}$ with $i \ge k$ corresponds either to an i-subspace or an $(i-1)$-subspace in $\mathcal{P}'$ and q^{k-t} divides $\begin{bmatrix} l \\ t \end{bmatrix}_q - \begin{bmatrix} l-1 \\ t \end{bmatrix}_q$ for $l \ge k$. □

In the following let $\mathcal{N}$ be a q^r-divisible set of t-subspaces in $\mathbb{F}_q^v$ with minimal v, i.e., $\mathcal{N}$ is not completely contained in any hyperplane. By a_i we denote the number of hyperplanes H of $\mathbb{F}_q^v$ with $\#\{N \in \mathcal{N} : N \le H\} = i$ and set $n := \#\mathcal{N}$. Double-counting the incidences of the tuples (H) and (B, H), where H is a hyperplane and $B \in \mathcal{N}$ with $B \le H$ gives:

$$\sum_{i=0}^{n-1} a_i = \begin{bmatrix} v \\ 1 \end{bmatrix}_q \quad \text{and} \quad \sum_{i=0}^{n-1} i a_i = n \cdot \begin{bmatrix} v-t \\ 1 \end{bmatrix}_q. \tag{3}$$

For two different elements B_1, B_2 of $\mathcal{N}$ their span $\langle B_1, B_2 \rangle$ has a dimension i between $t+1$ and $2t$. Denoting the number of corresponding ordered pairs by b_i, double-counting gives:

$$\sum_{i=0}^{n-1} i(i-1) a_i = \sum_{i=t+1}^{2t} b_i \begin{bmatrix} v-i \\ 1 \end{bmatrix}_q \quad \text{and} \quad \sum_{i=t+1}^{2t} b_i = n(n-1). \tag{4}$$

As a first non-existence criterion we state:

Lemma 4.3 *For a non-empty q^r-divisible set $\mathcal{N}$ of t-subspaces in $\mathbb{F}_q^v$, there exists a hyperplane H with $\#\{N \in \mathcal{N} : N \le H\} < n/q^t$, where $n = \#\mathcal{N}$.*

Proof. Let i be the smallest index with $a_i \ne 0$. Then, the equations of (3) are equivalent to $\sum_{j \ge 0} a_{i+q^r j} = \begin{bmatrix} v \\ 1 \end{bmatrix}_q$ and $\sum_{j \ge 0} (i + q^r j) \cdot a_{i+q^r j} = n \begin{bmatrix} v-t \\ 1 \end{bmatrix}_q$. Subtracting i times the first equation from the second equation gives $\sum_{j>0} q^r j a_{i+q^r j} = n \cdot \frac{q^{v-t}-1}{q-1} - i \cdot \frac{q^v-1}{q-1}$. Since the left-hand side is non-negative, we have $i \le \frac{q^{v-t}-1}{q^v-1} \cdot n < \frac{n}{q^t}$. □

The proof of Lemma 4.3 expresses the simple fact that a hyperplane with the minimum number of t-subspaces in $\mathcal{N}$ contains at most as many t-subspaces as the average number of t-subspaces in $\mathcal{N}$ per hyperplane. Lemma 4.3 excludes e.g. the existence of q-divisible sets $\mathcal{N}$ of t-subspaces in $\mathbb{F}_q^v$ of a cardinality $n \in [1, q-1]$.

Next we turn to constructions of q^r-divisible sets of t-subspaces. For $t = 1$ each k-subspace and each affine k-subspace, i.e., the difference of a $(k+1)$-subspace and a contained k-subspace, yields a q^{k-1}-divisible set. With this, the next lemma shows that a q^r-divisible set of t-subspaces of cardinality q^{r+1} exists for all integers $r, t \geq 1$.

Lemma 4.4 *Let $\mathcal{N}$ be a q^r-divisible set of t-subspaces in $\mathbb{F}_q^v$ such that q^r divides $\#\mathcal{N}$. Then, for each $s \in \mathbb{N}$ there exists a q^r-divisible set $\mathcal{N}'$ of $(t+s)$-subspaces in $\mathbb{F}_q^{v+s}$.*

PROOF. Assume $s \geq 1$, choose an s-subspace U in $\mathbb{F}_q^{v+s}$ such that $\mathbb{F}_q^v \oplus U = \mathbb{F}_q^{v+s}$, and set $\mathcal{N}' = \{U + N : N \in \mathcal{N}\}$. □

Lemma 4.5 *For integers $t \geq 1$ and $a \geq 2$ let $\mathcal{N}$ be a t-spread in $\mathbb{F}_q^{at}$, i.e., a set of $\frac{q^{at}-1}{q^t-1}$ disjoint t-subspaces. Then $\mathcal{N}$ is $q^{(a-1)t}$-divisible.*

PROOF. Since any point in $\mathbb{F}_q^{at}$ is contained in a unique member of $\mathcal{N}$ and $x \cdot \begin{bmatrix} t \\ 1 \end{bmatrix}_q + \left(\frac{q^{at}-1}{q^t-1} - x\right) \cdot \begin{bmatrix} t-1 \\ 1 \end{bmatrix}_q = \begin{bmatrix} at-1 \\ 1 \end{bmatrix}_q$ for $x = \frac{q^{(a-1)t}-1}{q^t-1}$, every hyperplane contains exactly x elements from $\mathcal{N}$. The divisibility follows from $\frac{q^{at}-1}{q^t-1} - \frac{q^{(a-1)t}-1}{q^t-1} = q^{(a-1)t}$. □

We remark that t-spreads exist for all values of t, a, and q. Examples can e.g. be obtained from the so-called *subfield construction*, i.e., taking all $\begin{bmatrix} a \\ 1 \end{bmatrix}_{q^t} = \frac{q^{at}-1}{q^t-1}$ points in $\mathbb{F}_{q^t}^a$ considering $\mathbb{F}_{q^t}$ as a t-dimensional vector space over $\mathbb{F}_q$.

Lemma 4.6 *For integers $t \geq 1$, $s \geq 0$, and $a \geq 2$ let $\mathcal{N}$ be a union of q^s disjoint t-spreads $\mathcal{S}_1, \ldots, \mathcal{S}_{q^s}$ in $\mathbb{F}_q^{at}$, i.e., $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$ for $i \neq j$. Then $\mathcal{N}$ is $q^{(a-1)t+s}$-divisible.*

PROOF. For each hyperplane H and each index $1 \leq i \leq q^s$ we have $\#\mathcal{S}_i \equiv \#\{U \in S_i : U \leq H\} \pmod{q^{(a-1)t}}$ due to Lemma 4.5. The result follows from $\#\mathcal{N} = q^s \cdot \#\mathcal{S}_1$ and $\#\{U \in \mathcal{N} : U \leq H\} \equiv q^s \cdot \#\{U \in S_1 : U \leq H\} \pmod{q^{(a-1)t+s}}$. □

In $\mathbb{F}_q^{at}$ there can be at most $\begin{bmatrix} at \\ t \end{bmatrix}_q \cdot \begin{bmatrix} t \\ 1 \end{bmatrix}_q / \begin{bmatrix} at \\ 1 \end{bmatrix}_q$ pairwise disjoint t-spreads, which is just the number of t-subspaces of $\mathbb{F}_q^{at}$ divided by the number of t-subspaces of a t-spread. If that upper bound is reached one speaks of a *t-parallelism*. These are known to exist for $(v = at, t, q)$ in $\{(2a, 2, 2), (2^i, 2, q), (6, 2, 3), (6, 3, 2)\}$ for all integers $a, i \geq 2$, see e.g. [14] and the cited references therein. So far, no non-existence result is known. If the stated upper bound on the number of t-spreads is not met, one speaks of a *partial t-parallelism*. For the maximum number $p(v, t, q)$ of pairwise disjoint t-spreads in $\mathbb{F}_q^v$, the lower bounds $p(2a, 2, q) \geq q^{2\lfloor \log(2a-1) \rfloor} + \cdots + q + 1$, $p(at, t, q) \geq 2^t - 1$, and $p(at, t, q) \geq 2$, where $a \geq 2$, are proven in [4] and [14].

Next we present a lower bound on the cardinality of a non-empty q^r-divisible set of t-subspaces:

Theorem 4.7 *Let $t \geq 2$ and $r \geq 1$ be integers and $\mathcal{N} \neq \emptyset$ be a q^r-divisible set of t-subspaces in $\mathbb{F}_q^v$, where v is minimal.*

(i) If q^r divides $\#\mathcal{N}$, then $\#\mathcal{N} \geq q^{r+1}$.

(ii) If $\#\mathcal{N}$ is not divisible by q^r, then $\#\mathcal{N} \geq q^t+1$ and $\#\mathcal{N} \geq q^r + \frac{q^{(\kappa-1)t}-1}{q^t-1} \cdot q^{r-(\kappa-1)t}$, where κ is the smallest positive integer satisfying $\frac{q^{\kappa t}-1}{q^t-1} \geq q^r$.

PROOF.

(i) Assume $\#\mathcal{N} = lq^r$ for some positive integer l. Setting $\Delta = q^r$, $y = q^{v-t-1}$, and $c_i = a_i(q-1)$ for all $0 \leq i \leq \#\mathcal{N} - 1$, the equations from (3) are equivalent to

$$\sum_{i=0}^{l-1} c_{i\Delta} = q^{t+1}y - 1 \quad \text{and} \quad \sum_{i=0}^{l-1} i(\Delta-1)c_{i\Delta} = l(\Delta-1)\left(qy-1\right).$$

From Equation (4) we conclude

$$l(l\Delta-1)(q^{-t+1}y-1) \leq \sum_{i=0}^{l-1} i(i\Delta-1)c_{i\Delta} \leq l(l\Delta-1)(y-1),$$

so that

$$l(\Delta-1)\left(qy-1\right) = \sum_{i=0}^{l-1} i(\Delta-1)c_{i\Delta} \leq \sum_{i=0}^{l-1} i(i\Delta-1)c_{i\Delta} \leq l(l\Delta-1)(y-1).$$

Since $l \geq 1$, we have $(\Delta-1)\left(qy-1\right) \leq (l\Delta-1)(y-1)$, so that $q\Delta + \Delta y + y \leq 2\Delta + qy$ for $l \leq q-1$. Since $q \geq 2$ and $\Delta \geq q$, we obtain $y \leq 0$, which is a contradiction. Thus, $l \geq q$ and $\#\mathcal{N} \geq q^{r+1}$.

(ii) Assume $\#\mathcal{N} = lq^r + x$ with $0 < x < q^r$ for some integers x, l. Lemma 4.3 gives $\#\mathcal{N} \geq q^t + 1$ and from the divisibility we conclude $l \geq 1$, so that we assume $l = 1$ in the following. With this, $\Delta = q^r$, and $y = q^{v-t}$, the equations from (3) are equivalent to

$$x(q-1)a_x = x\left(q^t y - 1\right) \quad \text{and} \quad x(q-1)a_x = (x+\Delta)(y-1),$$

so that $\Delta/y = x + \Delta - xq^t$ and $0 \leq v-t \leq r$. Isolating x gives $(q^t-1)\,x = (y-1)\cdot\frac{\Delta}{y} = \Delta\cdot\left(1-\frac{1}{y}\right)$, which implies that q^t-1 divides $y-1$, i.e., t divides v, and that x is increasing with y. So, let $v = \kappa \cdot t$ for some positive integer κ with $(\kappa-1)t \leq r$. Then, $x = \frac{q^{(\kappa-1)t}-1}{q^t-1} \cdot q^{r-(\kappa-1)t}$ is increasing with κ. Of course $\#\mathcal{N} \leq \begin{bmatrix} v \\ 1 \end{bmatrix}_q / \begin{bmatrix} t \\ 1 \end{bmatrix}_q$, so that $q^r \leq \frac{q^{\kappa t}-1}{q^t-1}$.

□

The construction of Lemma 4.4 and the remark before Lemma 4.4 show that (i) is tight. If $r \le t$, then the first part of (ii) is tight due to the construction of Lemma 4.5 with $a = 2$. If t-parallelisms exist for all parameters (the dimension v has of course to be divisible by t), then also the second part of (ii) is tight. The construction of Lemma 4.6 shows that also a weaker assumption suffices for this claim.

We remark that Theorem 4.7 generalizes a theorem on the so-called length of the tail of a vector space partition, originating from [17], for the special case $t = 1$, where the k-subspaces automatically are disjoint.

Theorem 4.8 *([35, Theorem 10]) For a non-empty q^r-divisible set $\mathcal{N}$ of pairwise disjoint k-subspaces in $\mathbb{F}_q^v$ the following bounds on $n = \#\mathcal{N}$ are tight.*

(i) We have $n \ge q^k + 1$ and if $r \ge k$ then either k divides r and $n \ge \frac{q^{k+r}-1}{q^k-1}$ or $n \ge \frac{q^{(a+2)k}-1}{q^k-1}$, where $r = ak + b$ with $0 < b < k$ and $a, b \in \mathbb{N}$.

(ii) Let q^r divide n. If $r < k$ then $n \ge q^{k+r} - q^k + q^r$ and $n \ge q^{k+r}$ otherwise.

Aiming at characterizations of all possible cardinalities of q^r-divisible sets of t-subspaces it is useful to collect some more constructions. Taking the set of all t-subspaces gives another construction of divisible sets of t-subspaces.

Lemma 4.9 *For integers $t \ge 1$ and $v \ge t + 1$ the set $\mathcal{N}$ of all t-subspaces of $\mathbb{F}_q^v$ is q^{v-t}-divisible.*

Proof. We have $\begin{bmatrix} v \\ t \end{bmatrix}_q - \begin{bmatrix} v-1 \\ t \end{bmatrix}_q = \begin{bmatrix} v-1 \\ t-1 \end{bmatrix}_q \cdot q^{v-t}$. □

The set of achievable cardinalities of q^r-divisible sets of t-subspaces is closed under addition:

Lemma 4.10 *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be q^r-divisible sets of t-subspaces in $\mathbb{F}_q^{v_1}$ and $\mathbb{F}_q^{v_2}$, respectively. Then, there exists a q^r-divisible set of t-subspaces in $\mathbb{F}_q^{v_1} \oplus \mathbb{F}_q^{v_2} \cong \mathbb{F}_q^{v_1+v_2}$ with cardinality $\#\mathcal{N}_1 + \#\mathcal{N}_2$.*

In many cases an ambient space of dimension smaller than $v_1 + v_2$ is sufficient.

5 Conclusion and open problems

Vector space t-partitions have many properties in common with ordinary vector space partitions, so that this class forms an interesting generalization. We have presented a few initial results on the existence of vector space t-partitions and their relaxation to q^r-divisible sets of t-subspaces. Only scratching the surface in this paper, we close with some open problems.

While Lemma 3.1 gives an upper bound on the cardinality of constant-dimension codes of dimension k in $\mathbb{F}_q^{2k}$ with subspace distance $2k - 2$ such that the codewords are disjoint from a $(k + 1)$-subspace U, the underlying question is more general. What about $t > 1$ in Lemma 3.1? If we forgo the link to vector space t-partitions via duality, we can ask for an upper bound on the cardinality of constant-dimension codes of dimension k in $\mathbb{F}_q^v$ with subspace distance d such that the codewords are disjoint from an s-subspace U. For the parameters $q = 2$, $v = 7$, $k = 3$, $d = 4$, and $s = 3$ the corresponding LMRD gives an example of cardinality 256. So far we are only able to prove an upper bound of 290.[3] So, we ask for tighter bounds in this specific case and for the general problem.

In Section 4 we have seen that the set of holes of a vector space t-partition has to be a q^r-divisible set of t-subspaces. This significantly restricts the possible types of vector space t-partitions and raises the question how tight the resulting restrictions are. For $q = 1$, the condition of q^r-divisibility is trivially satisfied in all cases. Indeed, we are not aware of any example of a hole configuration $\mathcal{N}$ of t-subsets which provably is not realizable as a vector space t-partition for $q = 1$, i.e., a partition of the set of t-subsets of a set V such that all parts of size t are given by $\mathcal{N}$.

Having determined the minimum possible cardinality of a q^r-divisible set of t-subspaces, for many parameters with $t \geq 2$, in Theorem 4.7, one can ask for the spectrum of possible cardinalities. For $t = 1$ the following is known:

Theorem 5.1 *([28, Theorem 12]) For the cardinality n of a q^r-divisible set $\mathcal{C}$ of 1-subspaces over $\mathbb{F}_q$ we have*

$$n \notin \left[(a(q-1)+b) \begin{bmatrix} r+1 \\ 1 \end{bmatrix}_q + a + 1, (a(q-1)+b+1) \begin{bmatrix} r+1 \\ 1 \end{bmatrix}_q - 1 \right],$$

where $a, b \in \mathbb{N}_0$ with $b \leq q-2$, $a \leq r-1$, and $r \in \mathbb{N}_{>0}$.

In other words, if $n \leq rq^{r+1}$, then n can be written as $a\left[\begin{smallmatrix} r+1 \\ 1 \end{smallmatrix}\right]_q + bq^{r+1}$ for some $a, b \in \mathbb{N}_0$.

For $q = 2$, $t = 2$, and $r = 1$ we remark that the possible cardinalities are given by $\mathbb{N}_{\geq 4}$. Examples of cardinality 4 and 6 are given by Lemma 4.4, Lemma 4.5 gives a construction for cardinality 5, and Lemma 4.9 gives a construction for cardinality 7, so that Lemma 4.10 continues these constructions to all cardinalities in $\mathbb{N}_{\geq 8}$. For other parameters there are gaps in the sets of possible cardinalities. For which parameters can these sets be completely determined? What is the second smallest cardinality? Can Theorem 5.1 be generalized, i.e., for which ranges do integer combinations of two base constructions explain all possible cardinalities? What is the largest cardinality n such that no q^r-divisible set of t-subspaces of cardinality n exists? This number was called *Frobenius number* for the special case $t = 1$ in [28]. Determine bounds on the Frobenius number.

Almost the same questions can be asked for vector space t-partitions. As for ordinary vector space partitions, the classification of all possible types is indeed a very hard problem if the dimension is not too small. However, for vector space t-partitions in $\mathbb{F}_2^7$ some improvements of the presented results seem to be achievable.

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[3] Since no vector space partition of $\mathbb{F}_2^6$ of type $3^1 2^{18} 1^2$ exists, every point P (outside of U) can be contained in at most 17 planes, which implies an upper bound of $\lfloor (127-7) \cdot 17/7 \rfloor = 291$. This upper bound can not be attained, since otherwise the argument from [30] gives a 4-divisible multiset of 3 points, which does not exist.

Triggered by the motivating example of $A_2(8,6;4) < 289$, we ask for a computer-free proof of $A_q(8,6;4) < (q^4+1)^2$. Nevertheless having just a very tiny numerical evidence, we state the following two rather strong conjectures in order to stimulate the search for counter examples.

Conjecture 5.2 $A_q(2k, 2k-2; k) = q^{2k} + 1$ *for each* $k \geq 4$.

We remark that the conjecture is true for the set case $q = 1$, while $A_1(6,4;3) = 2 = 1^6 + 1$ (slightly abusing notation).

Conjecture 5.3 *If* $\mathcal{P}$ *is a vector space* 2*-partition of* $\mathbb{F}_q^{2k-1}$ *of type* $k^{q^k+1}(k-1)^{m_{k-1}}2^\star$*, then* $m_{k-1} \leq q^{2k} - q^k$ *for all* $k \geq 4$.

Again the conjecture is true for the set case $q = 1$.

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